QUANTITATIVE METHODS FOR ECONOMICS I

Discipline Core Course

COURSE CODE: M23EC04DC

Postgraduate Programme in Economics

SELF LEARNING MATERIAL

SREENARAYANAGURU
OPEN UNIVERSITY

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The State University for Education, Training and Research in Blended Format, Kerala

SREENARAYANAGURU OPEN UNIVERSITY

Vision

To increase access of potential learners of all categories to higher education, research and training, and ensure equity through delivery of high quality processes and outcomes fostering inclusive educational empowerment for social advancement.

Mission

To be benchmarked as a model for conservation and dissemination of knowledge and skill on blended and virtual mode in education, training and research for normal, continuing, and adult learners.

Pathway

Access and Quality define Equity.

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DOCUMENTATION

Development of the Content
Dr. Anitha C.S, Dr. Christabell P. J.

Review
Content : Dr. Anitha C. S., Dr. G. Hariprakash
Format : Dr. I. G. Shibi
Linguistics : Dr. Anitha C. S.

Edit
Dr. Anitha C. S., Dr. G. Hariprakash

Scrutiny
Dr. Suraj G., Dr. Suchithra K.R., Yedu T. Dharan, Soumya V. D., Muneer K., Hima Chandran J.

Co-ordination
Dr. I. G. Shibi and Team SLM

Design Control
Azeem Babu T. A.

Cover Design
Jobin J.

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MESSAGE FROM VICE CHANCELLOR

Dear

I greet all of you with deep delight and great excitement. I welcome you to the Sreenarayanaguru Open University.

Sreenarayanaguru Open University was established in September 2020 as a state initiative for fostering higher education in open and distance mode. We shaped our dreams through a pathway defined by a dictum 'access and quality define equity'. It provides all reasons to us for the celebration of quality in the process of education. I am overwhelmed to let you know that we have resolved not to become ourselves a reason or cause a reason for the dissemination of inferior education. It sets the pace as well as the destination. The name of the University centres around the aura of Sreenarayanaguru, the great renaissance thinker of modern India. His name is a reminder for us to ensure quality in the delivery of all academic endeavours.

Sreenarayanaguru Open University rests on the practical framework of the popularly known "blended format". Learner on distance mode obviously has limitations in getting exposed to the full potential of classroom learning experience. Our pedagogical basket has three entities viz Self Learning Material, Classroom Counselling and Virtual modes. This combination is expected to provide high voltage in learning as well as teaching experiences. Care has been taken to ensure quality endeavours across all the entities.

The university is committed to provide you stimulating learning experience. The postgraduate programme in Economics is conceived to be a continuum of the UG programme in Economics as it has organic linkage with the content and the form of treatment. In fact is a progression of the finer aspects of theories and practices. The discussions are meant to arouse interest among the learners in understanding the discipline in the real context and therefore, the examples are drawn heavily from the real life experiences. The provision for empirical evidences integrated endeavour of the academic content makes this programme special and relevant. We assure you that the university student support services will closely stay with you for the redressal of your grievances during your studentship.

Feel free to write to us about anything that you feel relevant regarding the academic programme.

Wish you the best.

Regards,
Dr. P. M. Mubarak Pasha   01.02.2024

CONTENTS

MASTER OF ARTS ECONOMICS

Differential and Integral Calculus

BLOCK 1

UNIT 1
Concept of Sets

Learning Outcomes

After completing this unit, learner will be able to:

- understand the relation between sets
- analyse and interpret functions in real world applications
- apply linear functions in economics
- demonstrate concepts related to limits and continuity of functions

Background

The concept of sets serves as a fundamental building block in mathematics, providing a way to organise and categorise objects based on common properties. Understanding the relations between sets and operations performed on sets is essential for solving problems involving collections of elements.

Relations and functions introduce the idea of how elements in different sets are related to each other and how these relationships can be represented mathematically. This understanding lays the groundwork for analysing different types of functions, including constant, linear, quadratic, cubic, polynomial, exponential, and logarithmic functions. In economics, linear functions play a significant role in modelling relationships between variables such as demand, supply, cost, and revenue. Additionally, concepts such as limits, continuity, and derivatives are essential tools for analysing the behaviour of functions and optimising outcomes in economic models.

Keywords

Set, Function, Limits, Continuity, Derivative, Differentiation

Discussion

1.1.1 Concepts of Sets

- Representing collections of distinct elements

The concept of sets forms a fundamental foundation of mathematics, representing collections of distinct elements grouped together. A set can contain any kind of object, such as numbers, letters, or even other sets. Sets often serve as a basis for understanding of various mathematical structures and operations.

A well-defined collection of objects or elements is a set. A "well-defined collection" refers to a set where its members or elements are clearly defined and can be easily determined.

For example, consider the set of "Even Natural Numbers." This set is well-defined because it includes only those natural numbers that are divisible by 2 without leaving a remainder. Every member of this set is clearly determined: {2, 4, 6, 8, ...}.

On the other hand, a term like "Tall People" is not a well-defined collection by itself. What exactly constitutes "tall" is subjective and can vary from person to person, culture to culture. Hence, the set of "Tall People" lacks a clear and universally accepted definition, making it not well-defined in a mathematical sense.

- Used to model and analyse various economic phenomena

In economics, the concept of sets is applied extensively, particularly in the field of mathematical economics and microeconomics, where sets are used to model and analyse various economic phenomena. For example, sets are employed to represent consumer preferences, production possibilities, market demand, and supply curves, aiding in the formulation of economic models and theories that underpin decision-making and resource allocation.

1.1.2 Operations on Sets

Operations on sets encompass various procedures that can be applied to create new sets or modify existing ones. Common set operations include union, intersection, difference, and complement. These operations allow us to manipulate sets to extract meaningful information and insights, providing valuable tools in fields ranging from discrete mathematics to data analysis.

Union of Sets

If S_1 and S_2 are two sets then $S_1 \cup S_2$ is the set consists of all the elements that belong to either S_1 or S_2 or both.

Intersection of Sets

- Operations- Union, intersection, difference, complement of a set

If S_1 and S_2 are two sets then $S_1 \cap S_2$ is the set consists of all the elements that belong to both S_1 and S_2 .

For example, $S_1 = \{a, b, c\}$ and $S_2 = \{a, e, f\}$ then

$S_1 \cup S_2 = \{a, b, c, e, f\}$ and $S_1 \cap S_2 = \{a\}$.

Difference of Sets

If S_1 and S_2 are two sets then $S_1 - S_2$ is the set consists of all the elements that belong to S_1, but are not belong to S_2 .

For example, $S_1 = \{a, b, c\}$ and $S_2 = \{a, e, f\}$ then

$S_1 - S_2 = \{b, c\}$

Complement of a Set

The complement of a set is the set of all elements in a universal set that are not in the given set.

Let us denote the universal set as U, and the given set as S. The complement of set S, denoted as, S' or $\overline{S}$, consists of all elements that belong to U but do not belong to S.

For example, Consider the set $S = \{1,2,3,4,5,6\}$ as universe set and $S_1 = \{1,2,3\}$

Then the complement of $S_1 = \{4,5,6\}$

Ordered Pair

- Arranged in specific order

An ordered pair is a pair of objects or elements arranged in a specific order. The order in which the elements appear is significant and distinguishes one pair from another. Typically, ordered pairs are used to represent coordinates in a cartesian coordinate system.

Formally, an ordered pair (a, b) consists of two elements, a and b, where a is the first element and b is the second element. The order pair (a, b) is not the same as the order pair (b, a) unless $a = b$.

1.1.3 Relation between Sets

- Connections between elements of different sets

Relations between sets introduce connections or associations between elements from different sets. Relations are pivotal in analysing how elements interact and relate to each other in different contexts.

Equal Set

- Set contains identical elements

Two sets A and B are equal when they contain identical elements.

For example, $A = \{1, a, d, 4\}$ and $B = \{4, a, d, 1\}$ are equal sets.

Subset

The set A is a subset of B if every element of A is an element of B. i.e., A is a subset of B when $x \in A$ if and only if $x \in B$. Using the set inclusion symbol $\subset$ (contains in) $\supset$ (includes) we may write $A \subset B$ or $B \subset A$

For example, let $A = \{1,2\}$ is a subset of $B = \{1,2,4,5\}$

Cartesian Product

- Set of all ordered pairs

When considering two sets, A and B, the cartesian product (cross product) of A and B is defined as the set of all ordered pairs (a, b), where "a" belongs to set A and "b" belongs to set B. It is represented as $A \times B$. In simpler terms, $A \times B$ captures all possible combinations of elements from A and B in the form of ordered pairs.

For instance, if $A = \{a, b\}$ and $B = \{1, 2\}$, then the cartesian product $A \times B$ would consist of the following ordered pairs:

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

Similarly, if we consider the cartesian product $B \times A$, it would yield:

$$B \times A = \{(1,a),(2,a),(1,b),(2,b)\}$$

- Cartesian product involves ordered pairs

It is important to note that $A \times B \neq B \times A$. This distinction arises because the cartesian product involves ordered pairs, and thus, the order in which the sets are taken into account matters.

The cardinality of $A \times B$, denoted as $n(A \times B)$, is equal to the product of the cardinalities of sets A and B. In other words, the number of elements in $A \times B$ is given by $n(A) \times n(B)$.

1.1.4 Relations and Functions

Relation and function are two ways of establishing links between two sets.

Relation

Relations are a fundamental concept in mathematics that also finds practical application in various real-world scenarios. In economics, relations play a significant role in understanding how different variables and factors interact to influence economic outcomes.

- Interaction of different variables

Consider the concept of "Supply and Demand." The relationship between the quantity of a goods and services supplied by producers and the quantity demanded by consumers forms a critical aspect of market dynamics. This relationship is often represented graphically as the supply and demand curve, where the vertical axis represents the price, and the horizontal axis represents the quantity. The intersection point of these curves signifies the equilibrium price and quantity, where supply equals demand.

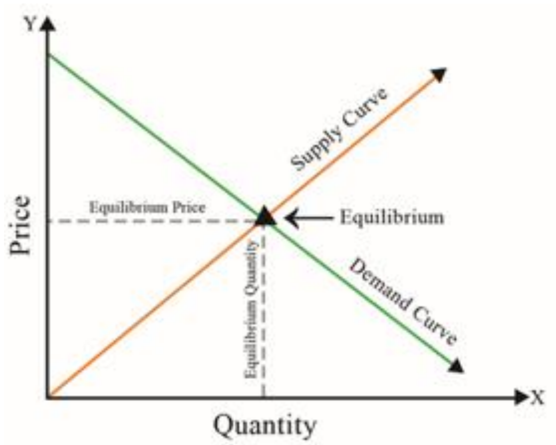

Fig 1.1.1 Supply and Demand Curve

If demand for a particular product increases due to changing consumer preferences, it can lead to a shortage at the current price, prompting producers to increase supply and potentially raise the price. Conversely, if supply decreases due to factors like scarcity of resources, the equilibrium price might rise, affecting consumer behaviour.

- Association between two or more sets of elements

In mathematics, a relation is a connection or association between two or more sets of elements. It describes how elements from one set are related to elements from another set based on a certain rule, property, or condition. Relations can involve various types of connections, such as comparisons, associations, or dependencies.

Let A and B are two sets, then any subset of $A \times B$ is called a binary relation from A to B. If $\mathcal{R}$ is a binary relation from A to B, then $\mathcal{R}$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Any subset of A$\times$ A is a binary relation on A.

Let $A = \{1,2,3,4\}$ and $B = \{w, x, y\}$

$\mathcal{R} = \{(1, \mathrm{w}), (2, \mathrm{y}), (3, y)\}$ is a relation from A to B.

Function

In mathematics, a function is a fundamental concept that describes a special type of relation between two sets, often denoted as A (domain) and B (codomain). A function assigns each element from the domain A to a unique element in the codomain B, following a specific rule or mapping. This rule ensures that each input element has exactly one corresponding output element.

- Special relation between domain and codomain

Formally, a function f from domain A to codomain B can be represented as $f: A \rightarrow B$,where $f(a)$ represents the output element corresponding to the input element a from domain A. The set of all images of A under the function f, denoted by $f(A)$, is called the range of f. For example, let $A = \{1,2,3\}$,

$B = \{a, b, c, d\}$.

A function $f: A \rightarrow B$ is $f = \{1, a) (2, c) (3, d)\}$.

But $f = \{1, a) (2, b)(2, c) (3, d)\}$ is not a function because the element 2 has more than one image in B.

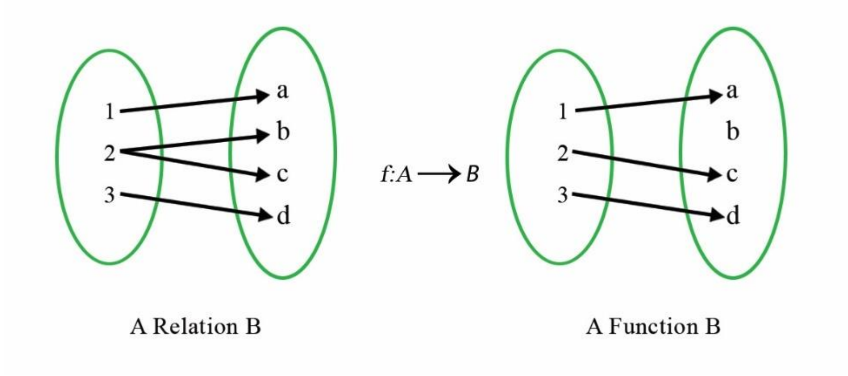

Fig 1.1.2 A Relation B   Fig 1.1.3 A Function B

- Extensively used function in Economics-Demand function

A is the domain, B is the co-domain and $\{a, c, d\}$ is the range of the function f. An important function which is extensively used in economics is the demand function which expresses quantity demanded of a commodity is a function of its price, other factors being held constant. If D_x is the quantity demanded of commodity x and P_x is its price, then $D_x x = f\ (P_x)$ is a function.

Example 1

Let $A = \{2, 3, 4\}$ and $B = \{4,5\}$.

Find a) $A \times B$ $b)\ B \times A$ $c)\ B^2$

Solution:

$a) A \times B = \{(2,4)\ (2,5)\ (3,4)\ (3,5)\ (4,4)\ (4,5)\}$

$b)\ B \times A = \{(4,2)\ (4,3)\ (4,4)\ (5,2)\ (5,3)\ (5,4)\}$

$c)\ B^2 = \{(4,4)(4,5)(5,4)(5,5).$

Example 2

Let A and B be sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A|$?

Solution:

Let $|A| = m$ and $|B| = n$, we know that, $|A \times B| = 2^{mn}$

$= 2^{3m} = 4096 = 2^{12}$ since $|B| = 3$. Equating the power of 2,

we get $3m = 12 \Rightarrow m = 4. \therefore\ |A| = 4.$

Example 3

Check whether the following are functions or not

1) $y = \sqrt{x}$ 2) $y = x^3 + 5x$

3) $R = \{(2,5)\ (5,2)\ (2,1)\}$

Solution:

1. It is not a function because if $x = 4$, then $y = \sqrt{4} = 2$ *or* -2 i.e., 4 is mapped into 2 and -2
2. It is a function since for each value of x, there is a unique value for y
3. R is not a function because the element 2 is associated with elements 5 and 1 of y.

1.1.5 Different Types of Functions

- Range consists of only one element

a. Constant Function

A function whose range consists of only one element is called a constant function.

For example, $f(x) = 7$ is a constant function.

- Mathematical function exhibits relationship between input and output

b. Linear Function

A linear function is a mathematical function that exhibits a specific type of relationship between its input (independent variable) and output (dependent variable). The defining characteristic of a linear function is that it has a constant rate of change, resulting in a straight-line graph when plotted on a cartesian coordinate system.

Mathematically, a function f is considered linear if it can be represented in the form:

$$y = f(x) = a + bx$$

where x is the input or independent variable, $f(x)$ is the output or dependent variable, a and b are constants. In real life it is used to find speed, distance and time of a moving object.

1.1.5.1 Linear Functions in Economics

Linear functions play a fundamental role in economics and are used to model various relationships and behaviours within economic systems. In economics, a linear function is a mathematical representation of a linear relationship between two variables.

Demand and supply functions are linear functions. The linear demand function with price as the only independent variable is written as $Q_d = a - bP$ where Q_d is the quantity demanded,

P is the price, a and b are constants. The minus sign before coefficient b indicates that quantity demanded of a commodity is negatively related with price of the commodity. That is, if price of a commodity falls, its quantity demand increases and vice versa. Supply function $Q_s = c + dP$, where Q_s is the quantity supplied, P is the price, a and b are constants, is a linear function.

- Linear relationship between two variables

Linear functions can represent cost functions, production functions, and revenue functions. For instance, the Total Cost (TC) of producing a certain quantity of goods may be represented as, $TC = a + bQ$, where a is the fixed cost, b is the variable cost per unit of production, and Q is the quantity produced.

In a similar vein, linear functions are invaluable for analysing revenue generation, as they allow for easy assessment of how total revenue and marginal revenue change with sales levels. Additionally, when modeling investment and saving functions with linear equations, it becomes more accessible to analyse the relationship between income and the decisions individuals or entities make regarding saving or investing their financial resources.

- More accessible to analyse relationship between variables

1.1.5.2 Quadratic Function

A quadratic function is a type of mathematical function that follows a specific algebraic form: $f(x) = ax^2 + bx + c$, where a, b, and c are constants with $a \neq 0$. Quadratic functions are named so because they involve the second power (square) of the variable x, making the highest power of x in the function. This results in a graph that typically forms a parabola.

$f(x) = ax^2 + bx + c$

The coefficients a, b, and c determine the behaviour and shape of the quadratic function. The coefficient a has a particularly significant role in shaping the parabolic graph.

When a is positive, the quadratic function opens upwards, resembling an upward-facing U-shape. This type of quadratic function is often referred to as a "convex" quadratic function. As x moves away from the vertex of the parabola (the lowest point), the function values increase, creating a bowl-like curve.

- Convex and Concave functions

On the other hand, when a is negative, the quadratic function opens downwards, forming a downward-facing U-shape. On the other hand, when a is negative, the quadratic function opens downwards, forming a downward-facing U-shape. This type of quadratic function is often called a "concave" quadratic function. As x moves away from the vertex, the function values decrease, resulting in a curve that resembles an inverted bowl.

- Quadratic function can model cost, revenue or profit

Quadratic functions find application in various fields, such as physics, engineering, and economics, for modeling various phenomena. In physics, they can represent trajectories of objects in projectile motion. In economics, quadratic functions can model cost, revenue, or profit as a function of quantity produced.

Quadratic functions are commonly used to model various cost functions in economics. For instance, the quadratic cost function $C(q) = aq^2 + bq + c$ can represent a firm's total cost, where q is the quantity of output produced. This helps in analysing how costs change with the level of production.

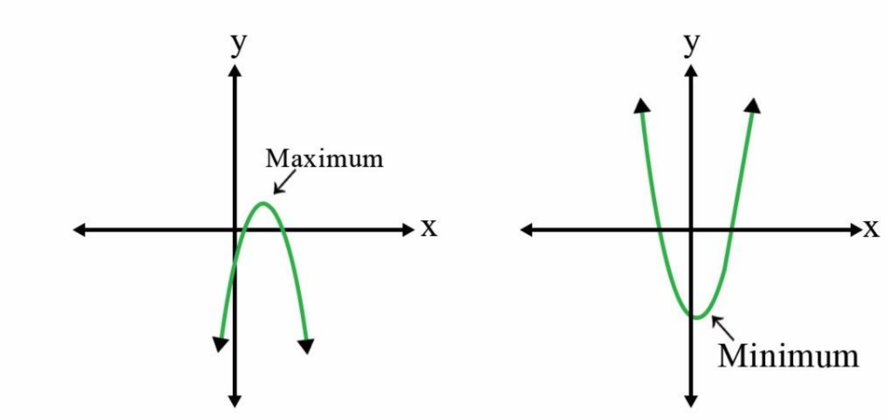

Fig 1.1.4 Concave Function   Fig 1.1.5 Convex Function

1.1.5.3 Polynomial Function

- Represented as the sum of a finite number of terms

A polynomial function in x is a mathematical function that can be represented as the sum of a finite number of terms, each in the form cx^n, where c is a constant and n is a non-negative integer. For instance, $f(x) = 4x^2 - 6$ is a polynomial function because it can be expressed as a sum of two terms $4x^2$ and -6.

Polynomial functions come in various degrees based on the highest exponent of x that appears in the terms. A third-degree polynomial function is called a cubic function because the highest exponent is 3.

For example, $g(x) = 3x^3 - 2x^2 + 5x - 1$ is a cubic polynomial.

- Used to model economic growth patterns

Polynomial function is used to find the slope of a hill, curve of a bridge etc. In economics the use of polynomial function is to model economic growth patterns, in stock market to see how the prices will vary over time.

1.1.5.4 Exponential Function

- Express relationship between input and output

An exponential function is a significant mathematical concept that takes the form $y = a^x$, where x serves as the independent variable or exponent, and a > 1 is a constant. The essence of an exponential function lies in its ability to express relationships between input and output by repeatedly multiplying an initial value to obtain the output corresponding to any given input.

Exponential functions find wide application in various real-world scenarios due to their remarkable properties. They are instrumental in understanding growth and decay phenomena across different domains, such as populations, financial investments, prices, and more.

- Base of an exponential function 'e'

The utilisation of a special number called $'e'$ as the base of an exponential function enhances the functionality of this concept. $'e'$, known as a transcendental number, is approximately equal to 2.71828. This number holds a unique property such that the slope of the exponential. Function $y = e^x$ at any point on its graph is precisely equal to the function's value at that point.

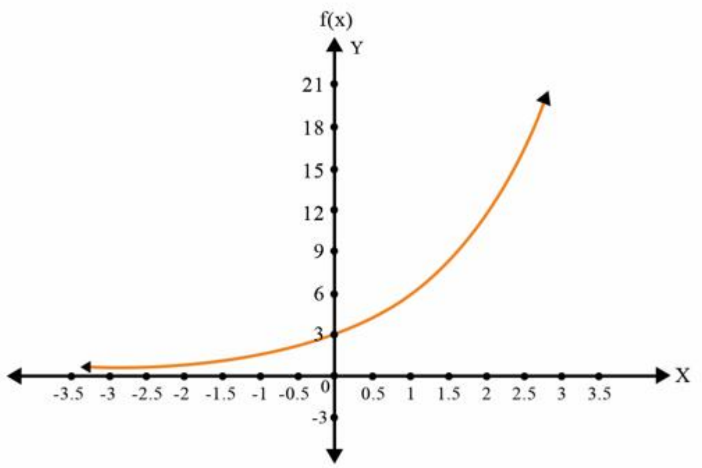

Fig 1.1.6 Exponential Function

The shape of the exponential function's curve varies based on the value of x. When x is positive, the curve exhibits rapid growth, while for negative x, it represents a decaying trend. The behaviour of exponential functions provides insight into how phenomena change over time, making them a fundamental component of mathematics with profound real-world implications.

- Provides insight into how phenomena changes over time

Exponential functions are invaluable tools in analysing growth and decay patterns. For example, when studying population growth, an exponential function can help to predict how a population will expand over time. Likewise, in the financial realm, exponential functions aid in understanding how investments can accumulate or diminish based on compounding interest or depreciation rates.

- Aid in understanding investment

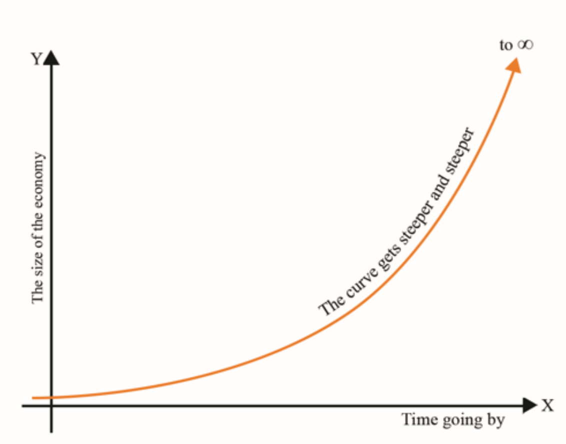

Fig 1.1.7 Exponential Growth Curve

An exponential growth curve gets steeper and steeper at an ever-increasing rate, has no limit, and approaches, but never reaches, infinity.

- Exponential growth curve never reaches infinity

1.1.5.5 Logarithmic Function

When a variable is expressed as a function of the logarithm of another variable, the function is the logarithmic function. To differentiate a logarithmic function, we will use the rules of differentiation and focus on the logarithm function.

The general form of a natural logarithmic function is $f(x) = \ln\big(g(x)\big)$ where $g(x)$ is the argument of the logarithm and the general form of common logarithmic function is

$f(x) = \log_{10} g(x)$.

The derivative of natural logarithmic function

$f(x) = \ln\big(g(x)\big)$ with respect to x is

$$\frac{d}{dx} \ln\,(g(x) = \frac{1}{g(x)}\ \frac{d}{dx}\big(g(x)\big) = \frac{g'(x)}{g(x)}$$

The derivative of common logarithmic function

$f(x) = \log_{10} g(x)$ with respect to x is

$$\frac{d}{dx} \log_{10} g(x) = \frac{1}{g(x)}\ \ \frac{d}{dx}\big(g(x)\big)\frac{1}{ln\ 10} = \frac{g'(x)}{g(x)}\ \frac{1}{ln\ 10}$$

The case of base b

$$\frac{d}{dx} ln_b\, g(x) = \frac{1}{g(x)}\ \frac{d\ g(x)}{dx}\frac{1}{lnb} = \frac{g'(x)}{g(x)}\ \frac{1}{ln\ b}$$

For example,

1) $g(x) = \ln(3x^2)$ the derivative of $g(x)$

$$= \frac{1}{3x^2}\frac{d(3x^2)}{dx} = \frac{6x}{3x^2} = \frac{2}{x}$$

2) $g(x) = log_2(3x^2)$ the derivative of $g(x)$

$$= \frac{1}{3x^2}\frac{d(3x^2)}{dx}\frac{1}{ln\ 2} = \frac{6x}{3x^2}\frac{1}{ln\ 2} = \frac{2}{x}\frac{1}{ln\ 2}$$

Note:

$$\frac{d}{dx}\ \log_{10} x = \frac{1}{x \ln 10}$$

$$\frac{d}{dx}\ \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\ \ln x = \frac{1}{x}$$

- Applications in modelling process

Logarithmic functions find applications in various economic contexts, notably in modelling processes with diminishing returns or growth rates, such as in analysing compound interest, population growth, or production functions with decreasing marginal returns to inputs.

The graph of the logarithmic function $log_b x$ illustrates a distinct behaviour based on the value of b, which serves as the base of the logarithm. The graph exhibits certain characteristics determined by the properties of logarithmic functions and the chosen base.

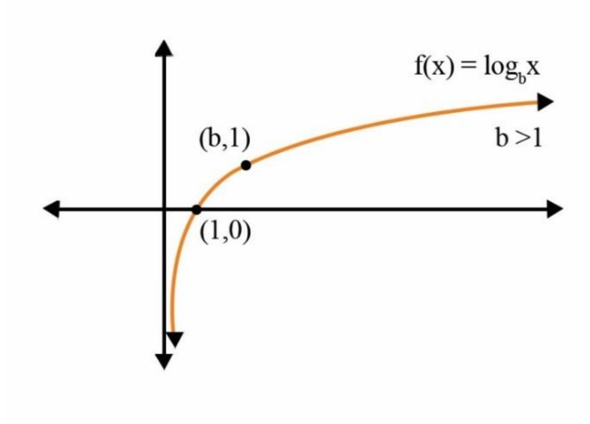

Fig 1.1.8 Logarithmic Function

1.1.6 Limit of a Function

- Describes the behaviour of the function

The limit of a function is a value of a function as the input of the function get closer or approaches to some number. It is a fundamental concept in calculus that describes the behaviour of the function as its input approaches a specific value. It quantifies how the function approaches a particular value or becomes arbitrarily close to it, without necessarily reaching that value.

Right Hand Limit

- Obtained from the right hand of the independent variable

When the limit of function is obtained from the right hand of the independent variable, then it is called Right Hand Limit $(R.H.L.)$

$$R\ H\ L = \lim_{x \to a^+} f(x) = l_1$$

Left Hand Limit

- Obtained from the left hand of the independent variable

When the limit of function is obtained from the left hand of the independent variable, then it is called Left Hand Limit (L.H.L.)

$$L\ H\ L = \lim_{x \to a^-} f(x) = l_1$$

If at $x = a$ both RHL and LHL exist and equal then the limit of the function will exist.

i.e., $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$

Here l is called the limit of the function and is expressed as $\lim_{x \to a} f(x) = l$

Working rules for finding Right Hand Limit and Left-Hand Limit

i) To obtain the limit of right and left hand, replace x variable with $(x + h)$ and $(x - h)$ respectively in the function.
ii) Thus, obtained function x, should be replaced with point (assume a)
iii) Now at $h \to 0$ determine the limit of function.

- Convenient technique for finding limits

Absolutely, we can often determine the limit of a function directly at a specific point using direct substitution. This approach involves simply plugging the value into the function and evaluating it to find the limit. For example, if you want to find the limit of a function $f(x)$ as x approaches a specific point c, you can do so by evaluating $f(c)$ directly. If the function is well-defined and continuous at that point, this method is straightforward and efficient, providing a quick way to calculate the limit without going through the process of calculating left and right limits. It is a convenient and intuitive technique for finding limits in many cases.

- Significant applications in economics

The concept of a limit in calculus finds significant application in economics, particularly in understanding the behaviour and dynamics of economic variables. For instance, when examining the demand for a specific product, the limit of the demand function as the price approaches zero helps to determine the elasticity of demand, a critical measure for pricing and market strategy. Additionally, limits are instrumental in analysing production functions and cost curves, aiding in assessing the efficiency and optimal input levels for a firm. As economic scenarios evolve, understanding the limits of various economic functions allows for a precise grasp of how variables interact and change, enabling more informed decision-making and policy formulation within the dynamic economic landscape.

Example 4

Find the value of

a. $\lim_{x \to 0} (7x^2 - 5x + 1)$

b. $\lim_{x \to 0} \frac{(e^x - 1)}{e^x}$

c. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

d. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 6x + 8}$

e. $\lim_{x \to \infty} \frac{9x^2 + 3x + 7}{5x^2 + 2x + 1}$

Solution

a. $\lim_{x \to 0} (7x^2 - 5x + 1) = 7 \times 0 - 5 \times 0 + 1 = 1$

b. $\lim_{x \to 0} \frac{(e^x - 1)}{e^x} = \lim_{x \to 0} \frac{e^x\left(1 - \frac{1}{e^x}\right)}{e^x} = \lim_{x \to 0} \left(1 - \frac{1}{e^x}\right) = 1 - \frac{1}{1}$

$= 1 - 1 = 0$

c. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{(x^2 + x + 1)}{(x+1)}$

$= \frac{(1^2 + 1 + 1)}{(1 + 1)}$

$= \frac{3}{2}$

d. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 6x + 8} = \lim_{x \to 2} \frac{(x-3)(x-2)}{(x-4)(x-2)} = \lim_{x \to 2} \frac{(x-3)}{(x-4)}$

$= \frac{(2-3)}{(2-4)} = \frac{1}{2}$

e. $\lim_{x \to \infty} \frac{9x^2 + 3x + 7}{5x^2 + 2x + 1} = \lim_{x \to \infty} \frac{x^2\left(9 + \frac{3}{x} + \frac{7}{x^2}\right)}{x^2\left(5 + \frac{2}{x} + \frac{1}{x^2}\right)} = \frac{9 + 0 + 0}{5 + 0 + 0} = \frac{9}{5}$

Example 5

$f(x)$ is defined as $f(x) = x$ $when\ 0 \leq x < \frac{1}{2}$ [LHL]

$= 0\ \ when\ x = \frac{1}{2}$

$= 1 - x\ ,when \frac{1}{2} < x \leq 1$[RHL]

Find $\lim_{x \to \frac{1}{2}} f(x)$

Solution

$$RHL = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

$$= \lim_{h \to 0} 1 - \left(\frac{1}{2} + h\right)$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$LHL = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}$$

$$\therefore \lim_{x \to \frac{1}{2}} f(x) = \frac{1}{2}$$

Example 6

$f(x)$ is defined as $f(x) = x \quad when\ 0 \leq x < 1$

$$= 2 \quad when\ x = 0$$

$$= 2 - x, \ when\ x \geq 1.$$

Find $\lim_{x \to 0} f(x)$

Solution

$$RHL = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} 2 - (1 + h)$$

$$= 2 - 1 = 1$$

$$LHL = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} (1 - h) = 1$$

$$\therefore \lim_{x \to 0} f(x) = 1$$

1.1.7 Continuity of a Function

Continuity of a function is a crucial concept in calculus that describes the smooth and unbroken behaviour of the function

- No interruptions or breaks in its graph of continuity function

without any abrupt jumps, gaps, or disruptions. A function is considered continuous at a specific point if its value at that point matches the limit of the function as it approaches that point. In simpler terms, a function is continuous when there are no interruptions or breaks in its graph.

Formally, a function $f(x)$is said to be continuous at $x = a$, if the following conditions are met:

- Conditions of continuity function

1. The function $f(x)$ is defined at $x = a$.

2. The limit of $f(x)$ as x approaches a exists.

3. $\lim_{x \to a} f(x) = f(a)$

For example, consider the function $f(x) = 2x + 3$. This function is continuous everywhere because it is a linear function, and there are no breaks or gaps in its graph. At any point x = a, the value of $f(x)$ matches the limit of the function as x approaches a, satisfying the conditions of continuity.

On the other hand, the function $g(x) = \frac{1}{x}$ is not continuous at $x = 0$. The reason is that $g(x)$ becomes unbounded as x approaches 0 from both sides (positive and negative). The function is not defined at $x = 0$, and the limit of $g(x)$ as x approaches 0 is infinite. Thus, the conditions for continuity are not satisfied at $x = 0$, making $g(x)$ discontinuous at that point.

Example 7

Prove that the function $f(x) = x^2 + 1$ is continuous at $x = 2$.

Solution

$$\lim_{x \to 2} f(x) = 2^2 + 1 = 5$$

$f(2) = 5$. $\therefore \lim_{x \to 2} f(x) = f(2)$. So, the function is continuous.

Example 8

Prove that the function $f(x) = \frac{1}{x-2}$ is discontinuous at $x = 2$

Solution

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{1}{x-2} = \frac{1}{0} = \infty$$

Limit does not exist $x = 2$.

$\therefore$ the function is discontinuous at $x = 2$.

Example 9

$f(x)$ is defined as $f(x) \ = 2 \ when\ x < 0$

$= x \ when\ x \geq 0.$

Prove that the function is discontinuous at $x = 0$.

Solution

$RHL = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} 0 + h = 0$

$LHL = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = 2$

Since $RHL \neq LHL$, limit does not exist, $x = 0$ $\therefore$ the function is discontinuous at $x = 0$.

Example 10

$f(x)$ is defined as $f(x) \ = x^2 \ if\ x \neq 1$

$= 2 \ if\ x = 1.$

Prove that the function is discontinuous at $x = 1$.

Solution

$RHL = \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} (1 + h)^2 = 1$

$LHL = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} (1 - h)^2 = 1$

Since $RHL = LHL$, $\lim_{x \to 1} f(x) = 1$ exists.

Now $f(1) = 2 \neq \lim_{x \to 1} f(x)$

$\therefore$ the function is not continuous at $x = 1$

Example 11

Find the value of k so that the function,

$f(x) \ = \frac{x^2-16}{x-4} \ when\ x \neq 4$

$= k \ \ when\ x = 4$ is continuous at $x = 4$.

Solution

$$RHL = \lim_{x \to 4^+} f(x) = \lim_{h \to 0} f(4+h) = \lim_{h \to 0} \frac{(4+h)^2 - 16}{(4+h) - 4}$$

$$= \lim_{h \to 0} \frac{16 + 8h + h^2 - 16}{h}$$

$$= \lim_{h \to 0} \frac{8h + h^2}{h} = \lim_{h \to 0} 8 + h = 8$$

$$LHL = \lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4-h)$$

$$= \lim_{h \to 0} \frac{(4-h)^2 - 16}{(4-h) - 4} = \lim_{h \to 0} \frac{16 - 8h + h^2 - 16}{-h}$$

$$= \lim_{h \to 0} \frac{-8h + h^2}{-h} = \lim_{h \to 0} 8 - h = 8$$

$RHL = LHL.$ $\lim_{x \to 4} f(x) = 8$ exists.

Given the function is continuous at $x = 4$

$\therefore RHL = LHL = f(4)$

Given $f(4) = k$, $\therefore k = 8$

1.1.8 Derivative of a Function

- Derivatives tell how quickly a function is changing

The derivative of a function is a fundamental concept in calculus that represents the rate at which the function's output changes with respect to its input. In simpler terms, it measures how the function value responds to a small change in the input variable. The derivative provides information about the slope of the function's graph at a particular point, indicating whether the function is increasing or decreasing and at what rate.

Differentiation is the process of finding the derivative of a function. Let y is a function of x where x is an independent variable. This is denoted as $y = f(x)$. Then the derivative of this function can be defined as follows.

- $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

Let Δy and Δx be small increment or change in y and x respectively, then $\frac{\Delta y}{\Delta x}$ is the incremental ratio. The value of incremental ratio when Δx is very small is called the differential coefficient or derivative of y with respect to x, and is denoted by $\frac{dy}{dx}$.

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

It is denoted as y' or $f'(x)$ or $\frac{dy}{dx}$.

Mathematically, the derivative is defined as the limit of the average rate of change as the interval between two points on the function's graph approaches zero. It is formally expressed as:

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists and h is a small increment in x.

Alternatively, if the function $f(x)$ is expressed as a formula, the derivative can be found using differentiation rules. These rules provide shortcuts for finding the derivative of different types of functions, including constants, powers, sums, products, quotients, and more.

- Tool for analysing and making decisions

Derivatives have various applications in mathematics, science, and engineering. They are used to analyse the rate of change, determine maximum and minimum points of functions, understand growth and decay processes, and much more. The concept of the derivative is foundational to calculus and plays a pivotal role in understanding the behaviour of functions and their relationships. The derivative of a function in economics is pivotal for understanding and optimizing behaviour in markets and production. It helps ascertain marginal concepts like marginal cost, revenue, and utility, guiding decision-making for firms and consumers, and ultimately influencing market equilibrium and resource allocation.

1.1.9 Rules of Differentiation

Rule 1. Differentiation of a power of a function

Derivative of the function $y = x^n$ is $\frac{dy}{dx} = nx^{n-1}$

Rule 2. Multiplication by a constant.

If $y = A\,x^n$, where A is a constant, then $\frac{dy}{dx} = A\,nx^{n-1}$

Rule 3. Derivative of a sum.

If $y = u + v$ where $u = f(x)$ and $v = g(x)$ are functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

For example, if $y = 2x^2 + 4x$, then

$$\frac{dy}{dx} = \frac{d\ (2x^2)}{dx} + \frac{d(4x)}{dx} = 2\,\frac{d(x^2)}{dx} + 4\frac{dx}{dx}$$

$$= 2 \times 2x + 4 = 4x + 4$$

Rule 4. Derivative of a constant.

If $y = f(x) = c,$ where c is a constant then $\frac{dy}{dx} = \frac{dc}{dx} = 0,$ derivative of a constant function is zero

Rule 5. Derivative of a product.

The derivative of the product $y = uv$, where $u = f(x)$ and $v = g(x)$ is

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = uv' + vu'$$

Rule 6. Derivative of a quotient.

The derivative of the quotient $y = \frac{u}{v}$, where $u = f(x)$ and $v = g(x)$

$$\frac{d(\frac{u}{v})}{dx} = \frac{v\,d(u) - u\,d(v)}{v^2}$$

Rule 7. Differentiation of a function of a function (chain rule).

If y is a function u, $y = f(u)$, and u is a function of x, $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 12

Find $\frac{dy}{dx}$ for the following functions

1) $y = x^5 - 4x^3 + 8x - 7$ 2) $y = 6x^{\frac{1}{3}} + 2e^x$

3) $y = x^6 + e^{3x}$

Solution

1) $\frac{dy}{dx} = 5x^4 - 4 \times 3 \times x^2 + 8 = 5x^4 - 12x^2 + 8$

2) $\frac{dy}{dx} = 6 \times \frac{1}{3} x^{\frac{1}{3}-1} + 2e^x = 2x^{-\frac{2}{3}} + 2e^x$

3) $\frac{dy}{dx} = 6 \times x^5 + 3e^{3x} = 6x^5 + 3e^{3x}$

Example 13

Find $\frac{dy}{dx}$ for the following functions

1) $y = \frac{e^x}{x}$ 2) $y = \frac{x}{x+5}$ 3) $y = \frac{5x^2+6x+7}{2x^2+3x+4}$

Solution

1) $\frac{dy}{dx} = \frac{xe^x - e^x}{x^2}$

2) $\frac{dy}{dx} = \frac{(x+5)-x}{(x+5)^2} = \frac{5}{(x+5)^2}$

3) $\frac{dy}{dx} = \frac{(2x^2+3x+4)(10x+6)-(5x^2+6x+7)(4x+3)}{(2x^2+3x+4)^2}$

$= \frac{3x^2+12x+3}{(2x^2+3x+4)^2}$

Example 14

Find $\frac{dy}{dx}$ for the following functions

1) $y = (3x^2 + 1)(x^3 + 2x)$

2) $y = x^2 e^{2x}$

3) $y = (ax^2 + bx)(cx^4 + dx)$

Solution

1) $\frac{dy}{dx} = (3x^2 + 1)(3x^2 + 2) + (x^3 + 2x)(6x)$

$= 14x^4 + 21x^2 + 2$

2) $\frac{dy}{dx} = x^2 e^{2x} \times 2 + e^{2x} \times 2x = e^{2x}(2x^2 + 2x)$

3) $\frac{dy}{dx} = (ax^2 + bx)(4cx^3 + d) + (2ax + b)(cx^4 + dx)$

$= 6acx^5 + 3adx^2 + 5bcx^4 + 2bdx$

Example 15

Find $\frac{dz}{dx}$ for the following functions

1) $z = y^2 + 1,\ y = x^2 + 3x$

2) $z = t^2 + 1,\ \ t = x + 1$

3) $z = 2w^2 + 1,\ w = 3y^2,\ \ y = 2x + x^3$

Solution

1) $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 2y(2x + 3) = 4xy + 6y$

$= 4x(x^2 + 3x) + 6(x^2 + 3x)$

$= 4x^3 + 18x^2 + 18x$

2) $\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx} = 2t \times 1 = 2t$

3) $\frac{dz}{dx} = \frac{dz}{dw}\frac{dw}{dy}\ \frac{dy}{dx} = 4w \times 6y(2 + 3x^2)$

$= 24wy(2 + 3x^2)$

$= 24 \times 3y^2 \times y(2 + 3x^2)$

$= 72(2x + x^3)^3(2 + 3x^2)$

Summarised Overview

A well-defined collection of objects or elements is a set. Common set operations include union, intersection, difference, and complement. A relation between two sets, A and B, is often represented as a set of ordered pairs (a, b), where "a" is an element from set A and "b" is an element from set B.

Function f from domain A to codomain B can be represented as $f: A \rightarrow B$,where $f(a)$ represents the output element corresponding to the input element a from domain A. Function f is considered linear if it can be represented in the form: $y = f(x) = a + bx$. A quadratic function is a type of mathematical function that follows a specific algebraic form: $f(x) = ax^2 + bx + c$, where a, b, and c are constants with $a \neq 0$.

A polynomial function in x is a function that can be represented as the sum of a finite number of terms, each in the form cx^n, where c is a constant and n is a non-negative integer. An exponential function is a significant mathematical concept that takes the for $y = a^x$, The inverse function of the exponential function $y = a^x$ with base a is the logarithmic function, If at $x = a$ both RHL and LHL exist and equal then the limit of the function will exist. A function $f(x)$is said to be continuous at $x = a$ if satisfies following conditions

1. The function $f(x)$ is defined at $x = a$.
2. The limit of $f(x)$ as x approaches a exists.
3. $\lim_{x \to a} f(x) = f(a)$

Differentiation is the process of finding the derivative of a function. Let y is a function of x where x is an independent variable.

Assignments

1. Find the value of $\lim_{x \to 0} \left(\frac{x^2 - 2ax + a^2}{x - a}\right)$
2. Examine the continuity of the function $f(x) = \frac{1}{x},\ x \neq 0$

   $= 3\ x = 0$
3. If $y = (ax)^m + \left(\frac{b}{x}\right)^n$, find $\frac{dy}{dx}$
4. Determine the differential coefficient of $2\sqrt{x} + 8 \log_e x\ + 2\ \log_a x$

5. If $y = x^y$ prove that $x\frac{dy}{dx} = \frac{y^2}{1-ylogx}$
6. If $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$, find $\frac{dy}{dx}$
5. If $ax^2 + 2hxy + by^2 = 0$, find $\frac{dy}{dx}$

Suggested Reading

1. Dowling T. Edward (2000), *Schaum's Outline of Introduction to Mathematical Economics*, McGraw Hill, New York.
2. Yamane, Taro. (2012). *Mathematics for Economists: An Elementary Survey*. New Delhi: Prentice Hall of India.

Reference

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
2. Chiang, A.C. (2008), *Fundamental Methods of Mathematical Economics*, McGraw Hill, New York.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

UNIT 2
Derivatives

Learning Outcomes

After completing this unit, learner will be able to:

- understand higher order derivatives
- apply higher order derivatives to analyse the behaviour of functions
- use higher order derivatives for finding economic concepts.

Background

Higher order derivatives refer to the successive differentiation of a function more than once, revealing insights into the rate at which the rate of change is changing. A fundamental understanding of higher order derivatives, particularly in the context of logarithmic and exponential functions as well as implicit functions, is essential for analysing various economic models and phenomena. Logarithmic differentiation is used in modelling diminishing returns or elasticities, where changes in variables are proportional rather than constant. Exponential differentiation is crucial for studying processes characterised by continuous growth or decay, such as population growth or depreciation of assets. Understanding implicit differentiation allows for the analysis of relationships between variables that are implicitly defined, offering insights into economic interdependencies that may not be explicitly stated. These mathematical tools enable economists to make precise predictions, optimise functions, and comprehend the intricate dynamics within economic systems, providing a foundation for informed decision-making and policy formulation.

Keywords

Higher Order Derivatives, Logarithmic Function, Exponential Function, Implicit Function

Discussion

1.2.1 Higher Order Derivatives

- Successive derivatives of a function

Higher-order derivatives refer to successive derivatives of a function beyond the first or second derivative. The first order derivative measures how a function changes at a specific point, representing the slope of the tangent line to the graph of the function at that point. Higher order derivatives, on the other hand, capture information about the rate at which the derivative itself changes with respect to the independent variable.

- Rate at which derivative itself changes

Let the function be $y = f(x)$, differentiating y with respect to x is first derivative of y, $\frac{dy}{dx}$ or $f'(x)$, with respect to the independent variable x. The second derivative of y with respect to x is $\frac{d^2y}{dx^2}$ or $f''(x)$. We can differentiate it again and is denoted as $\frac{d^3y}{dx^3}$ or $f'''(x)$ and so on. Higher-order derivatives are utilised to analyse how changes in various economic parameters, such as production, demand, or cost functions, affect the rates of change, curvature, and overall behaviour of economic models, offering valuable insights into optimisation, market dynamics, and decision-making processes.

For example, if $y = x^4$ its higher order derivatives are

$$\frac{dy}{dx} = 4x^3, \quad \frac{d^2y}{dx^2} = 12x^2, \quad \frac{d^3y}{dx^3} = 24x, \quad \frac{d^4y}{dx^4} = 24, \quad \frac{d^5y}{dx^5} = 0$$

Example 1

If $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, find, $\frac{d^3y}{dx^3}$

Solution

$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

$f'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$

$f''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$

$f'''(x) = 60ax^2 + 24bx + 6c$

Example 2

The total cost function of a firm is given as $TC = 100 + 10\,x^2$ and the total revenue function is given as $TR = 80x$. Find the profit maximising level of output?

Solution

Profit $P = TR - TC, \quad TR = 80x, \quad TC = 100 + 10x^2$

Profit $P = 80x - (100 + 10x^2) = 80x - 100 - 10x^2$

Profit is maximum, when $\frac{d^2p}{dx^2} < 0$

$\frac{dp}{dx} = \frac{d(80x-100-10x^2)}{dx}$

$= 80 - 20x = 0 \Rightarrow 20x = 80 \Rightarrow x = 4$

$\frac{d^2p}{dx^2} = \frac{d^2}{dx^2}(80 - 20x) = -20 < 0$

Profit maximising level of output is at $x = 4$.

Example 3

The cost function of a firm producing x commodities is given as $2x^2 - 60x$. How many units of x to be produced to minimise the cost?

Solution

$\frac{dTc}{dx} = \frac{d(2x^2-60x)}{dx} = 4x - 60 = 0 \Rightarrow x = \frac{60}{4} = 15$

Cost is minimised, when $\frac{d^2TC}{dx^2} > 0$

$\frac{d^2Tc}{dx^2} = \frac{d^2}{dx^2}(4x - 60) = 4 > 0$

The first and second order conditions for minimisation function are satisfied and therefore the cost minimising output is at $x = 4$

Example 4

If $y = \frac{7x^2}{x-1}$, prove that $\frac{d^2y}{dx^2} = \frac{14}{(x-1)^3}$

Solution

$$y = \frac{7x^2}{x-1}$$

Differentiating with respect to x by using quotient rule

$$\frac{dy}{dx} = \frac{(x-1)\times 14x - 7x^2}{(x-1)^2}$$

$$= \frac{14x^2 - 14x - 7x^2}{(x-1)^2}$$

$$= \frac{7x^2-14x}{(x-1)^2}$$

Differentiating again with respect to x

$$\frac{d^2y}{dx^2} = \frac{(x-1)^2 \times (14x-14) - (7x^2-14x)2(x-1)}{(x-1)^4}$$

$$= \frac{(x-1)[(x-1)\times(14x-14)] - (x-1)[(7x^2-14x)\times 2]}{(x-1)^4}$$

$$= \frac{(x-1)\times(14x-14) - 2\times(7x^2-14x)}{(x-1)^3}$$

$$= \frac{(14x^2-14x-14x+14) - (14x^2-28x)}{(x-1)^3}$$

$$= \frac{14}{(x-1)^3}$$

1.2.2 Differentiation of a Logarithmic Function

To differentiate a logarithmic function, we will use the rules of differentiation and focus on the natural logarithm (ln/log) function. When a variable is expressed as a function of the logarithm of another variable, the function is the logarithmic function.

- General form is $f(x) = \log(g(x))$

The general form of a logarithmic function is $f(x) = \ln\big(g(x)\big)$ where $g(x)$ is the argument of the logarithm. The derivative of $f(x) = \ln\big(g(x)\big)$ with respect to x using the chain rule is

$$\frac{d}{dx}\ln(g(x) = \frac{1}{g(x)}\frac{d\,g(x)}{dx}$$

For example, $g(x) = \ln(3x^2)$ the derivative of

$$g(x) = \frac{1}{3x^2}\frac{d(3x^2)}{dx} = \frac{6x}{3x^2} = \frac{2}{x}$$

The case of base b

$$\frac{d}{dx}\ln_b g(x) = \frac{1}{g(x)} \frac{d\,g(x)}{dx}\frac{1}{lnb}$$

Find the derivative of the following functions

1. $\sqrt{e^{2x} + 2x}$

$$\frac{d}{dx}(\sqrt{e^{2x} + 2x}) = \frac{1 \times e^{2x} \times 2}{2\sqrt{e^{2x}+2x}} + 2 = \frac{2(e^{2x}+1)}{2\sqrt{e^{2x}+2x}} = \frac{e^{2x}+1}{\sqrt{e^{2x}+2x}}$$

2. $3^x + 3x^2$

$$\frac{d}{dx}(3^x + 3x^2) = 3^x\, log3 + 6x$$

3. e^{3x-4}

$$\frac{d}{dx}(e^{3x^2-4}) = e^{3x^2-4} \times 6x = 6xe^{3x^2-4}$$

1.2.3 Differentiation of an Exponential Function

To differentiate an exponential function, we will use the rules of differentiation. The general form of an exponential function is $f(x) = a^x$, where a is a constant and x is the variable.

- General form is f(x) = a^x

The derivative of $f(x) = a^x$ with respect to x is $a^x\, ln\, a$, where ln represents the natural logarithm.

i.e., $\frac{d}{dx}(a^x) = a^x \ln a$

For example, $f(x) = 2^x$ the derivative of $f(x) = 2^x \ln 2$

Find the derivative of the following functions

1. $\log(4x^3 + x)$

$$\frac{d}{dx}(\log(4x^3 + x)) = \frac{1 \times (12x^2+1)}{4x^3+x} = \frac{12x^2+1}{4x^3+x}$$

2. $\log(\sqrt{5x-7})$

$$\frac{d}{dx}(\log(\sqrt{5x-7})) = \frac{1}{\sqrt{5x-7}} \times \frac{1 \times 5}{2\sqrt{5x-7}} = \frac{1}{5x-7}$$

3. $log_2(6x)$

$$\frac{d}{dx}(log_2(6x)) = \frac{1}{6x} \times 6 \times \frac{1}{log2} = \frac{1}{xlog2}$$

4. $log_3(3x+1)$

$$\frac{d}{dx}(log_3(3x+1)) = \frac{1 \times 3}{3x+1} \times \frac{1}{log3} = \frac{1}{(3x+1)log3}$$

5. $2\log(3x^2+1)$

$$\frac{d}{dx}(2\log(3x^2+1)) = 2 \times \frac{1}{3x^2+1} \times 6x = \frac{12x}{3x^2+1}$$

1.2.4 Implicit Function

- Dependent variable not expressed explicitly

An implicit function is a function where the dependent variable is not expressed explicitly in terms of the independent variable. When in a function the dependent variable is not explicitly isolated on either side of the equation then the function becomes an implicit function

Example 5

Find $\frac{dy}{dx}$ for $ax^2 + 2hxy + by^2 = 0$

Solution

$$ax^2 + 2hxy + by^2 = 0$$

Differentiating with respect to *x*

$$a \times 2 \times x + 2h(x\frac{dy}{dx} + y) + 2by\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2hx + 2by) = -2ax - 2hy$$

$$\frac{dy}{dx} = \frac{-2ax - 2hy}{2hx + 2by}$$

$$\frac{dy}{dx} = \frac{-ax - hy}{hx + by}$$

Example 6

Find $\frac{dy}{dx}$ for $x^2y - x + y = 0$

Solution

$$x^2y - x + y = 0$$

Differentiating with respect to x

$$x^2 \frac{dy}{dx} + y \times 2x - 1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 + 1) = -2xy + 1$$

$$\frac{dy}{dx} = \frac{-2xy + 1}{(x^2 + 1)}$$

Example 7

Find $\frac{dy}{dx}$ for $35x^3y^7 - 106 = 0$

Solution

$35x^3y^7 - 106 = 0$

Differentiating with respect to x

$$35(x^3 \times 7y^6 \frac{dy}{dx} + y^7 \times 3x^2) = 0$$

$$35(x^3 \times 7y^6 \frac{dy}{dx} + y^7 \times 3x^2) = 0$$

$$7x^3y^6 \frac{dy}{dx} = -3y^7x^2$$

$$\frac{dy}{dx} = \frac{-3y^7x^2}{7x^3y^6}$$

Summarised Overview

Higher-order derivatives refer to successive derivatives of a function beyond the first or second derivative. Higher order derivatives capture information about the rate at which the derivative itself changes with respect to the independent variable. The second derivative of y with respect to x is $\frac{d^2y}{dx^2}$ or $f''(x)$. The third derivative is denoted as $\frac{d^3y}{dx^3}$ or $f'''(x)$ and so on. Higher-order derivatives are utilised to analyse how changes in various economic parameters, such as production, demand, or cost functions, affect the rates of change, curvature. The derivative of $f(x) = \ln(g(x))$with respect to x using the chain rule is $\frac{d}{dx} \ln(g(x) = \frac{1}{g(x)} \frac{d\, g(x)}{dx}$. The derivative of a logarithmic function $f(x) = a^x$ with respect to x is $a^x \ln a$, where ln represents the natural logarithm. An implicit function is a function where the dependent variable is not expressed explicitly in terms of the independent variable.

Assignments

1. If $y = \frac{5x}{1-3x}$, find $\frac{d^2y}{dx^2}$ at $x = 2$
2. If $4x^2 - y^3 = 27$, find $\frac{d^2y}{dx^2}$
3. If $2x^3 + 5xy + 6y^2 = 78$, find $\frac{d^2y}{dx^2}$
4. If $x^2 + xy + y^2 = a^2$, prove that $\frac{d^2y}{dx^2} + \frac{6a^2}{(x+2y)^3} = 0$
5. If $x^3 + y^3 - 3axy$, prove that $\frac{d^2y}{dx^2} = \frac{2a^2xy}{(ax-y)^3}$
6. If $f(x) = ax^5 + 6x^4 + cx^3 + dx^4 + ex + f$, find f'''
7. If $y = (2x+3)(8x^2 - 6)$, find $\frac{d^3y}{dx^3}$
8. If $7x^4 + 3x^2y + 9xy^2 = 85$, find $\frac{d^2y}{dx^2}$

Suggested Reading

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
2. Chiang, A.C. (2008), *Fundamental Methods of Mathematical Economics*, McGraw Hill, New York.

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UNIT 3
Applications of Derivatives

Learning Outcomes

After completing this unit, learner will be able to:

- understand various marginal concepts
- apply derivatives in various economic concepts and decision-making process.

Background

The application of derivatives in economics is significant for understanding various economic phenomena. Effectively utilising derivatives in this context include a solid foundation in calculus, particularly the understanding of differentiation and integration. Proficiency in mathematical modelling is essential to represent economic relationships and dynamics accurately. A deep comprehension of economic theories is necessary to apply derivatives in contexts such as risk management, pricing options and futures, hedging strategies, and portfolio optimisation. Familiarity with statistical methods and probability theory is also valuable in analysing uncertainties and expected outcomes in economic scenarios.

Keywords

Marginal Utility, Marginal Revenue, Marginal Rate of Substitution, Production Function, Optimisation

Discussion

1.3.1 Application of Derivatives

- Helps in optimising functions

Derivatives, a fundamental tool of calculus, find extensive application in economics. In economics, derivatives are utilised to analyse and model various key concepts. For instance, the derivative of a utility function helps to determine the marginal utility of consuming an additional unit of a good, essential for understanding consumer behaviour and demand. The derivative of a cost function enables firms to optimise production levels by finding the marginal cost and maximising profit. Furthermore, derivatives play a crucial role in analysing market equilibrium by examining the slopes of demand and supply curves, aiding in price determination. Derivatives provide a powerful mathematical framework that allows economists to grasp the dynamics of markets, consumer preferences, production, and make informed decisions crucial for effective economic management and policy formulation.

1.3.2 Rate of Change

- Rate of change of one variable to other

The rate of change is a fundamental concept in mathematics. It refers to the speed at which one quantity changes in response to another quantity. The rate of change can be expressed as a ratio of the change in one variable to the change in another variable. In mathematical terms, if you have a function $y = f(x)$, then the rate of change of y concerning x at a specific point is given by the derivative of the function $f(x)$.

1.3.3 Slope of a Curve

- Rate at which the curve is changing at a particular point

The slope of a curve refers to the rate at which the curve is changing at a particular point. It is equivalent to the derivative of the curve's equation with respect to the variable(s) involved. In calculus, the slope of a curve at a specific point is given by the derivative of the function that represents the curve.

For a function $y = f(x)$, the slope of the curve at a point (x, y) is given by the derivative $f'(x)$. If the slope is positive, it means that the curve is increasing at that point. If it is negative, it means the curve is decreasing. If the slope is zero, it indicates a flat region (horizontal tangent) on the curve.

For instance, in a supply and demand curve, the slope of the demand curve represents the responsiveness of quantity demanded to changes in price. If the slope is steep (greater in absolute value), it implies that quantity demanded is highly responsive to price changes, indicating price elasticity. If the slope is shallow, it suggests a less responsive or inelastic relationship between price and quantity demanded.

- Similar to derivative

1.3.4 Marginal Concepts

Marginal concepts play a significant role in understanding demand and supply in economics, helping to analyse the behaviour of consumers and producers in market. Some of the important marginal concepts related to demand and supply is explained.

1.3.4.1 Marginal Utility (MU)

Total utility (TU) represents the combined satisfaction a consumer derives from consuming various units of a commodity. On the other hand, marginal utility (MU) reflects the extra satisfaction obtained from consuming one additional unit of the commodity. It is the additional utility or satisfaction gained from consuming one more unit of a good or service. Consumers make decisions based on marginal utility, maximising their overall satisfaction. Marginal utility signifies the increase in total utility attributed to consuming that extra unit. Another way to view marginal utility is as the decline in utility if one fewer unit of the commodity is consumed.

- Satisfaction obtained from consuming one additional unit of the commodity

Mathematically, it is expressed as the difference between the total utility of consuming n units, TU_n and consuming $n-1$ units, TU_{n-1} , denoted as $MU_n = TU_n - TU_{n-1}$. To determine marginal utilities for a utility function $U = f(X,Y)$, where X and Y are different goods, the partial derivatives are taken.

- Change in utility due to change in quantity

The marginal utility of X , MU_X , is the partial derivative of the utility function with respect to X.

i.e., $MU_X = \frac{\partial U}{\partial X}$

The marginal utility of Y, MU_Y, is the partial derivative of the utility function with respect to Y.

i.e., $MU_Y = \frac{\partial U}{\partial Y}$

1.3.4.2 Marginal Rate of Substitution (MRS)

- Exchange of one good to another

The MRS is typically defined as the amount of one good that a consumer is willing to give up in exchange for an additional unit of another good while maintaining the same level of satisfaction.

MRS is the ratio of the marginal utilities of commodities involved in the utility function.

$MRS_{xy} = \frac{MU_X}{MU_Y}$

1.3.4.3 Marginal Revenue (MR)

- Revenue from selling one additional unit

In the context of a firm and its supply, marginal revenue is the additional revenue obtained from selling one more unit of a good. It is vital for profit maximisation decisions. It is closely related to demand as it reflects the change in revenue resulting from a change in quantity demanded.

Total revenue (TR) represents the overall income or sales generated from selling a specific quantity of goods or services at a given price. It is the product of the quantity of goods sold (Q) and the price.

$TR = P \times Q$ where P = Price, Q = Quantity.

Average Revenue (AR) is total revenue per unit of output sold. It is obtained by dividing the total revenue by the number of outputs sold, AR = $\frac{TR}{Q}$

The derivative of the total revenue is the marginal revenue with respect to the output.

i.e., $MR = \frac{d(TR)}{dQ}$

1.3.4.4 Marginal Cost (MC)

Marginal Cost is the additional cost incurred by producing one more unit of a good or service. Firms use this concept to determine the profit-maximizing level of production. The total

- Additional cost of producing one more unit

cost refers to the aggregate money expenditure incurred by a firm to produce a given quantity of output. The total cost is measured in relation to the production function by multiplying the factor prices with their quantities. TC = f (Q) which means that the T.C. varies with the output.

In other words, a variation in TC is the result of variation in TVC since TFC is always constant in the short run.

Average cost is obtained by dividing the total cost by total output produced. AC = $\frac{TC}{Q}$

The derivative of the total cost is the marginal cost with respect to the output

i.e., $MC = \frac{d(TC)}{dQ}$

Example 1

Given the demand function $P = 30 - 2Q$. Find the marginal revenue function.

Solution

Demand function $P = 30 - 2Q$

Total revenue TR = $P \times Q = (30 - 2Q)Q = 30Q - 2Q^2$

$$MR = \frac{d(TR)}{dQ}$$

$$= \frac{d(30Q - 2Q^2)}{dQ}$$

$$= 30 - 4Q$$

Example 2

Find the marginal cost function for the following total function.

a) $TC = 3Q^2 + 7Q + 12$

b) $TC = 3Q^4 + 2Q^3 + Q^2 - 5Q + 15$. Evaluate them at $Q = 3$.

Solution

a) $TC = 3Q^2 + 7Q + 12$

$$MC = \frac{d(TC)}{dQ}$$

$$= \frac{d(3Q^2 + 7Q + 12)}{dQ}$$

$$= 6Q + 7$$

At $Q = 3,\ MC = (6 \times 3) + 7 = 25$

b) $TC = 3Q^4 + Q^3 + Q^2 - 5Q + 15$

$$MC = \frac{d(TC)}{dQ}$$

$$= \frac{d(3Q^4 + Q^3 + Q^2 - 5Q + 15)}{dQ}$$

$$= 12Q^3 + 3Q^2 + 2Q - 5$$

At $Q = 3, MC = (12 \times 3^3) + (3 \times 3^2) + (2 \times 3) - 5 = 352$

Example 3

Find the marginal revenue function for the following demand function

1) $Q = 36 - 2P$ 2) $42 - 3P - Q = 0$. Evaluate them at $Q = 3$

Solution

1) $Q = 36 - 2P$

$2P = 36 - Q$

$P = 18 - 0.5Q$

$TR = (18 - 0.5Q)Q$

$= (18Q - 0.5Q^2)$

$MR = \frac{d(18Q - 0.5Q^2)}{dQ} = 18 - 0.5 \times 2Q$

$MR = 18 - Q$

At $Q = 3,\ MR = 15$

2) $42 - 3P - Q = 0$

$3P = 42 - Q$

$$P = 14 - \frac{Q}{3}$$

$$TR = \left(14 - \frac{Q}{3}\right) Q$$

$$TR = \left(14Q - \frac{Q^2}{3}\right)$$

$$MR = \frac{d\left(14Q - \frac{Q^2}{3}\right)}{dQ} = 14 - \frac{2Q}{3}$$

At $Q = 3,\ MR = 14 - 2 = 12$

Example 4

Find the marginal cost function for the average cost function

$$AC = 15Q + 4 + \frac{46}{Q}$$

Solution

$$AC = 15Q + 4 + \frac{46}{Q}$$

$$TC = (AC)Q = \left(15Q + 4 + \frac{46}{Q}\right) Q$$

$$= (15Q^2 + 4Q + 46\,)$$

$$MC = \frac{d(15Q^2 + 4Q + 46\,)}{dQ}$$

$$MC = 30Q + 4$$

1.3.5 Maximisation and Minimisation (Optimisation)

- To determine the maximum or minimum values of a function

Optimisation is a fundamental approach to determining the maximum or minimum values of a function, crucial for decision-makers across various domains. Whether it is maximising utility, profit, or revenue, or minimising costs, achieving optimal outcomes is a key objective. When it comes to utility, profit, and revenue, the objective is to maximise these, whereas in the case of costs, the aim is minimisation. Calculus plays a pivotal role in optimization techniques, aiding decision-makers in finding the best choices. Optimisation can involve a function with a single independent variable or multiple independent variables. This section focuses on explaining optimisation with a single independent variable, providing a foundational understanding of this vital mathematical method.

- Maximisation or minimisation of functions

A function, denoted as $f(x)$, exhibits a maximum at $x = a$ when it reaches a point where its values stop increasing and start decreasing. Conversely, the function is characterised by a minimum at $x = a$ when it reaches a point where its values stop decreasing and commence an increase.

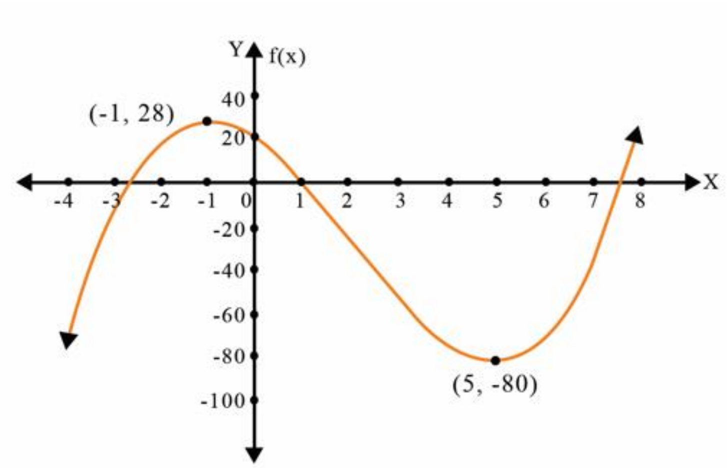

Fig 1.3.1 Maxima and Minima

In the figure (-1,28) is the maximum point and (5,-80) is the minimum point.

1.3.5.1 Condition for Maxima and Minima of a Function

Following are the conditions to find maximum and minimum of function $y = f(x)$ at $x = a$

1) First order condition or Necessary condition for both maximum and minimum is

- $\frac{dy}{dx} = 0$ or $f'(x) = 0$

$\frac{dy}{dx} = 0$ or $f'(x) = 0$. The point where the first derivative is zero is the critical point.

2) Second order condition or Sufficient condition for both maximum and minimum is

- for maximum, $\frac{d^2y}{dx^2} < 0$ or $f''(x) < 0$
- for minimum, $\frac{d^2y}{dx^2} > 0$ or $f''(x) > 0$

Example 5

Find the maximum and minimum value of the function

$y = x^3 - 2x^2 + x + 6$

Solution

$y = x^3 - 2x^2 + x + 6$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

Take $\frac{dy}{dx} = 0$, we get the critical points

$\frac{dy}{dx} = 3x^2 - 4x + 1 = 0$

$3x^2 - 4x + 1 = 0$

$3x^2 - 3x - x + 1 = 0$

$3x(x - 1) - (x - 1) = 0$

$(x - 1)(3x - 1) = 0$

$x = 1, \frac{1}{3}$ are critical points

Find $\frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2} = 6x - 4$

At $x = 1, \frac{d^2y}{dx^2} = 6 - 4 = 2 > 0$. Function has a minimum at $x = 1$

Minimum value at 1 = $(1)^3$- $2(1)^2$+ 1 + 6

Minimum value = $1 - 2 + 1 + 6 = 6$

At $x = \frac{1}{3}, \frac{d^2y}{dx^2} = 2 - 4 = -2 < 0$ Function has a maximum at $x = \frac{1}{3}$

Maximum value at $\frac{1}{3}$ = $(\frac{1}{3})^3$ - $2(\frac{1}{3})^2$ + $\frac{1}{3}$ + 6

Maximum value = $\frac{1}{27} - \frac{2}{9} + \frac{1}{3} + 6 = \frac{166}{27}$

Example 6

Maximise the total revenue for the total revenue function

$TR = 16Q - Q^2$.

Solution

$TR = 16Q - Q^2$

$$\frac{d(TR)}{dQ} = 16 - 2Q = 0 \Rightarrow Q = 8$$

$Q = 8$ is the critical point.

$$\frac{d^2(TR)}{dQ^2} = -2 < 0$$

The function has maximum at $Q = 8$.

Maximum value= $16 \times 8 - (8)^2 = 64$

Example 7

Find the average cost of the function if the total cost $TC = Q^3 - 4Q^2 + 15Q$. Also find the maximum or minimum average cost.

Solution

$$AC = \frac{TC}{Q} = \frac{Q^3 - 4Q^2 + 15Q}{Q} = Q^2 - 4Q + 15$$

First derivative of Average cost,

$(AC)' = 2Q - 4 = 0 \Rightarrow Q = 2$ is the critical point.

Second derivative of Average cost,

$(AC)'' = 2 > 0$, AC has a minimum at $Q = 2$

Minimum average cost = $2^3 - (4 \times 2^2) + (15 \times 2) = 22$

Example 8

Prove that the marginal cost (MC) must equal marginal revenue (MR) at the profit maximising level of output.

Solution

Profit $P = TR - TC$

$$\frac{dP}{dQ} = \frac{d(TR)}{dQ} - \frac{d(TC)}{dQ}$$

For critical point $\frac{dP}{dQ} = 0$

i.e., $\frac{d(TR)}{dQ} - \frac{d(TC)}{dQ} = 0$

$$\frac{d(TR)}{dQ} = \frac{d(TC)}{dQ}$$

$$MR = MC$$

Example 9

Let the demand function $P = 140 - 6Q$ and total cost $C = 60 + 20Q$. Calculate the optimum level of issue Q, Price P, Total revenue R, and Profit π.

Solution

$P = 140 - 6Q$

$C = 60 + 20Q$

Total Revenue, $R = P \times Q = 140Q - 6Q^2$

Profit $\pi = R - C$

$\pi = 140Q - 6Q^2 - (60 + 20Q)$

$= 120Q - 6Q^2 - 60$

For profit maximisation $\frac{d\pi}{dQ} = 120 - 12Q = 0$

$12Q = 120$

$Q = \frac{120}{12}$

$Q = 10$

Second condition for maximisation $\frac{d^2\pi}{dQ} = -12 < 0$

Putting Q in P, C, R $P = 140 - 60 = 80$

$C = 60 + 200 = 260$

$\pi = -600 + 1200 - 60 = 540$

Example 10

Following is the demand and cost function of two separate markets

$P_1 = 80 - 5Q_1, \qquad P_2 = 180 - 29Q_2,$

$C = 50 + 20(Q_1 + Q_2)$

Determine the price, production, marginal revenue and total profit of the production of both the markets.

Solution

$P_1 = 80 - 5Q_1, \qquad P_2 = 180 - 29Q_2,$

$C = 50 + 20(Q_1 + Q_2)$

Total Revenue of first market $R_1 = P_1Q_1 = 80Q_1 - 5Q_1^2$

Total Revenue of second market $R_2 = P_2Q_2 = 180Q_2 - 29Q_2^2$

$$MR_1 = \frac{dR_1}{dQ_1} = 80 - 10Q_1 = 0$$

$Q_1 = 8$

$$MR_2 = \frac{dR_2}{dQ_2} = 180 - 58Q_2 = 0$$

$$Q_2 = \frac{180}{58} = \frac{90}{29}$$

Total Profit $\pi = R_1 + R_2 - C$

$$\pi = R_1 + R_2 - C = 80Q_1 - 5Q_1^2 + 180Q_2 - 29Q_2^2 - 50 - 20(Q_1 + Q_2)$$

$$= 60Q_1 - 5Q_1^2 + 160Q_2 - 29Q_2^2 - 50$$

Put Q_1 and Q_2, $P_1 = 80 - 5 \times 8 = 40,$

$P_2 = 180 - 29 \times \frac{90}{29} = 90$

$MR_1 = 0, \ MR_2 = 0$

$\pi = 60Q_1 - 5Q_1^2 + 160Q_2 - 29Q_2^2 - 50$

$$= (60 \times 8) - (5 \times 8^2) + \left(160 \times \frac{90}{29}\right) - \left(29 \times \frac{90^2}{29^2}\right) - 50$$

$$= 87\frac{7}{29}$$

Summarised Overview

Derivatives is a fundamental tool of calculus and find great application in economics. The rate of change is related to derivatives. It refers to the speed at which one quantity changes in response to another quantity. The rate of change can be expressed as a ratio of the change in one variable to the change in another variable. This rate of change is expressed as Marginal concepts in economics which splay a significant role in understanding demand and supply in economics, helping to analyse the behaviour of consumers and producers in market. Some of the important marginal concepts are Marginal Utility, Marginal Revenue, Marginal Cost. Marginal utility is the extra satisfaction obtained from consuming one additional unit of the commodity. A production function defines the relationship between inputs (factors of production) and the output (quantity of goods or services) that can be produced. Marginal revenue is the additional revenue obtained from selling one more unit of a good. Marginal Cost is the additional cost incurred by producing one more unit of a good or service. Condition for Maxima of a function is $\frac{d^2y}{dx^2} < 0 \; or \; f''(x) < 0$ and condition for Minima is $\frac{d^2y}{dx^2} > 0 \; or \; f''(x) > 0$

Assignments

1. If the total cost $C = 15 + 10q - 9q^2 + q^3$, find the marginal cost.
2. If the production is in the form of $Q = AK^{\alpha}L^{\beta}$, find

   a) Marginal productivity of Capital K and labour L

3. If the demand is $p = 20 - 5q$ and average cost is q, find the maximum profit.
4. If the demand and supply of a firm is given as demand $Q = 30 - x$ and supply $S = x^2 + 6x + 7$ then at what level of production maximum profit can be earned and what would be the corresponding value of price profit and total revenue?
5. If following is the demand and cost $p = 100 - 0.5(q_1 + q_2)$, $C_1 = 5q_1$, $C_2 = 0.5q_2^2$

   Find the values of $q_1, q_2, p, \pi_1, and \; \pi_2$.

6. Find out the value of x for the maximum and minimum value of the function

   $x^4 - 8x^3 + 22x^2 - 24x$

7. Find out the maximum value of $\frac{log x}{x}$ where $0 < x < \infty$
8. Following is the demand and cost function of two separate markets

$$P_1 = 2 - Q_1, \quad P_2 = 9 - 6Q_2, \qquad C = Q_1 + Q_2$$

Determine the price, production, marginal revenue and total profit of the production of both the markets.

Suggested Reading

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
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UNIT 4
Integration

Learning Outcomes

After completing this unit, learner will be able to:

- understand the concepts of integration
- understand the rules of integration
- apply integration techniques to solve economic problems
- calculate consumer and producer surplus in various economic scenarios

Background

A solid foundation in calculus, specifically understanding differentiation and basic integration principles, lays the groundwork for comprehending the complexities of integration rules. Proficiency in algebraic manipulation and understanding the properties of functions are also essential for the study of integration. Integration by substitution involves a methodical approach to simplifying complex integrals through appropriate substitutions. Integration by parts, on the other hand, entails a structured technique for solving integrals by breaking them down into manageable parts. In the context of economics, these integration techniques are invaluable for calculating critical economic measures like consumer's and producer's surplus. Understanding and applying these methods enrich our ability to analyse market dynamics and economic welfare, making them vital tools for economists and mathematicians alike.

Keywords

Integration, Integration by Substitution, Integration by Parts, Consumer Surplus, Producer Surplus, Equilibrium Price

Discussion

1.4.1 Integration

Integration, a fundamental concept in calculus, plays a vital role in economics by aiding in the analysis of continuously changing variables. In economics, we often encounter dynamic processes and changing phenomena that evolve over time. Integration allows us to determine accumulated quantities or aggregate behaviours over continuous intervals. For instance, in economics, we can use integration to calculate total revenue, total cost, consumer surplus, producer surplus, and other important economic measures. By integrating the marginal functions, representing rates of change in economics, we can derive the total functions, providing valuable insights into the overall impact and outcomes of economic activities. Integration offers a powerful tool for understanding how small changes at each moment can accumulate and influence the overall economic situation.

- Inverse of differentiation

In differential calculus we measure the rate of change of function $f(x)$ as $f'(x)$. By reversing the process of differentiation and finding the original function from the derivative is the integration or anti differentiation. The original function is called the integral of $f'(x)$.

Suppose $f(x)$ be a function of x whose differential coefficient if $f'(x)$.

i.e., $\frac{d}{dx}f(x) = f'(x)$, then the integral of $f'(x)$ is $f(x)$. It can be written as $\int f'(x)\,dx = f(x) + C$ where C is a constant.

The constant of integration is a term that arises when solving indefinite integrals in calculus. When integrating a function, the process often involves finding the antiderivative, which is the reverse operation of differentiation. However, unlike differentiation, integration involves an indefinite integral, which means that there can be multiple functions that have the same derivative. Therefore, when integrating, we introduce a constant term known as the constant of integration.

- Integration involves an indefinite integral

1.4.2 Rules of Integration

- Integration rules depend on the rules of differentiation

The rules of integration depend on the rules of differentiation. In the coming section, we will discuss the basic rules of integration that will be obtained by reversing the corresponding rules of differentiation. Their accuracy is easily checked since the derivative of the integral must equal the integrand.

Rule 1. Constant Rule of Integration

The integral of a constant k is

$$\int k\,dx = kx + c$$

For example, $\int 2\,dx = 2x + c$

Rule 2. Power Rule of Integration

The integral of a power function x^n, where $n \neq -1$ is given by:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \qquad (n \neq -1)$$

For example, $\int x^5\,dx = \frac{x^{5+1}}{5+1} = \frac{x^6}{6} + c.$

Rule 3. Exponential Rule of Integration

The integral of an exponential function is

$$\int a^{kx}\,dx = \frac{a^{kx}}{k\log a} + c$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$$

For example,

$$\int 2^{3x}\,dx = \frac{2^{3x}}{3\log 2} + c$$

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$$

Rule 4. Logarithmic Rule of Integration

The integral of x^{-1} is (or $\frac{1}{x}$) is

$$\int x^{-1}\, dx = \log x + c \quad (x > 0)$$

For example,

$$\int \frac{2}{x}\, dx = \int 2\, x^{-1}\, dx = 2 \log x + c$$

Rule 5. The Constant Multiple Rule of Integration

The integral of a constant time a function equals the constant times the integral of the function

$$\int k\, f(x)\, dx = k \int f(x)\, dx$$

For example,

$$\int 3\, x^2\, dx = 3 \int x^2\, dx = 3 \times \frac{x^3}{3} + c = x^3 + c$$

Rule 6. The Sum and Subtraction Rule of Integration

The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

For example,

$$\int \left(x^6 + \frac{1}{x} + e^x - 1\right) dx$$

$$= \int x^6\, dx + \int \frac{1}{x}\, dx + \int e^x\, dx - \int dx$$

$$= \frac{x^7}{7} + logx + e^x - x + c$$

1.4.3 Integration by Substitution

- Transforms integrals into standard forms for simplification

In this operation, integration is done by changing the given integrand in standard form. For this any integration function of variable amount is put equal to another variable amount and substituting in integration, integration is changed in form of function of new variable amount in such a way that use of formula may be easily.

Example 1

Determine the integral $\int \sqrt{1-5x} + 9e^{3x}\, dx$

Solution

$$\int \sqrt{1-5x} + 9e^{3x}\, dx = \frac{(1-5x)^{\frac{3}{2}}}{\frac{3}{2} \times -5} + \frac{9e^{3x}}{3}$$

$$= -\frac{2}{15}(1-5x)^{\frac{3}{2}} + 3e^{3x} + c$$

Example 2

Find the value of $\int x^2\, logx\;\; dx$

Solution

$$\int x^2\, logx\;\; dx = logx\, \times \frac{x^3}{3} - \int \frac{1}{x} \times \frac{x^3}{3} dx$$

$$= logx \times \frac{n^3}{3} - \frac{1}{3}\int\; n^2\;\; dx$$

$$= logx \frac{x^3}{3} - \frac{1}{3} . \frac{x^3}{3} + c$$

$$= \frac{x^3}{3} logx - \frac{x^3}{9} + \text{c}$$

Example 3

Find $\int \left(x - \frac{1}{x}\right)^2\; dx$

Solution

$$\int \left(x - \frac{1}{x}\right)^2\; dx = \int x^2 - 2 + \frac{1}{x^2}\; dx = \frac{x^3}{3} - 2x - \frac{1}{x} + c$$

Example 4

Determine the following integrals 1) $\int \frac{5x+7}{x} + e^{3x}\, dx$

2) $\int \frac{(x+2)(4x^2-5)}{x}\ dx$

Solution

1) $\int \frac{5x+7}{x} + e^{3x}\ dx = \int 5 + \frac{7}{x} + e^{3x}\ dx = 5x + 7\ logx + \frac{e^{3x}}{3} + c$

2) $\int \frac{(x+2)(4x^2-5)}{x}\ dx = \int \frac{4x^3-5x+8x^2-10}{x}\ dx = \int (4x^2 - 5 + 8x - \frac{10}{x})\ dx$

$$= \frac{4x^3}{3} - 5x + \frac{8x^2}{2} - 10\ logx = \frac{4x^3}{3} - 5x + 4x^2 - 10\ logx + c$$

Example 5

Determine the following integrals

1) $\int \frac{x^8}{(1-x^3)^{\frac{1}{3}}}\ dx$

2) $\int \frac{logx}{x} dx$

3) $\int \frac{6\,x^2}{1-2x^3}\ dx$

4) $\int \frac{3\,x^2}{(x^3+4)^5}\ dx$

5) $\int \frac{e^x}{1+e^x}\ dx$

Solution

1. Put $1 - x^3 = u, \quad -3x^2\ dx = du, \quad x^2\ dx = -\frac{du}{3}$

   $x^3 = 1 - u,$

$$\int \frac{x^8}{(1-x^3)^{\frac{1}{3}}}\ dx = \int \frac{(x^3)^{\ 2}}{(1-x^3)^{\frac{1}{3}}}\ dx$$

$$= \int \frac{(1-u)^2}{u^{\frac{1}{3}}} \times -\frac{du}{3}$$

$$= \int \frac{1-2u+u^2}{u^{\frac{1}{3}}} \times -\frac{du}{3}$$

$$= -\frac{1}{3}\left[\int u^{-\frac{1}{3}} - 2u^{\frac{2}{3}} + u^{\frac{5}{3}}\right]$$

$$= -\frac{1}{3}\left[\frac{u^{\frac{2}{3}}}{\frac{2}{3}} - 2\frac{u^{\frac{5}{3}}}{\frac{5}{3}} + \frac{u^{\frac{8}{3}}}{\frac{8}{3}}\right] + c$$

$$= -\frac{1}{2}u^{\frac{2}{3}} + \frac{2}{5}u^{\frac{5}{3}} - \frac{1}{8}u^{\frac{8}{3}}$$

$$= -\frac{1}{2}\,(1-x^3)^{\frac{2}{3}} + \frac{2}{5}(1-x^3)^{\frac{5}{3}} - \frac{1}{8}(1-x^3)^{\frac{8}{3}}$$

2. Put $logx = u,\ \frac{1}{x}\,dx = du$

$\int \frac{logx}{x}dx \ = \int u\,du \ = \frac{u^2}{2} + c \ = \frac{(logx)^2}{2} + c$

3. Put $1 - 2x^3 = u, \qquad -6x^2\,dx = du,$

$$\int \frac{6\,x^2}{1-2x^3}\,dx = -\int \frac{du}{u}$$

$= -\log u + c = -log\,(1-2x^3) + c$

4. Put $x^3 + 4 = u,\ \ 3x^2\,dx = du,$

$$\int \frac{3\,x^2}{(x^3+4)^5}\,dx = \int \frac{du}{(u)^5} = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + c$$

$$= -\frac{1}{4}(x^3+4)^{-4} + c$$

5. Put $1 + e^x = u,\ \ e^x\,dx = du,$

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{du}{u} = \log u + c = \log\,(1+e^x) + c$$

1.4.4 Integration by Parts

If $f(x)$ and $g(x)$ are two functions of x, then the integration of $f(x).g(x)$ is done by integration by parts.

Consider $f(x)$ is the first function and $g(x)$ is the second function.

$$\int f(x).g(x) = f(x)\int g(x) - \int\left[\frac{df(x)}{dx}\int g(x)dx\right]dx$$

$i.e., \int f(x).g(x) = f(x)\int g(x) - \int[f'(x)\int g(x)dx\,]dx$

Integration of multiplication of two function

= First function × integral of second-integral of (differential of first function × integral of second function)

- LIATE order for integration by parts

In the context of using the integration by parts formula, the choice of which function to designate as the first function and which to designate as the second function is guided by a specific priority order, often referred to as "LIATE." Here is the order one should consider:

- Logarithmic functions (L)
- Inverse trigonometric functions (I)
- Algebraic functions (A)
- Trigonometric functions (T)
- Exponential functions (E)

Example 6

Determine the following integral

1) $\int logx\ \ dx$

2) $\int x^2\, logx\ \ dx$

3) $\int x\ \ e^{2x}\ \ dx$

4) $\int \frac{2x}{(x-8)^3}\ dx$

5) $\int 6x\ \ e^{(x+3)}\ \ dx$

6) $\int x(x+1)^5\, dx$

7) $\int 15x\ \ (x+4)^{\frac{3}{2}}\ \ dx$

Solution

1) Let $f(x) = logx, \quad g(x) = 1$

$$\int logx\ \ dx = \int logx.1\ \ dx$$

$$= logx \int 1\, dx - \int \left(\frac{d}{dx}(logx) \times \int 1.dx \right) dx$$

$$= logx \times x - \left(\int \frac{1}{x} \times x \right) dx$$

$$= x\, logx - \left(\int dx \right)$$

$$= x\, logx - x + c$$

2) Let $f(x) = x^2, \quad g(x) = logx$

$$\int x^2\, logx\;\, dx = x^2 \int logx - \int \left(\frac{d}{dx}(x^2) \int logx\, dx \right) dx$$

$$= x^2 \times \frac{1}{x} - \int 2x \times \frac{1}{x}\, dx$$

$$= x - \int 2\, dx$$

$$= x - 2x + c = -x + c$$

3) Let $f(x) = x, \quad g(x) = e^{2x}$

$$\int x\;\, e^{2x}\;\, dx = x \times \int e^{2x}\, dx - \int \left(\frac{d}{dx}(x) \int e^{2x}\, dx \right) dx$$

$$= x \times \frac{e^{2x}}{2} - \int 1 \times \frac{e^{2x}}{2}\, dx$$

$$= x \times \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

4) Let $f(x) = 2x, \quad g(x) = \frac{1}{(x-8)^3} = (x-8)^{-3}$

$$\int \frac{2x}{(x-8)^3}\, dx = \int 2x \times (x-8)^{-3}\, dx$$

$$= 2x \times \int (x-8)^{-3} - \int \left(\frac{d}{dx}(2x)\;\, \int (x-8)^{-3} dx \right) dx$$

$$= 2x \times \frac{(x-8)^{-2}}{-2} - \int 2 \times \frac{(x-8)^{-2}}{-2} dx$$

$$= -x \times (x-8)^{-2} - (x-8)^{-1} + c$$

$$= \frac{-x}{(x-8)^2} - \frac{1}{(x-8)} + c$$

5) Let $f(x) = 6x, \quad g(x) = e^{x+3}$

$$\int 6x \ e^{x+3} \ dx$$

$$= 6x \times \int e^{x+3} \, dx - \int \left(\frac{d}{dx}(6x) \int e^{x+3} \, dx \right) dx$$

$$= 6x \times e^{x+3} - \int 6 \times e^{x+3} \, dx$$

$$= 6xe^{x+3} - 6e^{x+3} + c$$

6) Let $f(x) = x, \quad g(x) = (x+1)^5$

$$\int x(x+1)^5 \, dx = x \times \int (x+1)^5 \, dx - \int \left(\frac{d}{dx}(x) \int (x+1)^5 \, dx \right) dx$$

$$= x \times \frac{(x+1)^6}{6} - \int 1 \times \frac{(x+1)^6}{6} \, dx$$

$$= x \times \frac{(x+1)^6}{6} - \frac{1}{6} \frac{(x+1)^7}{7} + c$$

$$= \frac{x\,(x+1)^6}{6} - \frac{(x+1)^7}{42} + c$$

7) Let $f(x) = 15x, \quad g(x) = \ (x+4)^{\frac{3}{2}}$

$$\int 15x \ (x+4)^{\frac{3}{2}} \ dx = 15x \times \int \ (x+4)^{\frac{3}{2}} dx - \int \left(\frac{d}{dx}(15x) \int \ (x+4)^{\frac{3}{2}} \, dx \right) dx$$

$$= 15\,x \times \frac{(x+4)^{\frac{5}{2}}}{\frac{5}{2}} - \int 15 \times \frac{(x+4)^{\frac{5}{2}}}{\frac{5}{2}} \, dx$$

$$= 6x \times (x+4)^{\frac{5}{2}} - 6 \times \frac{(x+4)^{\frac{7}{2}}}{\frac{7}{2}} + c$$

$$= 6x \times (x+4)^{\frac{5}{2}} - \frac{12}{7}(x+4)^{\frac{7}{2}} + c$$

1.4.5 Applicability of Integration in Economics

Integration plays a crucial role in economics by enabling the determination of total cost and total revenue functions from marginal cost. Additionally, it allows the calculation of consumer and producer surpluses based on demand and supply functions.

Consumer's Surplus

- Benefit from paying less than maximum willing price

Consumer surplus is the additional value that consumers gain from a transaction because they are able to purchase a product at a price lower than the maximum price, they are willing to pay. Graphically, it's the area between the demand curve D(x) and the horizontal line representing the equilibrium price y=P. This area visually represents the extra benefit or surplus consumers receive from the difference between what they are willing to pay and what they actually pay at the equilibrium price.

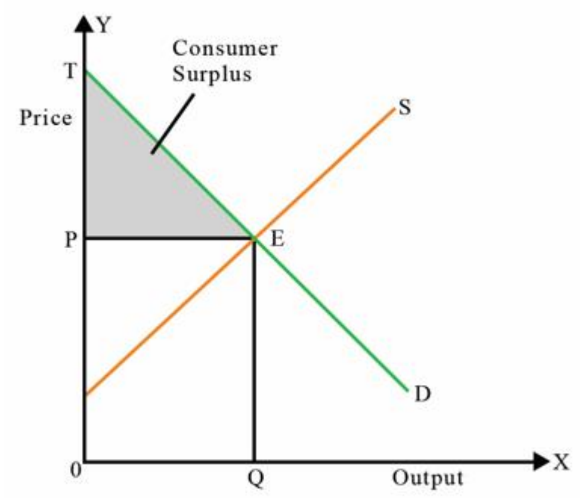

Fig 1.4.1 Consumer Surplus

Let p = D(x) describe the demand function for a product and p = S(x) be the supply for the same product. Equilibrium point is the point where D(x) = S(x).

The formula for consumer's surplus CS is

$$CS = \int_0^Q D(x)\,dx - PQ$$

Producer's Surplus

- Gap between equilibrium price and producer's minimum price

The producer's surplus is the difference between the equilibrium price of an item and the lower price at which a producer is willing to sell that item. This is also the area between the curves $S(x)$ and the horizontal line $y = P$.

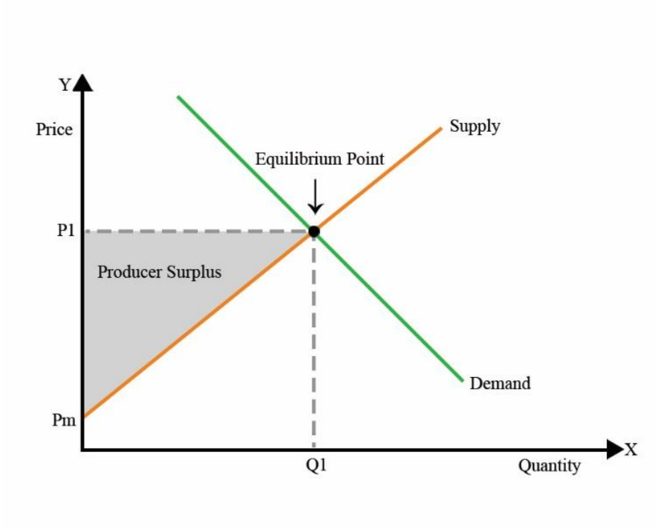

Fig 1.4.2 Producer Surplus

- Measure of Producer welfare

A supply function $S(x)$ represents the quantity that can be supplied at a price P. Let P be the market price for the corresponding supply Q. But there can be some producers who are willing to supply the commodity below the market price gain from the fact that the price is P. This gain is called the producer's surplus.

The formula for producer's surplus PS is

$$PS = PQ - \int_0^Q S(x)\, dx.$$

Example 7

The demand function of a commodity *is* $p = 36 - x^2$. Find the consumer's surplus for $p = 11$

Solution

When $p = 11, \quad 11 = 36 - x^2 \Rightarrow x^2 = 25 \Rightarrow x = 5$

$$CS = \int_0^Q D(x)\, dx - PQ$$

$$= \int_0^5 (36 - x^2) \, dx - 11 \times 5$$

$$= \left(36x - \frac{x^3}{3}\right)_0^5 - 55$$

$$= 180 - \frac{125}{3} - 55 = \frac{250}{3}$$

Example 9

Find the consumer's surplus and producer's surplus for the demand function

$D(x) = 25 - 3x$ and the supply function $S(x) = 5 + 2x$.

Solution

Equilibrium point is D(x) = S(x)

$25 - 3x = 5 + 2x$

$5x = 20 \Rightarrow x = 4$

$Q = 25 - 12 = 13$

$$CS = \int_0^Q D(x) \, dx - PQ$$

$$= \int_0^4 (25 - 3x) \, dx - 4 \times 13$$

$$= \left(25x - \frac{3x^2}{2}\right)_0^4 - 52$$

$$= 100 - 24 - 52 = 24$$

$$PS = PQ - \int_0^Q S(x) \, dx$$

$$= 4 \times 13 - \int_0^4 (2x + 5) \, dx$$

$$= 52 - \left(\frac{2x^2}{2} + 5x\right)_0^4$$

$$= 52 - 16 - 20$$

$$= 16$$

Example 9

For a certain item the demand curve is $D(x) = \frac{20}{x+1}$ and the supply curve $S(x) = x + 2$. Find the equilibrium price. Compute the consumer's surplus and producer's surplus.

Equilibrium point is D(x)=S(x)

$$\frac{20}{x+1} = x + 2$$

$$(x + 1)(x + 2) = 20$$

$$x^2 + 3x + 2 = 20$$

$$x^2 + 3x - 18 = 0$$

$$x = -6, 3$$

The positive solution $x = 3$ gives the Equilibrium quantity and the equilibrium price is

$$\frac{20}{4} = 5$$

$$CS = \int_0^3 D(x)\, dx - PQ$$

$$= \int_0^3 \left(\frac{20}{x+1}\right) dx - 3 \times 5$$

$$= 20(\log(x + 1)_0^3 - 15$$

$$= 20\, log4 - 15 = 12.73$$

$$PS = PQ - \int_0^Q S(x)\, dx$$

$$= 3 \times 5 - \int_0^3 (x + 2)\, dx$$

$$= 15 - \left(\frac{x^2}{2} + 2x\right)_0^3$$

$$= 15 - \frac{9}{2} - 6 = \frac{9}{2} = 4.5$$

Summarised Overview

Integration analyses continuously changing variables and provides a tool to determine accumulated quantities or aggregate behaviours over continuous intervals. For instance, in economics, integration is extensively used to calculate essential measures such as total revenue, total cost, consumer surplus, producer surplus, and more. By integrating marginal functions representing rates of change, economists can derive total functions. Differential calculus measures the rate of change, denoted as f'(x), and integration, or anti-differentiation, involves finding the original function from its derivative. The constant of integration arises in solving indefinite integrals due to multiple functions having the same derivative. The rules of integration are derived by reversing the corresponding rules of differentiation. Integration by substitution and integration by parts are techniques employed to simplify integrals. These methods involve transforming the integrand or breaking down complex integrals into simpler forms, respectively.

Assignments

1. Find the integral
    1. $\int x^3 e^{2x}$ dx
    2. $\int x^{\frac{1}{2}} + 3x^{-\left(\frac{1}{2}\right)}\ dx$
    3. $\int x^4(2x^5 - 5)^4\ dx$
    4. $\int \frac{6x^2+4x+10}{(x^3+x^2+5x)\ ^3}\ dx$
    5. $\int \frac{3x^2+2}{4x^3+8x}\ dx$
    6. $\int \frac{2x}{(x-8)^3}\ dx$
    7. $\int x^2 e^{x+3}\ dx$
    8. $\int x(x^3+1)^5\ dx$
2. Given the supply function $P = (Q+3)^3$. Find the producer's supply at $P_0 = 81$ and $Q_0 = 6$
3. Given the demand function $P = 25 - Q^2$ and supply function $S = 2Q + 1$. Find Consumer's surplus and *producer surplus*
4. Given the demand function $P = 42 - 5Q - Q^2$. Assuming the equilibrium price is 6, find the Consumer's surplus.
5. Given the demand function $P = 113 - Q^2$ and supply function $S = (Q+1)^2$. Find Consumer's surplus and producer's surplus

Suggested Reading

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
2. Chiang, A.C. (2008), *Fundamental Methods of Mathematical Economics*, McGraw Hill, New York.

Reference

1. Yamane, Taro. (2012). *Mathematics for Economists: An Elementary Survey*. New Delhi: Prentice Hall of India.
2. Dowling T. Edward (2000), *Schaum's Outline of Introduction to Mathematical Economics*, McGraw Hill, New York.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

MASTER OF ARTS ECONOMICS

Optimisation

BLOCK 2

UNIT 1
Optimisation of Functions

Learning Outcomes

After completing this unit, learner will be able to:

- understand the concepts of concavity and convexity.
- learn about optimisation techniques
- apply optimisation techniques to in real-world situations

Background

Imagine running a car company and wanting to make the most money possible. Well, that is where the cool parts of economics come into play – like having superpowers for making really smart decisions! But there is this sweet spot, where you make just the right number of cars to make the most profit. Now, here is the economic superpowers convexity, concavity, and optimisation are involved.

If your profit-making process is like a roller coaster, convexity means that making more cars can lead to a big jump in profits. It is like getting extra rewards for each additional car produced. On the other hand, concavity warns you that there is a limit. Making too many cars might not bring in as much extra profit – it is like the roller coaster starting to slow down. Now, optimisation is the superhero move here. It is the secret sauce for finding that sweet spot. It helps you figure out the perfect number of cars to make to get the maximum profit without going overboard.

So, for the car company, learning about convexity, concavity, and optimisation is like having a treasure map for making the best decisions. It is about finding that perfect balance in production – not too much, not too little – just right for making the most money

Keywords

Critical Points, Relative Maximum, Relative Minimum

Discussion

2.1.1 Concavity and Convexity

- Line between any two points stays above, below or on the graph

A convex function is a mathematical function characterised by the property that the line segment connecting any two points on its graph lies entirely above the function's graph. A concave function is characterized by a downward curve, with the property that any line segment connecting two points on its graph will be positioned below or on the graph itself.

- Downward curve connecting points positioned below or on graph

A function f(x) is considered concave at x = α when, in a nearby region around the point [α, f(α)], the function's graph is situated entirely beneath its tangent line. Conversely, a function is convex at x = α when, in it is close to [α, f(α)], the graph of the function lies completely above its tangent line.

The presence of a positive second derivative at x = α indicates convexity at that point, while a negative second derivative signifies concavity at α. The slope of the first derivative does not influence the assessment of concavity.

$$\boldsymbol{f''^{(\alpha)} > 0 : f(x) \textit{ is convex at } x = \alpha}$$

$$\boldsymbol{f''^{(\alpha)} < 0 : f(x) \textit{ is concave at } x = \alpha}$$

If $\boldsymbol{f''(x) > 0}$ for all x in the domain, *f(x)* is strictly convex.

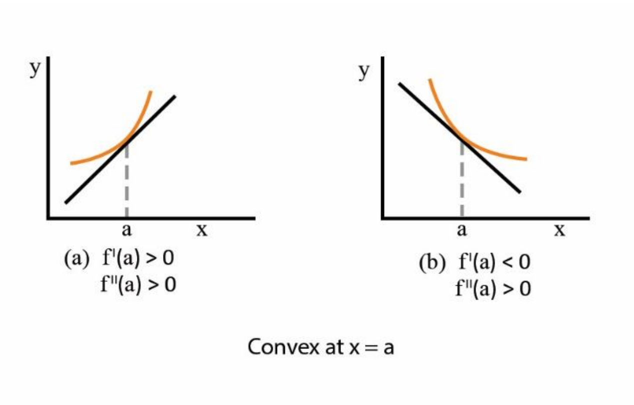

Fig 2.1.1 Convexity

If $f''(x) < 0$ for all x in the domain, *f(x)* is strictly concave.

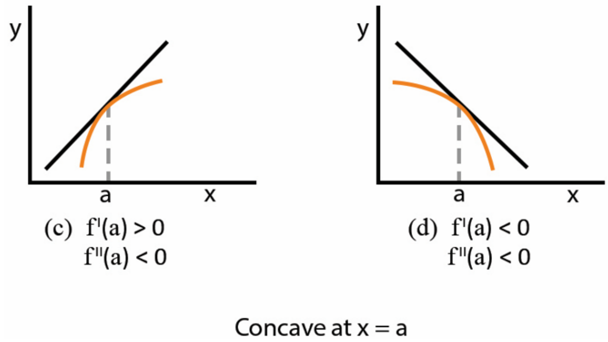

Fig 2.1.2 Concavity

Example 1

Consider the function $y = 2x^2 - 5x + 3$. Determine whether the function is concave or convex at x = 2.

Solution

First, let us find the second derivative of the function:

$y = 2x^2 - 5x + 3$

$y' = 4x - 5$

$y'' = 4$

Now, substitute x = 2 into the second derivative:

$y''(2) = 4$

Since the second derivative at x = 2 is positive ($y''(2) = 4$), the function is convex at x = 2.

In summary, for the function $y = 2x^2 - 5x + 3$, it is convex at x = 2.

Example 2

Given the function $y = -3x^2 + 4x - 1$, determine whether the function is concave or convex at x = 2.

Solution

Let us find the first and second derivatives of the function:

$y = -3x^2 + 4x - 1$

$y' = -6x + 4$

y'' = -6

The second derivative is a constant value, which means it does not depend on x. Since the second derivative is negative (y'' = -6), the function is concave downward everywhere.

Now, let us evaluate the concavity at x = 2:

y''(2) = -6

Since the second derivative is negative

(y''(2) = -6), the function is concave downward at x = 2.

In summary, for the function $y = -3x^2 + 4x - 1$, the function is concave downward at x = 2.

2.1.2 Optimisation

- Determining the points of relative maximum or minimum within a function

Optimisation involves the systematic exploration of a function to determine its relative maximum or minimum points. This process is essential for solving a wide range of real-world problems in economics. Whether it is identifying the highest profit a business can achieve or the lowest cost in manufacturing, optimisation seeks to uncover the optimal values of variables that lead to the best outcomes. By analysing the behaviour of functions and their derivatives, optimisation techniques allow us to make informed decisions and efficiently allocate resources to achieve desired goals.

2.1.2.1 Optimisation of Single Variable Function

Following are the steps to determine whether the function is in minimum or maximum

1. Determine the critical point(s) by taking the first derivative, setting it equal to zero, and solving. It is known as the first-order condition.
2. Determine the sign(s), assess the second derivative at the critical point(s), and repeat.

$f(\alpha)>0$: $convex, relative\ minimum$

$f(\alpha)<0$: $concave, relative\ maximum$

$f(\alpha)=0$: $The\ test\ is\ inconclusive.$

This step, the *second-derivative test*, is also called the *second-order condition*.

In sum,

Relative Maximum	**Relative Minimum**
$f'(\alpha) = 0$	$f'(\alpha) = 0$
$f''(\alpha)<0$	$f''(\alpha)>0$

Example 3

Optimise $f(x) = 2x^3 - 30x^2 + 126x + 59$.

Solution

a. Find the critical points by taking the first derivative, setting it equal to zero, and solving for *x*.

$f'(x) = 6x^2 - 60x + 126 = 0$

$6(x-3)(x-7) = 0$

critical points $are\ x = 3\ and\ x = 7$

b. Test for concavity by taking the second derivative, evaluating it at the critical points, and checking the signs to distinguish between a relative maximum and minimum.

$f(x) = 12\,x - 6$

$f(3) = 12(3) - 60 = -24 < 0$

$concave, relative\ minimum$

$f(7) = 12(7) - 60 = 24 > 0$

$convex, relative\ maximum$

The function is maximised at $x = 3$ and minimised at $x = 7$

2.1.2.2 Optimisation of Multivariable Functions

For a multivariable function like z = f(x, y) to reach a relative minimum or maximum, the following conditions are to be satisfied:

1. Solve the equations, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ to get the critical points (a, b)

2. Let $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$

Case 1: if $rt - s^2 > 0$ and $r > 0$ the function has a minimum at (a, b)

Case 2: if $rt - s^2 > 0$ and $r < 0$ the function has a maximum at (a, b)

Case 3: if $rt - s^2 < 0$, the critical point is saddle point

Case 4: if $rt - s^2 = 0$, the nature cannot determine by this method

Example 1

Find the relative maxima or minima for the function

$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

Solution

$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

$$\frac{\partial f}{\partial x} = 6x - 2y, \qquad \frac{\partial f}{\partial y} = -2x + 2y - 8,$$

Critical points are obtained by solving the equations $\frac{\partial f}{\partial x} = 0$,

$$\frac{\partial f}{\partial y} = 0$$

$$6x - 2y = 0 \ldots \ldots (1)$$

$$-2x + 2y - 8 = 0 \ldots \ldots (2)$$

$(1) + (2)$, $4x - 8 = 0 \Rightarrow \quad 4x = 8 \quad \Rightarrow x = \frac{8}{4} = 2$

Substituting $x = 2$ in (1) $6 \times 2 - 2y = 0 \Rightarrow 12 - 2y = 0 \Rightarrow$

$2y = 12 \Rightarrow y - \frac{12}{2} = 6$

(2,6) is the critical point.

The second derivatives are

$$r = \frac{\partial^2 f}{\partial x^2} = 6, \qquad t = \frac{\partial^2 f}{\partial y^2} = 2, \quad s = \frac{\partial^2 f}{\partial x \partial y} = -2$$

$rt - s^2 = 6 \times 2 - (-2)^2 = 12 - 4 = 8 > 0, \quad r = 6 > 0$

The function has a minimum at the critical point (2,6)

Example 2

Find the relative maxima or minima for the function

$f(x,y) = 2xy - x^3 - y^2$

Solution

$f(x,y) = 2xy - x^3 - y^2$

$$\frac{\partial f}{\partial x} = 2y - 3x^2, \qquad \frac{\partial f}{\partial y} = 2x - 2y,$$

Critical points are obtained by solving the equations $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$2y - 3x^2 = 0$

$2x - 2y = 0$

$2x - 2y = 0 \Rightarrow \; x = y$

Substituting in the equation, $2y - 3x^2 = 0$

$\Rightarrow 2x - 3x^2 = 0 \; \Rightarrow x(2 - 3x) = 0$

$$\Rightarrow x = 0, \qquad or \;\; 2 - 3x = 0 \;\; \Rightarrow x = \frac{2}{3}$$

$$\Rightarrow x = 0, \;\; x = \frac{2}{3}$$

When $x = 0$ then $y = 0 \Rightarrow (0,0)$ is a critical point

When $x = \frac{2}{3}$ then $y = \frac{2}{3} \Rightarrow (\frac{2}{3}, \frac{2}{3})$ is a critical point

$$r = \frac{\partial^2 f}{\partial x^2} = -6x, \qquad t = \frac{\partial^2 f}{\partial y^2} = -2, \;\; s = \frac{\partial^2 f}{\partial x \partial y} = 2$$

$rt - s^2 = 6x \times 2 - (2)^2 = 12x - 4$

At $x = (0,0)$, $rt - s^2 = -4 < 0 \;\Rightarrow (0,0)$ is a saddle point.

At $x = \left(\frac{2}{3}, \frac{2}{3}\right)$, $rt - s^2 = 12 \times \frac{2}{3} - 4 = 8 - 4 = 4 > 0$,

$(r)_{(\frac{2}{3}, \frac{2}{3})} = -6 \times \frac{2}{3} = -4 < 0$

$\Rightarrow ((\frac{2}{3}, \frac{2}{3}))$ is a maximum point.

Example 3

Find the relative maxima or minima for the function

$f(x, y) = 3xy - 6x - 3y + 7$

Solution

$f(x, y) = 3xy - 6x - 3y + 7$

$$\frac{\partial f}{\partial x} = 3y - 6, \qquad \frac{\partial f}{\partial y} = 3x - 3,$$

Critical points are obtained by solving the equations $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial x} = 0$

$3y - 6 = 0 \Rightarrow y = 2$

$2x - 2y = 0 \Rightarrow x = y = 2$

$\therefore\ y = 2, \quad x = 2$

$(1,2)$ is the critical point.

$$r = \frac{\partial^2 f}{\partial x^2} = 0, \qquad t = \frac{\partial^2 f}{\partial y^2} = 0, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 3$$

$rt - s^2 = -9$

When $x = (2,2)$, $rt - s^2 = -9 < 0 \Rightarrow (2,2)$ is a saddle point.

Example 4

Find the relative maxima or minima for the function $f(x, y) = x^3 + y^3 + 3xy$

Solution

$f(x, y) = x^3 + y^3 + 3xy$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y, \qquad \frac{\partial f}{\partial y} = 3y^2 + 3x$$

Critical points are obtained by solving the equations $\frac{\partial f}{\partial x} = 0$,

$$\frac{\partial f}{\partial y} = 0$$

$3x^2 + 3y = 0$

$3y^2 + 3x \ = 0$

$3x^2 + 3y = 0 \Rightarrow \ y = -x^2$

Substituting in the equation $3y^2 + 3x \ \Rightarrow 3(-x^2)^2 + 3x = 0 \ \Rightarrow x^4 + x = 0$

$\Rightarrow x(x^3 + 1) = 0 \ \Rightarrow \ \ x = 0, x^3 = -1 \ \ \ \Rightarrow \ x = 0, x = -1$

Substitute the value of x in the equation $y = -x^2$

When $x = 0, \quad y = 0 \Rightarrow (0,0)$ is a critical point

When $x = -1, \quad y = -(-1)^2 = -1 \Rightarrow (-1,-1)$ is a critical point

$$r = \frac{\partial^2 f}{\partial x^2} = 6x, \ \ t = \frac{\partial^2 f}{\partial y^2} = 6y, \ \ s = \ \frac{\partial^2 f}{\partial x \partial y} = 3$$

$rt - s^2 = 6x \times 6y - (3)^2 = 36xy - 9$

At $x = (0,0)$, $rt - s^2 = -9 < 0 \ \Rightarrow (0,0)$ is a saddle point.

At $x = (-1,-1)$, $rt - s^2 = 36 - 9 = 27 > 0,$

$(r)_{(-1,-1)} = -6 < 0$

$\Rightarrow (-1,-1)$ is a maximum point.

Example 5

Find the critical points and determine whether the function is at a relative maximum or minimum for the function:

$z = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$

Solution

To find the critical points, we need to calculate the first-order partial derivatives and set them equal to zero:

$\frac{\partial z}{\partial x} = \ 8x + 6y - 8 = 0$ …………. (1)

$\frac{\partial z}{\partial y} = 6x + 18y - 24 = 0$ ……….. (2)

Equation (1) multiplied by 3 $\Rightarrow 24\,x \ + \ 18\,y - \ 24 \ = \ 0$

Equation (2) $6x + 18y - 24 = 0$

Solving these equations simultaneously gives us the critical point: x = 0, y = $\frac{4}{3}$

Next, we need to analyse the second-order partial derivatives to determine whether the critical point is a relative maximum or minimum.

Calculate the second-order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = 8$$

$$\frac{\partial^2 z}{\partial y^2} = 18$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6$$

Now, evaluate these second-order partial derivatives at the critical point.

Since the product of the second-order direct partials is greater than the square of the cross partials ($8 \times 18 > 36$), and second order partial derivative with respect to 'x' ($8 > 0$) greater than zero, the function has a relative minimum at the critical point

(0, $\frac{4}{3}$).

Therefore, the critical point (0, $\frac{4}{3}$) corresponds to a relative minimum for the function

$$z = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$$

Summarised Overview

A function f(x) is labelled concave at x = α if its graph lies entirely beneath its tangent line in the nearby region around [α, f(α)]. Conversely, the function is deemed convex at x = α when its graph is situated entirely above its tangent line in proximity to [α, f(α)]. The presence of a positive second derivative at x = α indicates convexity, while a negative second derivative signifies concavity at α. The slope of the first derivative does not impact the concavity assessment. A positive second derivative ($f''(\alpha) > 0$) denotes convexity at x = α, and conversely, a negative second derivative ($f''(\alpha) < 0$) indicates concavity at x = α. For strictly convex or concave functions, the second derivative's sign holds true for the entire domain. The optimisation of single-variable functions involves finding critical points through the first derivative, using the second derivative to determine concavity or convexity, and applying the second-derivative test for relative maximum or minimum. Multivariable function optimisation requires satisfying conditions at critical points for relative minima or maxima, considering first-order partial derivatives and second-order direct and cross partial derivatives.

Assignments

1. Optimise the functions: (1) Find the critical value(s) at which the function is optimised and (2) Test for relative maximum or minimum.

   (a) $y = 7x^2 + 112x - 54$

   (b) $y = x^4 - 8x^3 - 80x^2 + 15$

   (c) $y = -9x^2 + 72x - 13$

2. Check out the functions, (1) find the critical points and (2) check whether at these points the function is maximised, is minimised, is at an inflection point, or is at a saddle point.

   a) $z = 2y^3 - x^3 + 147x - 54y + 12$

   b) $z = 3x^2 - xy + 2y^2 - 4x - 7y + 12$

   c) $f(x, y) = 60x + 34y - 4xy - 6x^2 - 3y^2 + 5$

   d) $z = 48y - 3x^2 - 6xy - 2y^2 + 72x$

   e) $f(x, y) = 5x^2 - 3y^2 - 30x + 7y + 4xy$

Suggested Reading

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
2. Chiang, A.C. (2008), *Fundamental Methods of Mathematical Economics*, McGraw Hill, New York.
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Reference

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UNIT 2
Lagrange Multiplier

Learning Outcomes

After completing this unit, learner will be able to:

- understand and solve real-world optimisation problems with constraints
- gain insight into the significance of the lagrange multiplier
- apply the lagrange multipliers method to optimise function
- solve systems of equations involving partial derivatives of the lagrangian function

Background

Imagine you are faced with a scenario where you want to find the best value for something, like maximising profit or minimising cost. This is where optimisation comes into play – the goal is to identify the highest or lowest point in a given situation. However, real-world problems often come with certain rules or conditions that must be followed while determining the optimal solution. This is where Lagrange Multipliers, introduced by the mathematician Joseph-Louis Lagrange, come in handy.

Lagrange Multipliers is a toolkit that allows you to navigate and solve these specialised optimisation puzzles. They help simplify complex problems by transforming them into more manageable ones. To illustrate, consider the example of baking a cake. You have a recipe to follow (constraints), and Lagrange Multipliers act as your tools to adjust the ingredients in such a way that you achieve the best possible cake, optimising the outcome while adhering to your recipe.

Now, let us apply this concept to planning a party with a budget constraint. Lagrange Multipliers function as your party planner in this scenario, assisting you in determining the optimal way to maximise fun without exceeding your budget.

They take into account your party plans, represented by the objective function (maximising fun), and the budget constraint, ensuring that you derive the utmost enjoyment within the financial limits set. Lagrange Multipliers serve as a powerful method for tackling optimisation problems with constraints. They enable you to strike a balance between achieving the best possible outcome and adhering to specific rules or limitations, making them an invaluable tool in various fields, from mathematics to real-world decision-making processes.

Keywords

Lagrange Multiplier, Constrained Optimisation, Critical Points

Discussion

2.2.1 Constrained Optimisation with Lagrange Multiplier

- Optimises constrained problems using derivative tests

Lagrange Multipliers method, named after Italian mathematician Joseph-Louis Lagrange, presents a brilliant approach in the realm of mathematical optimization. This technique converts a constrained problem into a format that enables the application of the derivative test for an unconstrained problem.

- Optimises functions with equality constraints, aiding precise solutions

The method of Lagrange multiplier is a fundamental tool for discerning local maxima and minima of a function in the format of $f(x, y, z)$, subject to equality constraints such as $g(x, y, z) = k$ or $g(x, y, z) = 0$. This methodology proves invaluable when dealing with scenarios where specific equations must be precisely satisfied by the values of the desired variables.

Differential calculus finds application in optimizing functions while adhering to constraints. When presented with a function $f(x, y)$ constrained by $g(x, y) = k$ (a constant), a novel function F can be created by (1) equating the constraint to zero, (2) multiplying it by the Lagrange multiplier (λ), and (3) adding the product to the original function:

$$F(x, y, \lambda) = f(x, y) + \lambda\,[k - g(x, y)].$$

The Lagrangian function $F(x, y, \lambda)$ comprises the original objective function $f(x, y)$ and the constraint $g(x, y)$. With the constraint set to zero, the product $\lambda[k - g(x, y)]$ also becomes zero, making it not alter the value of the objective function. To optimise the function, critical values x_0, y_0, and λ_0 are determined by calculating the partial derivatives of F concerning all three independent variables, equating them to zero, and solving the system of equations simultaneously:

$$F_x(x, y, \lambda) = 0$$

$$F_y(x, y, \lambda) = 0$$

$$F_l(x, y, \lambda) = 0$$

Example 1

Optimise the function:

$f(x, y) = 2x^2 - 5xy + 3y^2$

Subject to the constraint:

$x + 2y = 20$

Solution

We use the Lagrange Multiplier method to solve this problem. The Lagrangian function is given by

$L(x, y, \lambda) = f(x, y) = \lambda(g(x, y) - k)$

In this case, $f(x, y) = 2x^2 - 5xy + 3y^2$,

$g(x, y) = x + 2y$, and $k = 20$.

So the Lagrangian function becomes:

$L(x, y, \lambda) = 2x^2 - 5xy + 3y^2 + \lambda(x + 2y - 20)$

Now, we find the partial derivatives of L with respect to x, y, and λ, and set them equal to zero to find the critical points:

$$\frac{\partial L}{\partial x} = 4x - 5y + \lambda = 0$$

$$\frac{\partial L}{\partial y} = -5x + 6y + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = (x + 2y - 20) = 0$$

Solving this system of equations gives us the critical point

$$(x_0, y_0) = \left(\frac{160}{21}, \frac{130}{21}\right)$$

Finally, we substitute these values back into the original objective function to find the optimised value:

$$f\left(\frac{160}{21}, \frac{130}{21}\right) = 2\left(\frac{160}{21}\right)^2 - 5\left(\frac{160}{21}\right)\left(\frac{130}{21}\right) + 3\left(\frac{130}{21}\right)^2$$

$$= -4.76$$

So, the function is optimised to a value of -4.76 when $x = \frac{160}{21}$ and $y = \frac{130}{21}$, subject to the constraint

$x + 2y = 20.$

2.2.2 Significance of Lagrange Multiplier

- Lagrange multiplier estimates impact on objective function under constraints

The Lagrange multiplier (λ) holds significance as it serves to estimate the marginal influence on the objective function due to a minor alteration in the constant value of the constraint. This multiplier plays a pivotal role in optimization problems involving constraints. By introducing the Lagrange multiplier, the problem of maximizing or minimizing a function under certain constraints can be transformed into a more manageable form, enabling the determination of critical points where the function attains its extrema while adhering to the given constraints. The Lagrange multiplier (λ) serves as an approximation of the marginal impact exerted on the objective function due to a minor modification in the constant value of the constraint.

Example 2

To verify that a 1-unit change in the constant of the constraint will cause a change of approximately 348 units in Z, take the original objective function $z = 4x^2 + 3xy + 6y^2$ and optimise it subject to a new constraint $x + y = 57$ in which the constant of the constraint is 1 unit larger.

Z= $4x^2 + 3xy + 6y^2 + \lambda(57 - x - y)$

$$\frac{\partial z}{\partial x} = 8x + 3y - \lambda = 0$$

$$\frac{\partial z}{\partial y} = 3x + 12y - \lambda = 0$$

$$\frac{\partial z}{\partial \lambda} = 57 - x - y = 0$$

$\lambda = 8x + 3y$

$\lambda = 3x + 12y$

$\therefore 8x + 3y = 3x + 12y$

$5x = 9y$

$x = \frac{9}{5}y$

$57 - x - y = 0$

$57 - \frac{9}{5}y - y = 0$

$y(\frac{9}{5} + 1) = 57$

$y(\frac{14}{5}) = 57$

$y = \frac{57 \times 5}{14} = 20.36$

$x = \frac{9}{5} \times 20.36 = 36.64$

$\lambda = 8 \times 36.64 + 3 \times 20.36 = 354.2$

In the old constraint the constant of the constraint is 1 unit less than that present constant.

$\therefore$ old constraint is $x + y = 56$

and the objective function becomes

Z= $4x^2 + 3xy + 6y^2 + \lambda(56 - x - y)$

$$\frac{\partial z}{\partial x} = 8x + 3y - \lambda = 0$$

$$\frac{\partial z}{\partial y} = 3x + 12y - \lambda = 0$$

$$\frac{\partial z}{\partial \lambda} = 56 - x - y = 0$$

$$\lambda = 8x + 3y$$

$$\lambda = 3x + 12y$$

$$\therefore 8x + 3y = 3x + 12y$$

$$5x = 9y$$

$$x = \frac{9}{5}y$$

$$56 - x - y = 0$$

$$56 - \frac{9}{5}y - y = 0$$

$$y(\frac{9}{5} + 1) = 56$$

$$y(\frac{14}{5}) = 56$$

$$y = \frac{56 \times 5}{14} = 20$$

$$x = \frac{9}{5} \times 20.36 = 36$$

$$\lambda = 8 \times 36 + 3 \times 20 = 348$$

and $\therefore z = 9744$

Substituting these values in the Lagrangian function gives $\boldsymbol{Z} = \mathbf{10095}$ which is 351 larger than the old constrained optimum of 9744, a close approximation of the 348 increment suggested by $\boldsymbol{\lambda}$.

Example 3

Find the minimum value of $x^2 + y^2$ subject to the condition $2x + 3y = 6$

Solution

Let $L(x, y) = x^2 + y^2 + \lambda\,(2x + 3y - 6)$

$$\frac{\partial L}{\partial x} = 2x + 2\lambda, \qquad \frac{\partial L}{\partial y} = 2y + 2\lambda, \qquad \frac{\partial L}{\partial \lambda} = 2x + 3y - 6$$

Solve the equations $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$2x + 2\lambda = 0$, $2y + 3\lambda = 0, 2x + 3y - 6 = 0$

$2x + 2\lambda = 0 \Rightarrow \lambda = -x$

$2y + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}y$

Equating, $x = \frac{2}{3}y$

Substituting in $2x + 3y - 6 = 0 \Rightarrow 2 \times \frac{2}{3}y + 3y - 6 = 0$

$$\Rightarrow \frac{4}{3}y + 3y - 6 = 0 \Rightarrow \left(\frac{4}{3} + 3\right)y = 6 \Rightarrow \frac{13}{3}y = 6 \Rightarrow$$

$$y = \frac{6 \times 3}{13} = \frac{18}{13}$$

Substituting $y = \frac{18}{13}$ in the equation $x = \frac{2}{3}y$ we get

$$x = \frac{2}{3} \times \frac{18}{13} = \frac{12}{13}$$

$\left(\frac{12}{13}, \frac{18}{13}\right)$ is the critical point.

Minimum value = $\left(\frac{12}{13}\right)^2 + \left(\frac{18}{13}\right)^2 = \frac{36}{13}$

Example 4

Find the maximum value of $x^2 + 3xy - 6y$ subject to the condition $x + y = 18$

Solution

Let $L(x, y) = x^2 + 3xy - 6y + \lambda\,(x + y - 18)$

$$\frac{\partial L}{\partial x} = 2x + 3y + \lambda, \qquad \frac{\partial L}{\partial y} = 3x - 6 + \lambda,$$

$$\frac{\partial L}{\partial \lambda} = x + y - 18$$

Solve the equations $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$2x + 3y + \lambda = 0 \qquad 3x - 6 + \lambda = 0 \qquad x + y - 18 = 0$

$2x + 3y + \lambda = 0 \;\; \Rightarrow \; \lambda = -2x - 3y$

$3x - 6 + \lambda = 0 \;\;\; \Rightarrow \; \lambda = 6 - 3x$

Equating, $-2x - 3y = 6 - 3x \;\; \Rightarrow -2x + 3x = 6 + 3y$

$\Rightarrow \; x = 6 + y$

Substituting in $\;\; x + y - 18 = 0 \;\;\; \Rightarrow 6 + 3y + y - 18 = 0$

$\Rightarrow 4y = 18 - 6 \;\;\; \Rightarrow \;\; 4y = 12 \Rightarrow y = 3$

When $y = 3$, $\qquad x = 6 + 3 \times 3 = 6 + 9 = 15$

$(15, 3)$ is the critical point.

Maximum value = $(15)^2 + 3 \times 15 \times 3 - 6 \times 3 = 342$

Example 5

Find the minimum value of $x^2 - xy + y^2$ subject to the condition $x + y = 20$

Solution

Let $L(x, y) = \; x^2 - xy + y^2 + \lambda\,(x + y - 20)$

$\frac{\partial L}{\partial x} = 2x - y + \lambda, \qquad \frac{\partial L}{\partial y} = -x + 2y + \lambda,$

$\frac{\partial L}{\partial \lambda} = x + y - 20$

Solve the equations $\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial \lambda} = 0$

$2x - y + \lambda = 0 \;\;\; \Rightarrow \; \lambda = -2x + y$

$-x + 2y + \lambda, = 0 \;\; \Rightarrow \; \lambda = x - 2y$

Equating $\;\; -2x + y = x - 2y \;\;\;\; \Rightarrow 3x = 3y \;\;\;\; \Rightarrow x = y$

Substituting in $\;\;\; x + y - 20 = 0 \;\;\; \Rightarrow x + x = 20$

$\Rightarrow 2x = 20 \;\;\; \Rightarrow \;\; x = 10$

When $x = 10, \; y = 10$

$(10, 10)$ is the critical point.

Minimum value = $(10)^2 - 10 \times 10 + (10)^2 = 100$

Example 6

Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$

Solution

Let $L(x, y) = x^2 + y^2 + z^2 + \lambda\,(x + y + z - 3a)$

$$\frac{\partial L}{\partial x} = 2x + \lambda, \qquad \frac{\partial L}{\partial y} = 2y + \lambda, \qquad \frac{\partial L}{\partial z} = 2z + \lambda,$$

$$\frac{\partial L}{\partial \lambda} = x + y + z - 3a$$

Solve the equations $\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial z} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$

$2x + \lambda = 0\,, \qquad 2y + \lambda, = 0, \qquad 2z + \lambda, = 0$

$2x + \lambda = 0 \Rightarrow \lambda = -2x,$

$2y + \lambda = 0 \Rightarrow \lambda = -2y,$

$2z + \lambda = 0 \Rightarrow \lambda = -2z,$

Equating $-2x = -2y = -2z \quad \Rightarrow x = y = z$

Substituting in $x + y + z - 3a = 0 \quad \Rightarrow x + x + x =$

$3a \quad \Rightarrow 3x = 3a \quad \Rightarrow \quad x = a$

When $x = a, \qquad y = a, \; z = a$

(a, a, a) is the critical point.

Minimum value = $a^2 + a^2 + a^2 = 3a^2$.

Summarised Overview

The Lagrange Multipliers method, is a powerful tool for mathematical optimisation, particularly in constrained problems. This technique transforms constrained problems into a format suitable for applying the derivative test in unconstrained problems. The approach involves creating a Lagrangian function that combines the original objective function and the constraint, enabling the determination of critical points by setting partial derivatives equal to zero. The significance of the Lagrange Multiplier lies in its ability to estimate the marginal influence on the objective function due to changes in the constant value of the constraint. It plays a crucial role in transforming optimisation problems with constraints into more manageable forms, aiding in the identification of critical points where the function attains extrema while adhering to given constraints.

Assignments

1. Use Lagrange multipliers to optimise the functions subject to the given constraint, and calculate the effect on the value of the function from a 1-unit change in the constant

   a) $f(x, y) = 26x - 3x^2 + 5xy - 6y^2 + 12y$, subject to $3x + y = 170$

   b) $f(x, y, z) = 4xyz^2$, subject to $x + y + z = 56$

   c) $f(x, y, z) = 5xy + 8xz + 3yz$, subject to $2xyz = 1920$

2. Optimise the function

   $z = 4x^2 + 3xy + 6y^2$, subject to the constraint $x + y = 56$

3. Optimise the function

   $z = 4x^2 - 2xy + 6y^2$, Subject to $x + y = 72$

Suggested Reading

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
2. Yamane, Taro. (2012). *Mathematics for Economists: An Elementary Survey*. New Delhi: Prentice Hall of India.
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UNIT 3
Economic Applications

Learning Outcomes

After completing this unit, learner will be able to:

- analyse the rate of change in economic variables
- comprehend the impact of marginal concepts on demand, supply, cost, revenue, and production functions
- identify maxima and minima points in economic functions
- understand the real-world implications of economic models

Background

Economists and individuals alike use optimisation techniques to achieve the best possible outcomes, whether it is maximising satisfaction for consumers or maximising profits for producers. Utility refers to the satisfaction or well-being derived from consuming goods and services. For consumers, the goal is to maximise utility within the constraints of their budget. This is where optimisation comes into play. Imagine you have a limited budget, and you want to make the most out of it when purchasing different goods. Optimisation techniques help you find the optimal combination of goods that provides the highest level of satisfaction while staying within your budgetary limits. It is like finding the sweet spot that brings you the most happiness for every rupee you spend. For producers, optimisation revolves around cost minimisation and profit maximisation. In the production process, firms aim to minimise costs while simultaneously maximising profits. This delicate balance is achieved through strategic decision-making and the application of optimisation techniques. Consider a manufacturing scenario where a firm produces two goods. The firm's objective is to minimise the costs associated with production while maximising overall profit. Optimisation helps in determining the most efficient levels of production, ensuring that the resources are utilised effectively to yield the highest possible profit margins.

Understanding optimisation in economics is about making choices that lead to the best possible outcomes. Whether you are a consumer aiming for maximum satisfaction or a producer striving for optimal financial performance, the principles of optimisation guide decision-making processes in the dynamic world of economics.

Keywords

Producer, Consumer, Utility Maximisation, Cost Minimisation, Profit Maximisation

Discussion

2.3.1 Economic Applications

The tools of optimisation can be applied in the several contexts of economic analysis. When a consumer would like to maximise the utility, the optimisation techniques come to the forefront. On the other hand, when a producer would like to maximise profit by minimising cost, there also the tools of optimisation help an economist.

2.3.2 Utility Maximisation

- Consumers optimise well-being by allocating resources efficiently

Utility maximisation is a core principle in economics, capturing the essence of how individuals optimise their well-being within the constraints of limited resources. Rooted in the idea of rational decision-making, consumers seek to allocate their budget across various goods and services to achieve the highest level of satisfaction. The process involves weighing the marginal utility gained from each additional unit of a good against its price, aiming for an equilibrium where the marginal utility per rupee spent is consistent across all choices.

For a consumer, the utility is maximised with a constraint, i.e., the budget. The examples give an idea on how the utility can be maximised with a budget constraint.

Example 1

a. Maximise utility $u = Q_1Q_2$ when $P_1 = 1, P_2 = 4$, and budget $B = 120$.

Solution

a. The budget constraint is $Q_1 + 4Q_2 = 120$. Forming a new function to incorporate the constraint,

$U = Q_1Q_2 + \lambda(120 - Q_1 - 4Q_2)$

Thus,

$U_1 = Q_2 - \lambda = 0$

$U_2 = Q_1 - 4\lambda = 0$

$U_\lambda = 120 - Q_1 - 4Q_2 = 0$

and $\acute{Q}_1 = 60, \acute{Q}_2 = 15$, and $\acute{\lambda} = 15$

Maximum utility is u = 60 × 15 = 900

Example 2

a. Maximise utility $u = Q_1Q_2$, subject to $P_1 = 10, P_2 = 2$, and $B = 240$.

b. What is the marginal utility of money?

Solution

a. Form the Lagrangian function

$U = Q_1Q_2 + \lambda(240 - 10Q_1 - 2Q_2)$.

$U_1 = Q_2 - 10\lambda = 0$

$U_2 = Q_1 - 2\lambda = 0$

$U_\lambda = 240 - 10Q_1 - 2Q_2 = 0$

Thus, $\acute{Q}_1 = 12, \acute{Q}_2 = 60$, and $\acute{\lambda} = 6$

b. The marginal utility of money at $\acute{Q}_1 = 12$ and $\acute{Q}_2 = 60$ is 6.

Example 3

Maximise utility $u = Q_1Q_2 + Q_1 + 2Q_2$,

subject to $P_1 = 2, P_2 = 5$, and $B = 51$.

Form the Lagrangian function

$U = Q_1Q_2 + Q_1 + 2Q_2 + \lambda(51 - 2Q_1 - 5Q_2)$

$$\frac{\partial y}{\partial Q_1} = U_1 = Q_2 + 1 - 2\lambda = 0$$

Maximum utility

$u = (13 \times 5) + 13 + (2 \times 5)$

$u = 65 + 13 + 10$

$u = 88$

So, the maximum utility is 88 at the optimal values of Q_1=13 and Q_2=5 with the given budget constraint.

$$\frac{\partial y}{\partial U_2} = U_2 = Q_1 + 2 - 5\lambda = 0$$

$$U_\lambda = 51 - 2Q_1 - 5Q_2 = 0$$

Thus, $\acute{Q}_1 = 13, \acute{Q}_2 = 5$, and $\acute{\lambda} = 3$

2.3.3 Cost Minimisation and Profit Maximisation

- Cost minimisation optimises expenses, while profit maximisation boosts revenue

Cost minimisation and profit maximisation are essential objectives for businesses seeking efficient operations and financial success. Cost minimisation involves strategies to reduce production and operational expenses, optimising resource allocation and efficiency. By streamlining processes, negotiating favourable supplier contracts, and adopting technology, businesses aim to minimise costs while maintaining quality. On the other hand, profit maximisation focuses on increasing revenue and net income. This involves identifying optimal pricing strategies, expanding market share, and enhancing product or service offerings to meet consumer demands. Striking a balance between cost minimisation and profit maximisation is crucial for sustainable business growth, ensuring competitiveness in dynamic markets.

For a producer, optimisation happens once the profit is maximised. In order to maximise profit, cost minimisation is a must. The examples give an idea on how the profit maximisation is possible using optimisation techniques.

Example 4

A firm's costs are related to its output of two goods x and y functional relationship is

$TC = x^2 - 0.5xy + y^2$

The additional cost of a slight increment in output x will be given by the differential

$$dTC = (2x - 0.5y)dx$$

The costs of larger increments can be approximated by multiplying the partial derivative with respect to x by the change in x. Mathematically,

$$\Delta TC \approx \frac{\partial TC}{\partial x}\Delta x$$

Since $\frac{\partial TC}{\partial x}$ = the marginal cost (MC_x) of x, we can also write ΔTC as,

$$\Delta TC \approx MC_x \Delta x$$

If initially $x = 100, y = 60$, and $\Delta x = 3$, then

$$\Delta TC \approx [2(100) - 0.5(60)] \cong 510$$

Example 5

A firm producing two goods x and y has the profit function

$$\pi = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

Find the profit-maximising level of output.

Solution

To find the profit-maximising level of output for each of the two goods and test to be sure profits are maximised:

1. Take the first-order partial derivatives, set them equal to zero, and solve for x and y simultaneously.

$$\pi_x = 64 - 4x + 4y = 0$$
$$\pi_y = 4x - 8y + 32 = 0$$

When solved simultaneously, $\acute{x} = 40$ and $\acute{y} = 24$

2. Take the second-order direct partial derivatives and make sure both are negative, as is required for a relative maximum.

$$\pi_{xx} = -4\ ,\ \pi_{yy} = -8, \pi_{xy} = 4 = \pi_{yx}$$

3. Take the cross partials to make sure

   $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$

4. (-4)(-8) > $(4)^2$

   32 > 16

   π_{xx} = -4 < 0

Profits are indeed maximised at $\acute{x} = 40$ and $\acute{y} = 24$. At that point, $\pi = 1650$.

Example 6

Given the profit function $\pi = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18$ for a firm producing two goods x and y, (a) maximise profits, (b) test the second-order condition, and (c) evaluate the function at the critical values $\acute{x}$ and $\acute{y}$.

Solution

a) $\pi_x = 160 - 6x - 2y = 0\pi_y = -2x - 4y + 120 = 0$

When solved simultaneously, $\acute{x} = 20$ and $\acute{y} = 20$

b) Taking the second partials,

$\pi_{xx} = -6,\ \pi_{yy}$ = -4

$\pi_{xy} = (-2) = \pi_{yx}$

$-6 \times -4 > (-2)^2$ and π_{xx} = -6 < 0

$\pi_{xx}\pi_{yy} > (\pi_{xy})^2, \pi$ is maximised at $\acute{x} = \acute{y} = 20$

c) $\pi = 2782$

Example 7

Find the profit-maximising level of (a) output, (b) price, and (c) profit for the monopolistic producer with the demand functions

$Q_1 = 49\frac{1}{3} - \frac{2}{3}P_1$

$Q_2 = 36 - \frac{1}{2}P_2$

and the joint cost function $c = Q_1^2 + 2Q_1Q_2 + Q_2^2 + 120$.

Solution

a) From (Q_1) and (Q_2),

$$Q_1 = \frac{148}{3} - \frac{2}{3}P_1$$

$$Q_2 = 36 - \frac{1}{2}P_2$$

$3Q_1$ = 148 - $2P_1$

$2P_1$= 148 - $3Q_1$

$P_1 = 74 - 1.5Q_1$

Q_2 = 36 - $\frac{1}{2}P_2$

- $\frac{1}{2}P_2$ = 36 - Q_2

P_2 = 72 - $2Q_2$

Substituting P_1 and P_2 in $\pi = P_1Q_1 + P_2Q_2 - c$,

$$\pi = (74 - 1.5Q_1)Q_1 + (72 - 2Q_2)Q_2 - (Q_1^2 + 2Q_1Q_2 + Q_2^2 + 120)$$

$74Q_1 - 2.5Q_1^2 + 72Q_2 - 3Q_2^2 - 2Q_1Q_2 - 120$

The first-order condition for maximizing (π) is

$\pi_1 = 74 - 5Q_1 - 2Q_2 = 0, \pi_2 = 72 - 6Q_2 - 2Q_1$

Thus, $\acute{Q}_1 = 11.54$ and $\acute{Q}_2 = 8.15$. Testing the second-order condition, $\pi_{11} = -5, \pi_{22} = -6,$ and $\pi_{12} = -2.$ Thus, $\pi_{11}, \pi_{22} < 0; \pi_{11}\pi_{22} > (\pi_{12})^2$, and π is maximised.

b) Substituting the critical values in (P_1) and (P_2),

$P_1 = 74 - 1.5(11.54) = 56.69$

$P_2 = 72 - 2(8.15) = 55.70$

c) $\pi = 600.46$

Summarised Overview

The optimisation tools are essential for economists to analyse economic scenarios effectively. Utility maximisation, a focal point in consumer choices, involves understanding how individuals can make optimal decisions to maximise their satisfaction or well-being given budget constraints. The Lagrangian function, is a mathematical tool that allows economists to efficiently solve complex optimisation problems related to utility maximisation. This method provides a systematic approach to finding the optimal combination of goods or services that a consumer should choose to achieve the highest level of satisfaction within their budget limitations.

Cost minimisation is crucial for optimising resources and ensuring efficient production processes. The concept of marginal cost helps to determine the additional cost incurred with each incremental unit of output. Understanding marginal cost is essential for producers aiming to make informed decisions about production levels and resource allocation. Profit maximisation, another key aspect for producers, involves finding the optimal level of output that results in the highest profit. The step-by-step process of identifying the profit-maximising levels of output for goods, showcasing how economic principles can guide firms in making strategic decisions to enhance their financial performance.

Optimisation tools in economics, offer a systematic approach for both consumers and producers to make informed decisions. Whether it is maximising utility within budget constraints for consumers or optimising costs and profits for producers, these mathematical techniques provide a valuable framework for analysing and improving economic scenarios.

Assignments

1. Maximise utility $u = xy + 3x + y$ subject to $P_x = 8, P_y = 12$, and $B = 212$.
2. Find the profit-maximizing level of (a) output, (b) price, and (c) profit when

$$Q_1 = 5200 - 10P_1$$
$$Q_2 = 8200 - 20P_2$$

and $c = 0.1Q_1^2 + 0.1Q_1Q_2 + 0.2Q_2^2 + 325$

Suggested Reading

1. Allen, R.G.D. (2008). *Mathematical Analysis for Economists*. New Delhi: AITBS Publishers.
2. Yamane, Taro. (2012). *Mathematics for Economists: An Elementary Survey*. New Delhi: Prentice Hall of India.
3. Chiang, A.C. (2008), *Fundamental Methods of Mathematical Economics*, McGraw Hill, New York.
4. Dowling T. Edward (2000), *Schaum's Outline of Introduction to Mathematical Economics*, McGraw Hill, New York.

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1. Michael Hoy and L. John (2004) *Mathematics for Economics*, PHI, New Delhi.
2. Sydsaeter, Knut, and Peter J. Hammond. (2005). "*Mathematics for Economic Analysis*." Pearson Education Inc., Delhi

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

UNIT 4
Linear Programming

Learning Outcomes

After completing the unit, learner will be able to

- understand the fundamental characteristics and nature of linear programming problems
- learn to formulate linear programming problems
- solve linear programming problems using the simplex method
- analyse duality and understand the relationship between the primal and dual problems

Background

Linear programming is a valuable tool in economics, operations research, and many other fields enabling efficient decision-making in resource-constrained scenarios. Consider a business aiming to produce two distinct products while facing constraints like limited labour, raw materials, and time. Linear programming serves as a strategic framework to optimise the allocation of these resources, ultimately working towards financial goals such as profit maximisation or cost minimisation. In essence, linear programming facilitates the optimal utilisation of limited resources to achieve specific objectives while adhering to linear relationships and constraints. Its applications extend across various fields, empowering businesses, and organisations to make data-driven, informed decisions. It is similar to an analytical compass guiding economic agents toward resource efficiency and strategic economic decision-making.

Keywords

Linear Programming, Simplex Method, Duality, Constraints, Feasible Solutions

Discussion

2.4.1 Linear Programming

A bakery owner produces two types of cakes: chocolate cakes and fruit cakes. The owner has a budget of ₹800 for ingredients and packaging materials, and the available oven space allows for the production of up to 100 cakes. A chocolate cake costs ₹10 to make, while a fruit cake costs ₹15. The bakery owner estimates that selling one chocolate cake results in a profit of ₹5, while selling one fruit cake yields a profit of ₹7. The owner aims to determine the optimal quantity of chocolate and fruit cakes to produce within the given constraints to maximise their total profit, assuming all produced cakes can be sold.

- Constrained optimisation by maximising or minimising a linear objective function

The class of problems that aim to maximise or minimise factors like profit, cost, or resource usage fall under a broader category known as optimisation problems. Linear programming is a significant subset of optimisation problems, focusing on finding the best solutions within linear constraints. These problems hold great significance due to their extensive applicability across various fields of economics making them a valuable tool for decision-making and problem-solving.

Standard form

1. The objective function can be expressed as a linear function to be maximised. If it is minimised convert into maximised by multiplying by -1
2. Constraints are expressed in linear equalities.
3. All variables are non-negative.

Canonical form

1. The objective function can be expressed as a linear function to be maximised. If it is minimised convert into maximised by multiplying by -1
2. Constraints are less than or equal to (≤) type.
3. All variables are non-negative.

2.4.2 Characteristics of Linear Programming Problem (LPP)

Following are the characteristics of LPP

- Decision variables are used to represent the quantities of items

1. Decision variables will decide the output:

In a Linear Programming Problem (LPP), decision variables are used to represent the quantities of items that are to be determined to achieve a particular objective. The decisions made about these variables will directly affect the outcome or output of the problem. For example, in a production problem, the decision variables might represent the number of products to manufacture.

- Objective function typically involves the decision variables

2. The objective function should be specified quantitatively:

The objective function in an LPP defines the goal of the problem in quantitative terms. It is a mathematical expression that needs to be either maximised or minimised. The objective function typically involves the decision variables and reflects the goal of the optimization process. For instance, if the goal is to maximise profit, the objective function would involve profit values associated with the decision variables.

- Constraints are expressed as mathematical equations

3. Constraints should be expressed in mathematical form:

Constraints in an LPP represent limitations or restrictions on the available resources. These constraints are expressed as mathematical inequalities or equations involving the decision variables. They reflect the practical restrictions that need to be adhered to while optimizing the objective function. For example, constraints can represent limitations on resource availability, production capacities, or budget constraints.

- Linear relationship between decision variables, coefficients and constraints

4. Relationships between two or more variables should be linear:

In an LPP, the relationships between the decision variables, as well as the coefficients in the objective function and constraints, must be linear. This means that the variables are raised to the power of one and are not multiplied or divided with each other. The objective function and constraints should involve variables and coefficients in a linear fashion, such as

$Ax + By = C$

5. The values of the variables should always be non-negative or zero:

In real-world applications, decision variables often represent quantities that cannot be negative (e.g., number of products

- Non-negative values of decision variables

produced) or cannot take negative values (e.g., time spent on a task). Therefore, the values of decision variables are constrained to be non-negative or zero in LPPs.

6. There should always be finite and infinite inputs and output numbers:

- Finite and infinite values to determine optimal solutions

LPPs consider finite quantities of resources and items, as well as finite outputs (such as profits or costs). However, in some cases, infinite values (e.g., profit per unit) can be part of the problem formulation. These finite and infinite values are used to determine optimal solutions.

7. For a Linear Programming problem, Decision Variable, Objective Function, and Constraint function should always be linear functions:

- Simple relationships between variables and coefficients

In an LPP, the linearity of decision variables, objective function, and constraints is a fundamental requirement. Linear functions ensure that the relationships between variables and coefficients remain simple and follow the principles of linear algebra. This linearity is essential for efficiently solving LPPs using various optimization techniques.

2.4.3 Formulation of LPP

Suppose A and B represent the quantity of cakes and pastries to be produced at a bakery, respectively. These quantities are non-negative and are referred to as *non-negative constraints*.

Given the bakery has a total of 120 hours of baking time available and needs to produce up to 30 items, we have:

$$3A + 2B \leq 120$$

$$A + B \leq 30$$

These inequalities are referred to as linear constraints.

If P represents the profit from baking A cakes and B pastries, we have:

P = 50A + 40B

Our objective is to maximise the profit P, making it our objective function.

2.4.4 Solution of LPP using Simplex method

- The simplex method is the most popular method used for the solution of Linear Programming Problems (LPP).
- The Simplex method is a search procedure that shifts through the set of basic feasible solutions, one at a time until the optimal basic feasible solution is identified.
- It can be used for two or more variables as well (always advisable for more than two variables to avoid lengthy graphical procedure).
- The simplex method is not used to examine all the feasible solutions.
- It deals only with a small and unique set of feasible solutions, the set of vertex points (i.e. extreme points/corner points) of the convex feasible space that contain the optimal solution.
- All the resource values or constraints should be non-negative.
- All the inequalities of the constraint should be converted to equalities with the help of slack or surplus variables.

- Simplex method finds optimal solutions by exploring vertex points

Example 1

Use simplex method to solve the following L.P.P.

Maximise $z = 7x_1 + 5x_2$

subject to the constraints:

$x_1 + 2x_2 \leq 6$

$4x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0$

Solution

Convert the LPP into standard form

Maximise

$z = 7x_1 + 5x_2 + 0x_3 + 0x_4$

$x_1 + 2x_2 + x_3 = 6$

$4x_1 + 3x_2 + x_4 = 12$

$x_1, x_2, x_3, x_4 \geq 0$

Find an initial basic feasible solution.

Since the number of constraints = 2

and number of variable = 4 , $4 - 2 = 2$

variables are non basic variables. Let the non basic variables be x_1 and x_2 and its values are $x_1 = 0,\ x_2 = 0,$

The remaining variables are, x_3, x_4 are basic variables and its values from the equations are

$x_3 = 6, \quad x_4 = 12,$

The optimal value is $z = 7 \times 0 + 5 \times 0 = 0$

Since the optimality is not attained, the basic variable which leaves the basis and the non basic variable which enters the basis so as to form a new basic feasible solution. Repeat the process till the optimality attained.

Table 2.4.1 Simplex Table 1- Example 1

	C →	7	5	0	0	
c_B	x_B	x_1	x_2	x_3	x_4	Ratio
0	$x_3 = 6$	1	**2**	1	0	$\frac{6}{1} = 6$
0	$x_4 = 12$	**4***	3	0	1	$\frac{12}{4} = 3$
$z_j - c_j$	$z = 0$	-7	−5	0	0	

In the simplex table, C_B is the coefficient of the basic variable x_3, x_4 in the objective function. Here the values are zeros.

C is the coefficient of the objective function.

x_B is the basic variables.

$z_j - c_j = \sum c_B . x_i - C$ is the net evaluation.

Here, $z_1 - c_1 = 0 \times 1 + 0 \times 4 - 7 = -7$

$z_2 - c_2 = 0 \times 2 + 0 \times 3 - 5 = -5$

Now, since more than one $z_j - c_j$ are negative, we choose the most negative of these, viz., -7, which lies in the column x_1. Since all the components of x_1 are positive, the vector x_1 will enter the basis x_B.

To select the vector which should leave the basis x_B, we compute $\left\{\frac{x_{B_i}}{x_{i_1}}, x_{i_1} > 0, i = 1,2\right\}$ and choose the minimum of these ratios, viz., $\frac{12}{4} = 3$. Thus the vector x_4 leaves the basis.

The leading common element (pivotal element) is 4, which becomes the leading element for the next iteration. The pivotal element has been shown in the simplex table in bold type with a star.

4 is the pivotal element, x_1 enters in the basis and x_4 leaves from the basis.

Convert the leading element to unity by dividing that row by 4

Table 2.4.2 Simplex Table 2- Example 1

	$C \rightarrow$	7	5	0	0
	x_B	x_1	x_2	x_3	x_4
	$x_3 = 6$	1	**2**	1	0
	$x_1 = \frac{12}{3} = 4$	**1**	$\frac{3}{4}$	0	$\frac{1}{4}$
$z_j - c_j$	$z = 0$	-7	-5	0	0

All other members of the column x_1 to zero by using elementary transformation $R_1 \rightarrow R_2 - R_1$, $R_3 \rightarrow 7R_2 + R_3$ Compute again the net evaluations. $z_j - e_j$.

Table 2.4.3 Simplex Table 3- Example 1

$C \rightarrow$	7	5	0	0
x_B	x_1	x_2	x_3	x_4
$x_3 = 2$	0	$-\frac{5}{4}$	1	$\frac{1}{4}$

$x_1 = 3$	1	$\frac{3}{4}$	0	$\frac{1}{4}$
$z = 21$	0	$\frac{1}{4}$	0	$\frac{7}{4}$

Since all the net evaluations are non-negative, the solution is optimal.

Thus the optimal solution is $x_1 = 3,\ \ x_2 = 0,\ \ max\ z = 21$

Example 2

Use simplex method to solve the following L.P.P.

Maximise, $z = 3x_1 + 5x_2$

subject to the constraints:

$x_1 + x_2 \leq 2$

$2x_1 + 5x_2 \leq 10$

$8x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0$

Solution

Convert the LPP into standard form

Maximise

$z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$

$x_1 + x_2 + x_3 = 2$

$2x_1 + 5x_2 + x_4 = 10$

$8x_1 + 3x_2 + x_5 = 12$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Since the number of constraints = 3 and number of variable = 5, $5 - 3 = 2$ variables are non basic variables. Let the non basic variables be x_1 and x_2 and its values are $x_1 = 0,\ x_2 = 0.$

The remaining variables are x_3, x_4, x_5 and its values from the equations are

$x_3 = 2, \quad x_4 = 10, \quad x_5 = 12$
The optimal value is $z = 3 \times 0 + 5 \times 0 = 0$

Since the optimality is not attained, the basic variable which leaves the basis and the non basic variable which enters the basis so as to form a new basic feasible solution.

Table 2.4.4 Simplex Table 1- Example 2

	C→	3	5	0	0	0	
c_B	x_B	x_1	x_2	x_3	x_4	x_5	Ratio
0	$x_3 = 2$	1	**1***	1	0	0	$\frac{2}{1} = 2$
0	$x_4 = 10$	2	5	0	1	0	$\frac{10}{5} = 2$
0	$x_5 = 12$	8	3	0	0	1	$\frac{12}{3} = 4$
$z_j - c_j$	$z = 0$	-3	−5	0	0	0	

Now, since more than one $z_j - c_j$ are negative, we choose the most negative of these, viz., -5, which lies in the column x_2. Since all the components of x_2 are positive, the vector x_2 will enter the basis x_B.

To select the vector which should leave the basis $\mathbf{x_B}$, we compute $\left\{\frac{\mathbf{x_{B_i}}}{\mathbf{x_{i_1}}}, \mathbf{x_{i_1}} > \mathbf{0}, \mathbf{i} = \mathbf{1}, \mathbf{2}\right\}$ and choose the minimum of these ratios, viz., $\frac{\mathbf{2}}{\mathbf{1}} = \mathbf{2}$. Thus the vector $\mathbf{x_3}$ leaves the basis. (If two ratios are equal we can arbitrary choose one vector)

1 is the pivotal element, x_2 enters in the basis and x_3 leaves from the basis.

Convert the leading element to unity by dividing that row by 1.

Table 2.4.5 Simplex Table 2- Example 2

	C→	3	5	0	0	0	
	x_B	x_1	x_2	x_3	x_4	x_5	Ratio
	$x_2 = 2$	1	**1**	1	0	0	$\frac{2}{1} = 2$
	$x_4 = 10$	2	5	0	1	0	$\frac{10}{5} = 2$
	$x_5 = 12$	8	3	0	0	1	$\frac{12}{3} = 4$
$z_j - c_j$	$z = 0$	-3	−5	0	0	0	$z = 0$

All other members of the column x_2 to zero by using elementary transformation.

$R_2 \to -5R_1 + R_2,\ R_3 \to -3R_1 + R_3$

$R_4 \to 5R_1 + R_4$. Compute again the net evaluations. $z_j - e_j$.

Table 2.4.6 Simplex Table 3- Example 2

x_B	x_1	x_2	x_3	x_4	x_5
$x_2 = 2$	1	1	1	0	0
$x_4 = 10$	-3	0	-5	1	0
$x_5 = 12$	5	0	-3	0	1
$z = 10$	2	0	5	0	0

Since all the net evaluations are non-negative, the solution is optimal.

Thus the optimal solution is $x_1 = 0,\ \ x_2 = 2,\ \ max\ z = 10$

Example 3

Use simplex method to solve the following L.P.P.

Minimise $z = -2x_1 - 3x_2$

$2x_1 + x_2 \leq 6$

$x_1 + 2x_2 \leq 8$

$x_1 - x_2 \leq 1$

$x_1 \leq 2$
$x_1, x_2 \geq 0$

Solution

Convert the LPP into standard form

Maximise

$z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$

subject to the constraints:

$2x_1 + x_2 + x_3 = 6$

$x_1 + 2x_2 + x_4 = 8$

$x_1 - x_2 + x_5 = 1$

$x_1 + x_6 = 2$

$x_1, x_2, x_3, x_4, x_{5,}x_6 \geq 0$

Since the number of equations = 4 and number of variable = 6, $6 - 4 = 2$ variables are non basic variable.

Take $x_1 = 0,\ x_2 = 0$ as non basic variables.
$x_3 = 6,\quad x_4 = 8,\quad x_5 = 1,\quad x_6 = 2$ are basic variable.

Table 2.4.7 Simplex Table 1- Example 3

	C →	2	3	0	0	0	0	
c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	
0	$x_3 = 6$	2	1	1	0	0	0	$\frac{6}{1} = 6$
0	$x_4 = 8$	1	**2***	0	1	0	0	$\frac{8}{2} = 4$
0	$x_5 = 1$	1	-1	0	0	1	0	
0	$x_6 = 2$	1	0	0	0	0	1	
$z_j - c_j$	$z = 0$	-2	−3	0	0	0	0	x_2 enters, x_4 leaves

Table 2.4.8 Simplex Table 2- Example 3

	$c \rightarrow$	2	3	0	0	0	0
	x_B	x_1	x_2	x_3	x_4	x_5	x_6
	$x_3 = 6$	2	1	1	0	0	0
	$x_2 = 4$	1/2	1	0	1/2	0	0
	$x_5 = 1$	1	-1	0	0	1	0
	$x_6 = 2$	1	0	0	0	0	1
$z_j - c_j$	$z = 0$	-2	-3	0	0	0	0

all other members of the column x_2 to zero by using elementary transformation

$R_1 \rightarrow R_1 - R_2,\ R_3 \rightarrow R_2 + R_3,\ \ R_5 \rightarrow 3R_2 + R_5$

Compute again the net evaluations. $z_j - e_j$.

Table 2.4.9 Simplex Table 3- Example 3

$c \rightarrow$	2	3	0	0	0	0	
x_B	x_1	x_2	x_3	x_4	x_5	x_6	
$x_3 = 2$	$\frac{3}{2}$*	0	1	$-\frac{1}{2}$	0	0	$\frac{2}{\frac{3}{2}} = \frac{4}{3}$
$x_2 = 4$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{4}{\frac{1}{2}} = 8$
$x_5 = 5$	$\frac{3}{2}$	0	0	$\frac{1}{2}$	1	0	$\frac{5}{\frac{3}{2}} = \frac{10}{3}$
$x_6 = 2$	1	0	0	0	0	1	$\frac{2}{1} = 2$
$z = 12$	$(-\frac{1}{2})$	0	0	$\frac{3}{2}$	0	0	x_1 enters, x_3 leaves

Table 2.4.10 Simplex Table 4- Example 3

c →	2	3	0	0	0	0	
x_B	x_1	x_2	x_3	x_4	x_5	x_6	
$x_1 = \frac{4}{3}$	**1**	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{\frac{2}{3}}{2} = \frac{4}{3}$
$x_2 = \frac{10}{3}$	0	1	$-\frac{1}{3}$	$\frac{2}{2}$	0	0	$\frac{4}{\frac{1}{2}} = 8$
$x_5 = 3$	0	0	-1	1	1	0	$\frac{5}{\frac{3}{2}} = \frac{10}{3}$
$x_6 = \frac{2}{3}$	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{2}{1} = 2$
$z = \frac{38}{3}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$	0	0	

Since all the net evaluations are bon-negative, the solution is optimal.

Thus the optimal solution is $x_1 = \frac{4}{3}$, $x_2 = \frac{10}{3}$, $max\ z = \frac{38}{3}$, $min\ z = -\frac{38}{3}$

Example 4

Use simplex method to solve the following L.P.P.

Minimise $z = x_1 - 3x_2 + 2x_3$

$3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Solution

Convert the LPP into standard form

Maximise

$z = -x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$

subject to the constraints:

$3x_1 - x_2 + 2x_3 + x_4 = 7$

$-2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Since the number of equation =3 and number of variable = 6 ,

$6 - 3 = 3$ variables are non basic

Take $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ as non basic variables.

Then $x_4 = 7, x_5 = 12$, $x_6 = 10$ basic variable. solution

$z = 0$

Construct a table as follows

Table 2.4.11 Simplex Table 1- Example 4

	c	-1	3	-2	0	0	0	
c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	
0	$x_4 = 7$	3	-1	2	1	0	0	
0	$x_5 = 12$	-2	**4***	0	0	1	0	$\frac{12}{4} = 3$
0	$x_6 = 10$	-4	3	8	0	0	1	$\frac{10}{3} = 3.33$
	z = 0	1	−3	2	0	0	0	$z_j - c_j$

4 is the pivotal element

Table 2.4.12 Simplex Table 2- Example 4

	c =	-1	3	-2	0	0	0
c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6
0	$x_4 = 7$	3	-1	2	1	0	0
0	$x_5 = 3$	$-\frac{1}{2}$	**1**	0	0	$\frac{1}{4}$	0

0	$x_6 = 10$	-4	3	8	0	0	1
	z = 0	1	−3	2	0	0	0

Table 2.4.13 Simplex Table 3- Example 4

c =	-1	3	-2	0	0	0	
x_B	x_1	x_2	x_3	x_4	x_5	x_6	
$x_4 = 10$	$\frac{5}{2}*$	0	2	1	$\frac{1}{4}$	0	$\frac{10}{\frac{5}{2}} = 4$
$\boldsymbol{x_2 = 3}$	$\frac{-1}{2}$	1	0	0	$\frac{1}{4}$	0	
$x_6 = 1$	$\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	
z' = 9	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	$z_j - c_j$

$\frac{5}{2}$ is the pivotal element

Table 2.4.14 Simplex Table 4- Example 4

	c =	-1	3	-2	0	0	0	
c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	
0	$x_4 = 4$	**1**	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	
0	$x_2 = 3$	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	
0	$x_6 = 1$	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	
	z' = 9	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	$z_j - c_j$

Table 2.4.15 Simplex Table 5- Example 4

c	-1	3	-2	0	0	0	
x_B	x_1	x_2	x_3	x_4	x_5	x_6	
$x_1 = 4$	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	
$x_2 = 5$	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	
$x_6 = 11$	0	0	10	$\frac{2}{5}$	$-\frac{1}{1}$	1	
z' = 11	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	$z_j - c_j$

optimum basic feasible solutions to the given L.P.P. is

$x_1 = 4, x_2 = 5, x_3 = 0 \max z' = \min z = -11$

2.4.5 Duality

- Linear programming problems have primal and dual counterparts

Every maximisation (minimisation) problem in linear programming has a corresponding minimisation (maximisation) problem. The original problem is called the primal, the through the use of the is called the dual. The relationship between the two can best be expressed through the use of the parameters they share in common.

Table 2.4.16 Primal and Dual

Primal	**Dual**
Objective function Maximise	Objective function Minimise
n variable and m constraints	m variable and n constraints
Constraints are $\leq$ type	Constraints are $\geq$ type
Coefficient of objective function	Constant
Constant	Coefficient of objective function

2.4.6.1 Rules of Transformation to Obtain the Dual

In the formulation of a dual from a primal problem,

1. The direction of optimisation is reversed. Maximisation in the primal becomes minimisation in the dual and vice versa.
2. The inequality signs of the technical constraints are reversed, but the nonnegativity restraint on decision variables is always maintained.
3. The rows of the coefficient matrix of the constraints in the primal are transposed to columns for the coefficient matrix of constraints in the dual.
4. The row vector of coefficients in the objective function in the primal is transposed to a column vector of constants for the dual constraints.
5. The column vector of constants from the primal constraints is transposed to a row vector of coefficients for the objective function in the dual.
6. Primal decision variables $\boldsymbol{x_j}$ are replaced by dual decision variables $\boldsymbol{z_i}$.

Example 1

Obtain the dual problem of the following L.P.P:

Maximise subject to the constraints:

$$f(x) = 2x_1 + 5x_2 + 6x,$$

$$5x_1 + 6x_2 - x_3 \leq 3$$

$$-2x_1 + x_2 + 4x_3 \leq 4$$

$$x_1 - 5x_2 + 3x_3 \leq 1$$

$$-3x_1 - 3x_2 + 7x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Also verify that the dual of the dual problem is the primal problem.

Solution

Coefficient matrix in the primal is

$$A = \begin{bmatrix} 5 & 6 & -1 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & -2 & 1 & -3 \\ 6 & 1 & -5 & -3 \\ -1 & 4 & 3 & 7 \end{bmatrix}$$

Let w_1, w_2, w_3, w_4 be the dual variable.
The dual is
Minimise $g(x) = 3w_1 + 4w_2 + w_3 + 6w_4$

subject to the constraints: $5w_1 - 2w_2 + w_3 - 3w_4 \geq 2$

$6w_1 + w_2 - 5w_3 - 3w_4 \geq 5$

$-w_1 + 4w_2 + 3w_3 + 7w_4 \geq 6$

$w_1, w_2, \quad w_3, w_4 \geq 0$

Dual of the dual problem

$$A = \begin{bmatrix} 5 & -2 & 1 & -3 \\ 6 & 1 & -5 & -3 \\ -1 & 4 & 3 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 6 & -1 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{bmatrix}$$

Minimise, $f(x) = 2x_1 + 5x_2 + 6x_3$

subject to the constraints: $5x_1 + 6x_2 - x_3 \leq 3$

$-2x_1 + x_2 + 4x_3 \leq 4$

$-x_1 - 5x_2 + 3x_3 \leq 1$

$-3x_1 - 3x_2 + 7x_3 \leq 6$

$x_1, x_2, \; x_3 \geq 0$

Dual of the dual is the primal.

Example 2

Construct the dual problem of the following L.P.P:

Maximise $z = 3x_1 + 17x_2 + 9x_3$

subject to the constraints: $x_1 - x_2 + x_3 \geq 3$

$-3x_1 + 2x_2 \leq 1$

$x_1, x_2, \; x_3 \geq 0$

Solution

The primal is not in the canonical form. Convert it to canonical form

Maximise $z = 3x_1 + 17x_2 + 9x_3$

subject to the constraints: $-x_1 + x_2 - x_3 \leq -3$

$-3x_1 + 2x_2 \leq 1$

$x_1, x_2,\ x_3 \geq 0$

$$A = \begin{bmatrix} -1 & 1 & -1 \\ -3 & 2 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & -3 \\ 1 & 2 \\ -1 & 0 \end{bmatrix}$$

The dual is

Minimise $z^* = -3w_1 + w_2$

subject to the constraints: $-w_1 - 3w_2 \geq 3$

$w_1 + 2w_2 \geq 17$

$-w_1 \geq 9$

$w_1, w_2, \quad w_3, w_4 \geq 0$

Summarised Overview

Linear Programming (LP) is a critical subset of optimisation problems within economics, focused on maximising or minimising objectives like profit, cost, or resource utilisation while adhering to linear constraints. Key characteristics of Linear Programming Problems (LPP) include decision variables that directly influence outcomes, quantitatively defined objective functions, mathematical expression of constraints, linearity of variable relationships, non-negativity of decision variables, and consideration of finite and infinite values. Formulating an LPP involves defining non-negative constraints and linear constraints to maximise an objective function, such as profit. The Simplex method is the primary technique for solving LPP, involving a search for the optimal basic feasible solution by considering a unique set of feasible solutions, known as vertex points, within a convex feasible space. It is essential to ensure non-negative resource values and convert constraint inequalities into equalities when using the Simplex method. These concepts form the foundation of Linear Programming in economics, facilitating rigorous decision-making and problem-solving.

Duality in linear programming is a crucial concept where every maximisation problem has an associated minimisation problem, referred to as the primal and dual, respectively. The relationship between them is defined by specific rules of transformation. In this process, the direction of optimisation is reversed, and inequality signs of technical constraints change while maintaining non-negativity constraints on decision variables. Coefficient matrices and vectors in the primal are transposed to form the dual. This duality concept allows for a fresh perspective on problem-solving, offering insights into both maximising and minimising aspects of a problem, contributing to informed economic decision-making within the constraints and objectives of a given scenario.

Assignments

1. A manufacturer produces tables x_1 and desks x_2. Each table requires 2.5 hours for assembling A,3 hours for buffing B, and 1 hour for crating C. Each desk requires 1 hour for assembling, 3 hours for buffing, and 2 hours for crating. The firm can use no more than 20 hours for assembling, 30 hours for buffing, and 16 hours for crating each week. Its profit margin is ₹3 per table and ₹4 per desk. Determine the maximum point of profit obtained by the manufacturer.

2. The dual of the linear programming problem

   Maximise

$$\Pi = 5x_1 + 3x_2$$

   subject to

$$6x_1 + 2x_2 \leq 36$$
$$5x_1 + 5x_2 \leq 40$$
$$2x_1 + 4x_2 \leq 28 x_1, x_2 \geq 0$$

   is

   Minimise

$$c = 36z_1 + 40z_2 + 28z_3$$

   Subject to

$$6z_1 + 5z_2 + 2z_3 \geq 5$$
$$2z_1 + 5z_2 + 4z_3 \geq 3$$
$$z_1, z_2, z_3 \geq 0$$

Suggested Reading

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Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

MASTER OF ARTS ECONOMICS

Estimation Theory

BLOCK 3

UNIT 1
Statistical Inference

Learning Outcomes

After completing this unit, learner will be able to:

- understand the concepts associated with statistical inference
- apply statistical tools to analyse and interpret data effectively
- present findings in a clear, concise, and understandable way

Background

Statistical Inference is a branch of statistics that involves drawing conclusions about a population based on a sample of data from that population. The entire group of individuals that share a common characteristic is referred to as the population. In statistical inference, the goal is often to make statements or predictions about this broader population. As it is often impossible to study an entire population, researchers select a subset, known as a sample, to represent the larger group. Statistical inference relies on the idea that a well-chosen sample can provide meaningful insights into the characteristics of the population from which it was drawn.

A key concept in statistical inference is the sampling distribution. This distribution represents the probability distribution of a sample statistic (such as the mean or standard deviation) over all possible samples of a given size from the same population. Understanding sampling distributions is crucial for making inferences, as it helps in estimating the variability and uncertainty associated with sample statistics. Important parameters, such as the mean and standard deviation of the sampling distribution, plays a pivotal role in statistical inference. This principle states that as the size of a sample increases, the sample mean approaches the true population mean. In other words, larger samples tend to produce more accurate estimates of the population parameters.

Keywords

Population, Sample, Sampling Distribution, Central Limit Theorem, Standard Error

Discussion

3.1.1 Statistical Inference

Inferential statistics plays a crucial role in economic analysis in the way of applying both quantitative and qualitative techniques in economics. These techniques allow us to make informed observations, predictions, and comments about the characteristics of the entire population based on the data we gather from a representative sample.

- Crucial role in analysing economic phenomena

We will begin by examining illustrative examples that give an understanding of the applications of statistical inference in the field of economics. These examples emphasise that statistical tools are vital, highlighting their crucial role in analysing economic phenomena. The price of petrol is influenced by factors, including supply, demand, production costs, government policies, and consumer preferences. Even when these factors are taken into account, there can still be variations in the commodity's price. This variability can occur because additional factors not initially considered might also impact the price. They are changes in global economic conditions, geopolitical events, unexpected shifts in consumer behaviour, or technological advancements that influence fluctuations in the price of petrol. This brings in the complexity of economic interactions and the need for statistical analysis to comprehend and predict such intricate dynamics.

Imagine an economist investigate the impact of a new tax policy on consumer spending patterns in a region. The economist collects data from a random sample of households before and after the implementation of the new tax policy. The data include information about household income and expenditure on various goods and services. The economist's goal is to determine whether the new tax policy has significantly affected consumer spending. She starts by calculating the average change in expenditure for different categories of goods and services before and after the policy change. In order to make meaningful conclusions about the entire population, statistical inference is necessary.

Consider another economist studying the relationship between education levels and income in a country. The economist collects data from a representative sample of individuals across different education levels and records their annual incomes. The economist's objective is to determine if there is a statistically significant difference in average income between individuals with different levels of education. He suspects that higher education might lead to higher average incomes, but they need to validate this hypothesis through statistical inference. Statistical inference stands as a foundation of statistical analysis, enabling a deep connection between observed samples and the underlying parameters of a population. It paves the way for drawing meaningful conclusions about a sample, even when our primary interest lies in understanding the broader population from which it originates.

- Statistical inference draws conclusions about a population based on sample data

3.1.2 Concept of Population and Sample

The Reserve Bank of India (RBI) estimates the inflation rate for the entire population in the country. To calculate the inflation rate, RBI needs to examine the price changes of all goods and services consumed by every individual in the country. It is not feasible to gather data from the entire population. Instead, RBI collects a representative sample of prices for a basket of goods and services. Assume that a municipal government has decided to implement a new social welfare program aimed at providing financial assistance to low-income families in a city. To determine the eligibility criteria and the level of assistance, the government needs to understand the income distribution of the population. The authorities will collect income data from a representative sample of households. This could provide initial insights to make calculations and predictions about the total population eligible for financial assistance.

- Population is the pool of individuals from a statistical sample is drawn for a study

As illustrated above, in practical scenarios, the need often arises to draw meaningful and reliable conclusions about a substantial collection of individuals or objects. However, attempting to examine the entire population, can present challenges due to its huge size or inherent complexities. To overcome these challenges, investigating only a smaller subset of this population

- Sample is the subset of data points selected from a population emerges, and this subset is referred to as a sample. The primary objective behind working with samples is to make informed assessments and inferences about the larger population based on the findings derived from the sample.

The term population within the context of statistical analysis takes on a specialised connotation. In statistics, the idea of a population extends beyond the mere count of individuals or objects. It pertains to the complete set of observations or measurements that hold significance for a particular inquiry. Sampling entails selecting and studying a representative portion of the population which becomes crucial in statistical analysis. It serves as a practical approach to grasp insights and generalisations about the population without the need to analyse the entirety of it. By judiciously selecting and studying samples, economists can infer meaningful insights, such as trends, patterns, and correlations, which can subsequently be applied to the entire population.

- Statistic is a summary calculated from a sample of data

The inflation rate estimated out of a representative sample of goods and services reflects the price changes for the entire population's consumption patterns in a country. Statistical techniques, such as index number calculations, helps to estimate inflation based on the sample data collected. The inflation rate is crucial for monetary policy decisions, as it helps the RBI to manage inflation.

A statistic refers to a numerical summary derived from a sample, such as the sample mean or sample standard deviation. On the other hand, a parameter is a fixed numerical value that characterises aspects of the population, such as the population mean or population variance. It is often unknown and needs to be estimated using sample statistics

- Parameter is a numerical value that describes the characteristics of a population

3.1.3 Sampling Distributions

A key concept that emerges in this context is the sampling distribution of the statistic. This distribution elucidates the range of possibilities for the statistic's value and the associated probabilities of each outcome. The sampling distribution serves as a window into the behaviour of the sample statistic across numerous hypothetical samples, capturing the intricate interplay between randomness and the computed result.

- Sampling distribution which reveals the spectrum of possible value for a statistic

Alternatively, a perspective can be adopted wherein we envision constructing every conceivable sample of size n that can be drawn from the population. For each of these samples, we compute the corresponding statistic. By following this exhaustive approach, we systematically construct the distribution of the statistic, which takes into account the vast array of potential samples that can be generated from the population. This meticulously assembled distribution, often referred to as the sampling distribution of the statistic, provides a comprehensive outlook on the range of values that the statistic can adopt across the spectrum of possible samples.

3.1.4 Law of Large numbers

The law of large numbers is a fundamental principle in statistics that states that as the size of a sample or the number of observations increases, the sample mean or average approaches the true population mean, providing more accurate and reliable estimations. It is confirming our intuition that larger data sets provide more accurate results. While this law applies to various sample statistics, it is most easily understood as a principle about averages. Consider the sample mean, which is essentially an average. The law of large numbers, when applied to the sample mean, affirms that as the sample size increases, the sample mean converges toward the true population mean. In more precise terms, as the sample size approaches infinity ($N \to \infty$), the sample mean approaches the population mean ($\bar{x} \to \mu$).

- Sample mean stay close to the mean of the distribution

3.1.5 Central Limit Theorem

The Central Limit Theorem is stated as follows:

Let $\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots, \boldsymbol{X_n}$ be a sample from a population having mean μ and standard deviation σ. For *n* large, the sum

$\boldsymbol{X_1} + \boldsymbol{X_2} + \cdots + \boldsymbol{X_n}$ will approximately have a normal distribution with mean $n\mu$ and standard deviation $\boldsymbol{\sigma\sqrt{n}}$.

- For large sample- sampling distribution is normal

This theorem stands as a cornerstone concept, with far-reaching implications that significantly shape our understanding of probability distributions and statistical behaviour.

Let us observe the implication of the theorem using the example. A research study aims to estimate the average monthly income of employees in a large corporation. The corporation has 5,000 employees, and the researchers want to determine the probability that the total monthly income of all employees exceeds ₹6 million.

Let "Y_i" represents the monthly income of employee "i" where "i" ranges from 1 to 5000. According to the central limit theorem, the total monthly income will approximately follow a normal distribution with a mean of 5000 × μ (mean income) and a standard deviation of $\sigma\sqrt{(5000)}$.

Given that the mean income is ₹1500 and the standard deviation is ₹300, we have:

Mean of Y = 5000 × 1500 = 7.5×10^6

Standard deviation of Y = $300\sqrt{5000} \approx 2131.67$

To find the probability that the total monthly income exceeds ₹6 million, we calculate the *Z*-score:

$$Z = \frac{(6 \times 106 - 7.5 \times 106)}{2131.67} \approx -0.0745$$

Using the standard normal distribution table, we find the probability corresponding to Z = − 0.706 is approximately 0.2403.

Therefore, there is a probability of approximately 0.2403, or 24.03%, that the total monthly income of all employees will exceed ₹6 million. This result indicates that in this scenario, there is a relatively high chance that the total monthly income will fall below the given threshold.

- Helps to interpret and make predictions

By understanding the intricacies of the Central Limit Theorem, we gain an invaluable tool that transcends specific datasets and contexts. This theorem empowers us to interpret and make predictions about the distribution of sums, averages, and aggregates in scenarios where we may not have explicit knowledge about the underlying distribution of individual variables. It is a cornerstone of modern statistical thinking, offering us a lens through which to understand the patterns that emerge from the accumulation of random variables in diverse and complex scenarios.

3.1.6 Standard Error

- Mean aligns closely with the true population parameter

One of the key attributes we can compute for a sampling distribution is the mean, which provides a measure of the central tendency of the distribution of sample statistics. This mean often aligns closely with the true population parameter being estimated, reflecting the principle that well-constructed sample estimators tend to converge to the true population value as sample size increases.

- Variance captures the deviation from the mean

Another significant measure that arises in the context of a sampling distribution is the variance. The variance captures the spread or dispersion of the sample statistics around the mean of the distribution. It gives us an understanding of how much individual sample statistics deviate from the mean statistic, shedding light on the level of variability inherent in the estimation process.

Furthermore, we have the standard deviation, a measure closely linked to the variance. The standard deviation, sometimes referred to as the standard error in the context of a sampling distribution, provides a convenient way to express the dispersion of the sample statistics in the same units as the original data. Essentially, the standard deviation or standard error quantifies the average amount of deviation between individual sample statistics and the mean statistic of the distribution.

- Standard error specifically relates to the variability of sample statistics

The term "standard error" holds particular significance in the context of sampling distributions. While the standard deviation quantifies variability within a set of data points, the standard error specifically relates to the variability of sample statistics. It provides an estimation of how much the sample statistic is expected to deviate from the true population parameter, considering the inherent variability introduced by sampling. In essence, the standard error informs us about the precision of our sample-based estimates and plays a crucial role in confidence intervals and hypothesis testing.

Example 1

A school is trying to determine its students' reaction to a proposed syllabus change. To do so, the school selected a random sample of 50 students and questioned them. If 20 were in favour of the proposal, then

a) Estimate the proportion of all students who are in favour.
b) Estimate the standard error of the estimate.

Solution

a) The estimate of the proportion of all students who are in favour of the syllabus change is $\frac{20}{50} = 0.40$.

b) The standard error of the estimate is $\sqrt{\frac{p(1-p)}{50}}$

where p is the actual proportion of the entire population that is in favour. Using the estimate for p of 0.4, we can estimate this standard error by $\sqrt{\frac{0.4(1-0.4)}{50}} = 0.007$

In summary, a sampling distribution is a treasure trove of insights into the behaviour of sample statistics in relation to population parameters. By computing metrics such as the mean, variance, standard deviation, and moments, we gain a comprehensive understanding of the variability, spread, and shape of the distribution of sample-based estimators. The term "standard error" adds a nuanced layer of understanding, encapsulating the variability introduced by sampling and its implications for the precision of our estimations. These concepts collectively empower statisticians and researchers to make informed decisions, draw accurate conclusions, and communicate the reliability of their findings based on sample data.

- Helps to make informed decisions

Summarised Overview

Statistical inference is the process of drawing conclusions or making predictions about a population based on sample data. A sample is a smaller portion of a comprehensive dataset to make inferences about the population from which the sample is selected. A parameter represents a numerical value that characterises an entire population, whereas a statistic is a numerical value derived from a sample. The law of large numbers asserts that as the size of a sample or number of trials increases, the average or mean of the observed outcomes will approach the expected value or true population parameter. The central limit states that the distribution of sample means approaches a normal distribution as the sample size increases. Standard error is a measure of the variability or spread of sample statistics, such as the sample mean or sample proportion

Assignments

1. Define the following
   (i) Central Limit Theorem
   (ii) Standard Error
   (iii) Sampling Distribution
   (iv) Statistical Inference
   (v) Parameter
2. To learn the percentage of members who are in favour of increasing annual dues, a large social organisation questioned a randomly chosen sample of 20 members. If 13 members were in favour, what is the estimate of the proportion of all members who are in favour? What is the estimate of the standard error?
3. A random sample of 1000 construction workers revealed that 122 are presently unemployed.
   a) Estimate the proportion of all construction workers who are unemployed.
   b) Estimate the standard error of the estimate in part (a).

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UNIT 2
Sampling Distribution

Learning Outcomes

After completing this unit, learner will be able to:

- solve diverse research challenges
- gain better intuition for data interpretations
- utilise various sampling distributions like the normal distribution, t-distribution, and chi-square distribution

Background

If we want to understand the average height of students in your university, measuring everyone is impractical. So, we randomly select a smaller group, a sample, to represent the entire students. However, this single sample might not perfectly reflect the whole population. It could be taller or shorter by chance. This unavoidable fluctuation in samples opens the door to the world of sampling. This chapter deals with the probabilistic behaviour of statistics calculated from samples. Just like individual student heights vary, the average height across different samples drawn from the same population will fluctuate. This distribution, often bell-shaped and centered around the population mean, tells us how likely it is to observe different average heights from different samples. Moving on, we will examine the sample variance, a measure of how data points within a sample are dispersed. Its sampling distribution reflects the trade-off between sample size and variability. Larger samples tend to have more stable variances, while smaller ones exhibit wider ranges.

By understanding diverse sampling distributions, we can interpret sample statistics with greater confidence. This chapter equips the learners with the statistical lens to see beyond the single sample and disclose the hidden patterns within the entire population.

Keywords

Sample Mean, Sample Variance, Chi Square Distribution, Student's *t* Distribution, *F* Distribution

Discussion

3.2.1 Distributions of the Sample Mean

- Provides insights into the distribution of values within a dataset

The concept of a sampling distribution is a fundamental and crucial element in statistics that provides valuable insights into the distribution of values within a dataset. It serves as a bridge between the data we observe and the broader population it represents. By understanding the properties of different sampling distributions, such as the chi-square, Student's t, and F distributions, we gain not only a deeper comprehension of the data's inherent characteristics but also insights into the appropriate statistical tests that can be applied to the dataset.

Let us consider a scenario where a certain population is characterised by a probability distribution function, denoted as *f(x)*. Within this context, we get on the process of selecting samples from this population, and these samples are of a fixed size, '*n*.' This distribution summarises the complex tapestry of probabilities associated with the sample statistic $\boldsymbol{\overline{X}}$, which represents the sample mean. It is aptly referred to as the "sampling distribution of means" or the "sampling distribution for the sample mean." The following theorems provide a deeper understanding of the distribution:

Theorem 1: The Central Tendency of the Sampling Distribution of Means

- Provides a framework for understanding how the means of various samples are distributed

The mean value characterising the sampling distribution of means is symbolised as μ and $\bar{x}$. The mean of the population is often represented as μ. This theorem clearly declares that the anticipated value of the sample mean perfectly mirrors the population mean, underscoring a fundamental connection between the sample-based estimates and the broader population's central tendency.

The mean of the sampling distribution of means, denoted by $\boldsymbol{\mu_{\bar{x}}}$, is given by

$$E(\overline{X}) = \mu_{\overline{x}} = \mu$$

where $\boldsymbol{\mu}$ is the mean of the population.

Theorem 2: Variability in the Sampling Distribution of Means under Different Conditions

The principle of this theorem explains the nature of the population and the sampling process. In instances where a population spans an infinite expanse or sampling is executed with replacement, the variance within the sampling distribution of means, symbolised by $\sigma_{\overline{x}}^2$, can be ascertained as the population variance σ^2 divided by the sample size n. This theorem offers a formula that quantifies how the variability within the sample means expresses the characteristics of the larger population.

- $E\left[(\overline{X} - \mu)^2\right] = \sigma_{\overline{x}}^2$

If a population is infinite or if sampling is with replacement, then variance of the sampling distribution of means, denoted by $\boldsymbol{\sigma_{\overline{x}}^2}$, is given by,

$$E\left[(\overline{X} - \mu)^2\right] = \sigma_{\overline{x}}^2 = \frac{\sigma^2}{n}$$

where $\boldsymbol{\sigma^2}$ is the variance of the population.

Theorem 3: Sampling Variability and Finite Populations

- Incorporates a correction factor

The situation changes when the population is finite, and sampling occurs without replacement. This theorem illuminates that the variance in the sampling distribution of means $\boldsymbol{\sigma_{\overline{x}}^2}$ transcends a simple division by n. It incorporates a correction factor, interwoven with the population size N and the sample size n. This theorem acknowledges the interplay between the sample characteristics, the population size, and the absence of replacement during sampling, resulting in a nuanced expression that delves into finite population dynamics.

Theorem 4: Normal Distribution and Sample Mean

- mean μ and variance σ^2

When the population follows a normal distribution with mean μ and variance σ^2, this theorem establishes that the sample mean preserves the normal distribution's hallmark characteristics. In essence, the theorem establishes a direct correspondence between the sample mean's behaviour and the inherent properties of a normally distributed population.

Theorem 5: Asymptotic Normality and General Probability Distributions

- $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$

It introduces the concept of the standardised variable Z, which emerges from the sample mean's deviation from the population mean, normalised by the standard deviation. This theorem, rooted in the principles of asymptotic behaviour, reveals that as the sample size approaches infinity, this standardised variable Z assumes an asymptotically normal distribution. This result, expressed as a limit, underscores the universal tendency of such standardised variables to converge toward a standard normal distribution as the sample size grows indefinitely.

Then the standardised variable associated with $\boldsymbol{\overline{X}}$, given by

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

is *asymptotically normal*, i.e.

$$\lim_{n\to\infty} P(Z \leqq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du$$

$\boldsymbol{Z}$ is the test statistic.

These theorems constitute a pivotal framework within the statistical theory. Each theorem illuminates a distinct facet of the relationships between sample means, population parameters, variability, and distributional characteristics.

3.2.2 Sample Variance

- Measure of dispersion or spread of data values within a sample

Consider a scenario where we are dealing with a sample comprising random variables denoted as $X_1, X_2, \ldots, X_n$, with a total size of 'n.' Within this context, we wish to quantify the variability within this sample, leading us to the concept of sample variance.

The sample variance, symbolised as $\boldsymbol{S^2}$, emerges as a measure that characterises the variability present in the sample. Mathematically, it is defined as the sum of squared deviations of each individual data point from the sample mean, $\overline{X}$, divided by the sample size 'n':

$$\frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2}{n}$$

Within the framework of Theorem 1, where we established that the expected value of the sample mean matches the population mean, we yearn for a similar relationship involving the sample variance. A statistic is deemed an unbiased estimator when its expected value corresponds to the respective population parameter. In this case, we desire a situation where the expected value of the sample variance is equal to the population variance.

Whenever the expected value of a statistic is equal to the corresponding population parameter we call the statistic an unbiased estimator, and the value an unbiased estimate, of this parameter. It turns out that

$$E(S^2) = \pi_{S^2} = \frac{n-1}{n}\sigma^2$$

which is very nearly σ^2 only for large values of n (say $n \geqq 30$). The desired unbiased estimator is defined by

$$\hat{S}^2 = \frac{n}{n-1}S^2$$

$$= \frac{(X_1 - \overline{X})^2 + (X_2 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2}{n-1}$$

so that $E(\hat{S}^2) = \sigma^2$

Because of this, some statisticians choose to define the sample variance by $\hat{S}^2$ rather than S^2 and they simply replace n by n-1 in the denominator.

This observation implies that $\hat{S}^2$ serves as an unbiased estimator of the population variance, a quality that appeals to statisticians seeking accurate estimations of population parameters from sample data.

Example 1

Suppose an economist is studying the average monthly expenditure on consumer goods in a city. The population of expenditure values follows a normal distribution with a mean of ₹800 and a standard deviation of ₹150. The economist takes a random sample of size 150 households from this population.

What is the probability that the sample mean expenditure will be between ₹780 and ₹820?

Solution:
Given:
Population mean (μ) = \$800
Population standard deviation (σ) = \$150
Sample size (n) = 150
Desired range of sample mean = between ₹780 and ₹820
First, let's find the standard error of the sample mean ($\sigma\bar{x}$) using the formula:

$$\sigma\bar{x} = \frac{\sigma}{\sqrt{n}}$$

$$\sigma\bar{x} = \frac{150}{\sqrt{150}} \approx 12.247$$

Next, we convert the desired range of sample mean into z-scores using the formula:

$$z = \frac{(\bar{x} - \mu)}{\sigma\bar{x}}$$

For ₹780:

$$z1 = \frac{(₹780 - ₹800)}{12.247} \approx -1.632$$

For ₹820:

$$z2 = \frac{(₹820 - ₹800)}{12.247} \approx 1.632$$

Now, we use the z-scores to find the probabilities from the standard normal distribution table:

$P(-1.632 < Z < 1.632)$

Using a standard normal distribution table or a calculator, we find:

$P(-1.632 < Z < 1.632) \approx 0.9071$

Therefore, the probability that the sample mean expenditure will be between ₹780 and ₹820 is approximately 0.9071, or about 90.71%.

Example 2

Suppose a bank is interested in understanding the average savings account balance of its customers. The population of savings account balances follows a normal distribution with a mean of ₹5,000 and a standard deviation of ₹1,200. The bank takes a random sample of 150 savings account balances from its customer base. What is the probability that the sample mean savings account balance will be within ₹300 of the population mean?

Solution:

Given:

Population mean (μ) = ₹5,000

Population standard deviation (σ) = ₹1,200

Sample size (n) = 150

Desired range of sample mean = within ₹300 of the population mean

First, calculate the standard error of the sample mean (σ $\sigma\bar{x}$) using the formula:

$$\sigma\bar{x} = \frac{\sigma}{\sqrt{n}}$$

$$\sigma\bar{x} = \frac{1200}{\sqrt{150}} \approx 98.331$$

Next, convert the desired range of sample mean into z-scores using the formula:

$$z = \frac{(\bar{x} - \mu)}{\sigma\bar{x}}$$

For ₹4,700:

$$z_1 = \frac{(₹4{,}700 - ₹5{,}000)}{98.331} \approx -3.04$$

For ₹5,300:

$$z_2 = \frac{(₹5{,}300 - ₹5{,}000)}{98.331} \approx 3.04$$

Using a standard normal distribution table or a calculator, find the probabilities:

$P(-3.04 < Z < 3.04)$

The standard normal distribution is symmetric, so we can find the probability for the interval ($-3.04 < Z < 3.04$) as twice the probability of Z being less than 3.04:

$P(Z < 3.04) \approx 0.9988$

Therefore, the probability that the sample mean savings account balance will be within ₹300 of the population mean is approximately 0.9988, or about 99.88%.

In the above example, the principles of statistical inference and properties of normal distributions are applied to determine the probability of the sample mean falling within a specified range of the population mean. This type of analysis can help the bank understand the likely variation in sample mean savings account balances and make decisions about customer service and financial planning.

3.2.3 Chi-Square Distribution

The χ^2 distribution, often pronounced as "chi-square" distribution, is a distinct probability distribution that pertains to continuous random variables. This distribution is notably associated with the sampling distribution of variance and standard deviation. Thus we can define, Chi-square distribution is a probability distribution in relation to the sampling distribution of variance and standard deviation

3.2.4 Student's t Distribution

- Student's t distribution is used in statistics to model sample

Another distribution of paramount significance is the student's *t* distribution. This distribution holds particular relevance in scenarios involving the examination of a standardised form of the sample mean, especially when the underlying distribution adheres to a normal distribution pattern.

3.2.5 *F* Distributions

The *F* distribution emerges from the division of sums of squares in situations involving sampling from a normal distribution. As a result, this distribution holds significance in estimation, the context of the two-sample normal model, and hypothesis testing within the framework of the two-sample normal model.

Suppose that *U* has the chi-square distribution with $\boldsymbol{n} \in (\mathbf{0}, \infty)$ degrees of freedom, and that *U* and *V* are independent. The distribution of

$X = \frac{\frac{U}{n}}{\frac{V}{d}}$ is the *F* distribution with *n* degrees of freedom in the number and *d* degrees of freedom in the denominator.

In short, a grasp of sampling distributions equips economists with the tools to extract meaningful insights from the data while considering the inherent variability and uncertainty present in real-world observations.

Summarised Overview

The distribution of the sample mean is a probability distribution that describes the possible values of the mean of various samples drawn from a population. Sample variance is a statistical measure that quantifies the variability or dispersion of data points within a sample from their mean. The chi-square distribution is a probability distribution that arises in statistical inference and is used to analyse the distribution of the sum of squared standard normal deviates. The F distribution is a probability distribution used in statistical analysis to assess variability between sample variances and for comparing multiple groups

Assignments

1. Suppose the mean time it takes for a business to receive a refund for Goods and Services Tax (GST) payments is 20 days, with a standard deviation of 4 days. A manager is interested in knowing the probability that in a sample of 60 GST refund requests, the mean processing time will be more than 22 days.

2. Frequent fliers of a particular airline fly a random number of miles each year, having mean and standard deviation (in thousands of miles) of 23 and 11. respectively. As a promotional gimmick, the airline has decided to randomly select 20 of these fliers and give them, as a bonus, a check of ₹10 for each 1000 miles flown. Approximate the probability that the total amount paid out is

   (a) Between ₹4500 and ₹5000

   (b) More than ₹5200

3. Suppose that the number of miles that an electric car battery function has mean μ and standard deviation 100. Using the central limit theorem, approximate the probability that the average number of miles per battery obtained from a set of n batteries will differ from μ by more than 20 if

   (a) $n = 10$ (b) $n = 20$
   (c) $n = 40$ (d) $n = 100$

Suggested Reading

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UNIT 3
Estimation of Parameters

Learning Outcomes

After completing this unit, learner will be able to:

- learn the properties of small and large sample distribution
- gain an understanding of various sampling distributions
- apply estimations of population parameters in different contexts

Background

For studying the growth of a population, we might only have access to the population size at different points in time. By employing statistical techniques like least squares or maximum likelihood, we can estimate the underlying growth rate, a key parameter governing the population's route. This estimated parameter allows us not only to understand the past but also to forecast future population sizes, informing crucial decisions about resource allocation and infrastructure development.

Beyond simple examples, estimation plays a pivotal role in diverse fields. In economics, estimating the parameters of a market model helps us understand the interplay of supply and demand, guiding economic policy. In engineering, estimating the parameters of a physical system allows us to design robust structures and predict their behaviour under stress. Even in fields like psychology and sociology, estimation techniques shed light on human behaviour and social dynamics by uncovering the hidden parameters underlying observed patterns. Throughout this chapter, we will equip ourselves with the theoretical and practical tools for navigating the estimation. We will explore various estimation methods, their strengths, and limitations. It will help to interpret the results of our estimations, assessing the accuracy, and drawing meaningful conclusions from the extracted parameters.

Keywords

Point Estimation, Interval Estimation, Unbiasedness, Consistency, Efficiency, Sufficiency

Discussion

3.3.1 Large Sample Properties of Z

In statistical analysis, estimating population parameters is pivotal, particularly when direct observation of certain numerical characteristics is not possible. Instead, through the observation of related random variables, we seek to develop methods that utilize sample data to gain insights into these unobservable characteristics.

- Estimating hidden population traits through random samples

Starting with a random sample of size "n" from a population, estimation theory assumes the observations to be random, allowing us to identify the probability distribution dependent on parameters of interest for further analysis. Understanding and determining these parameters are essential as they define the distribution and guide the estimation process.

Large sample mean is used when the sample size is greater than or equal to 30 ($n \geq 30$).

The Z distribution, also referred to as the Standard Normal distribution, is a specific form of the Normal distribution characterized by a mean of zero ($\mu = 0$) and a variance of unity ($\sigma^2 = 1$).

- Z-distribution is a specific normal curve for large samples exceeding 30 individuals

To introduce the Z distribution, it is essential to look into the broader concept of the Normal distribution. Consider a scenario where the heights of a substantial number of adult males or females are measured, and the outcomes are visualized through a histogram. The x-axis represents height in inches, while the y-axis indicates the frequency of individuals falling within each height range. Notably, there would be a significant concentration of people around the middle of the histogram compared to the extremes. This concentration arises because a larger number of individuals cluster around the mean height, with only a few outliers being significantly taller or shorter. As the sample size increases and the intervals on the x-axis narrow, the histogram's bars approach infinitesimal width, causing the height data to essentially form a continuous curve. This curve follows the characteristic shape of the Normal distribution.

3.3.2 Small Sample Properties of t

The student probability density function f with $\boldsymbol{n} \in (\mathbf{0}, \infty)$ degrees of freedom have the following properties:

1. f is symmetric about $t = 0$.
2. f is increasing and then decreasing with mode
3. f is concave upward, then downward, then upward again with inflection points at $\pm\sqrt{n/(n+1)}$
4. $f(t) \to 0$ as $t \to \infty$ and as $t \to -\infty$.

We use t test, when we have small samples ($n < 30$).

In this case we use the t distribution to obtain confidence levels. For example, if $-\boldsymbol{t}_{.975}$ and $\boldsymbol{t}_{.975}$ are the values of T for which 2.5% of the area lies in each "tail" of the t distribution, then a 95% confidence interval for T is given by

$$-t_{.975} < \frac{(\bar{X}-\mu)\sqrt{n}}{\hat{S}} < t_{.975} \quad \text{------------------------} \quad (1)$$

from which we see that μ can be estimated to lie in the interval

$$\bar{X} - t_{.975}\frac{\hat{S}}{\sqrt{n}} < \mu < \bar{X} + t_{.975}\frac{\hat{S}}{\sqrt{n}} \quad \text{---------------------------} \quad (2)$$

with 95% confidence. In general, the confidence limits for population means are given by

$$\bar{\boldsymbol{X}} \pm \boldsymbol{t_c}\frac{\hat{S}}{\sqrt{n}} \quad \text{---------------------------} \quad (3)$$

where the value $\boldsymbol{t_c}$, can be read from the table

3.3.3 Properties of Chi-square

The properties of the χ^2 distribution density curve are as follows:

1. Right Skewed: The density curve of the χ^2 distribution originates from zero and tilts to the right.
2. Center and Spread: The central tendency and dispersion of a χ^2-distribution are defined by the degrees of freedom (*df*). The mean of the distribution is equal to the degrees of

freedom (*df*), and the standard deviation is represented by the square root of twice the degrees of freedom ($\sqrt{2df}$).

3. Non-Negativity: χ^2 distributed variables cannot assume negative values, as they are restricted to positive values only.

4. Normality with Increasing Degrees of Freedom: As the degrees of freedom increase, particularly when *df* surpasses 50, the χ^2 distribution progressively approximates a normal distribution. Figure 1 illustrates the χ^2 distributions for different degrees of freedom: 2, 4, 10, and 30.

5. Total Area Under the Curve: The entire area beneath the χ^2-distribution curve adds up to 1, which corresponds to 100% of the probabilities.

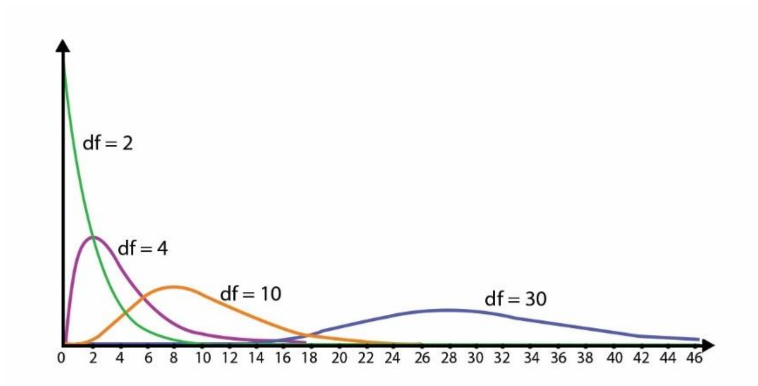

Fig 3.3.1 Chi Square Distribution

- Chi-square distribution is crucial for statistical analyses involving variances and standard deviations

In summary, the χ^2 distribution exhibits specific characteristics such as its skewness, dependence on degrees of freedom for central and dispersion properties, and its tendency to approach normality with increased degrees of freedom. This distribution plays a crucial role in various statistical analyses, particularly those involving variances and standard deviations.

3.3.4 Properties of F Distribution

Probability density function *f* satisfies the following properties:

1. If $0 < n < 2$, *f* is decreasing with $f(t) \to \infty$ as $x \downarrow 0$.

2. If n = 2, *f* is decreasing with mode at $x = 0$.

3. If n > 2, *f* is increases and then decreases, with mode at

$$x = \frac{(n-2)d}{n(d+2)}$$

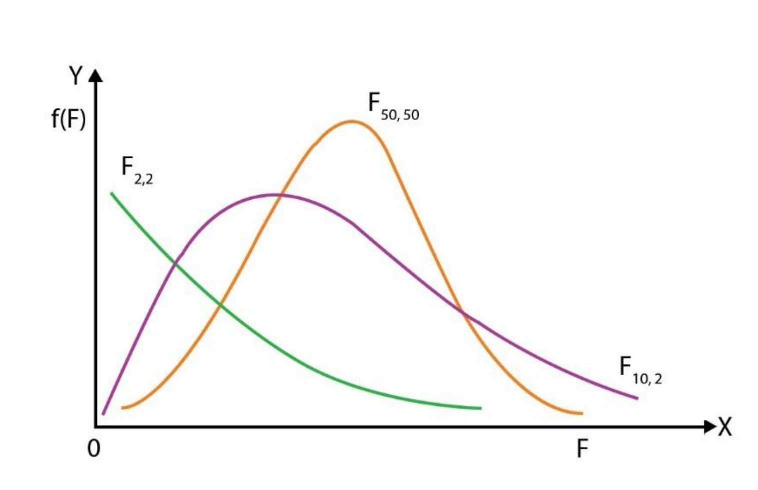

Fig 3.3.2 F Distribution

3.3.4 Use of F-test

F test is used to test for equality of population variances. Let X_1, X_2, …, X_m be a random sample of size m from a normal population with mean $\boldsymbol{\mu_X}$ and variance $\boldsymbol{\sigma_X^2}$ and let Y_1, Y_2, …, Y_n be a random sample of size n from a normal population with mean $\boldsymbol{\mu_Y}$ and variance $\boldsymbol{\sigma_Y^2}$. Assume that these 2 samples are independent and are drawn from populations that are normally distributed. Suppose we want to find out if the variance of the normal populations is the same, *i.e.* whether $\boldsymbol{\sigma_X^2 = \sigma_Y^2}$.

Since, we cannot directly observe the two population variances, let us get their estimators:

$$s_X^2 = \sum \frac{\left(X_i - \overline{X}\right)^2}{m-1}, s_Y^2 = \sum \frac{\left(Y_i - \overline{Y}\right)^2}{n-1}$$

Put a larger s^2 in the numerator.

$F = \frac{s_x^2}{s_y^2} F(m-1, n-1)$ → F value should be close to 1, if the two variances are close to each other

Since F distribution is often used to compare the variances of two populations, it is also known as variance ratio distribution.

3.3.5 Estimations of Populations Parameters

- Maximise profit and reduce waste

Imagine an automobile manufacturer, aiming to understand the characteristics with respect to consumer preferences for vehicle features. It includes which vehicles are preferred more compared to the other and what features determine the preference towards a particular model of a vehicle and so on. The manufacturer is to optimise production to meet demand precisely. Overproducing certain vehicle features, such as luxury add-ons or fuel-efficient engines, that do not align with consumer preferences could lead to surplus inventory and financial loss. On the other hand, underproducing popular features would mean missing out on potential profits.

- To conquer market demands, manufacturers must understand population

In order to face this challenge effectively, the manufacturer adopts a strategy of producing different quantities of vehicles with varying features based on demand. To measure demand accurately, he needs to comprehend the frequency of preferences within the population. This, in turn, necessitates a grasp of the population parameters such as mean and variability providing the foundation for informed decision-making. Having this knowledge could provide your company with a competitive edge, assuming all vehicles are of similar quality.

- Parameters to navigate the market effectively

In the realm of economics, concrete populations exist, and it becomes essential to comprehend their parameters. Population parameters are characteristics that pertain to a distribution. For instance, distributions are characterized by various attributes, such as means and standard deviations. The mean itself stands as a parameter of the distribution, as does the standard deviation.

3.3.6 Point Estimation

Suppose a retail company is considering setting the price for a new product they are about to launch. The population in this scenario would encompass all similar products in the market, both from competitors and those with similar features. The marketing team of the company decides to collect a sample of recent pricing data for similar products from five competitors in the market. They then adjust these prices based on features, brand reputation, and other factors to align them closely with the new product they plan to introduce.

Subsequently, the marketing team calculates the mean of the adjusted prices in their sample and recommends a pricing strategy of ₹99 for the new product. By doing so, they are making an inference about the mean price of the population. To assess the reliability of this inference, the marketing team should consider various factors: the relatively small sample size, potential fluctuations in market pricing over time, or the possibility that the new product might have unique features that set it apart from the sampled products. This thorough evaluation will help to ensure that the pricing decision is based on a well-founded and informed assessment of the population's characteristics. An estimate of a population parameter given by a single number is called a point estimate of the parameter.

- To determine the optimal price for a new product, a retail company analyses competitor pricing data

3.3.7 Interval Estimation

In the context of the earlier example, interval estimation refers to the process of determining a range of values within which a population parameter is likely to fall. In this case, the company wants to estimate the true mean price of similar products in the market to help set the price for their new product.

Instead of providing a single point estimate, such as suggesting that the mean price is exactly ₹99, interval estimation offers a range of values that is expected to contain the true population mean with a certain level of confidence. For instance, the marketing team might use statistical methods to calculate a confidence interval, which is a range of prices within which they believe the true mean price of similar products lies.

- Use interval estimation to determine the price of their new product effectively

Let us say the marketing team calculates a 95% confidence interval for the mean price to be ₹95 to ₹103 based on their sample data and adjustments. This means that they are 95% confident that the true mean price of similar products in the market falls within this interval.

An estimate of a population parameter given by two numbers between which the parameter may be considered to lie is called an interval estimate of the parameter. Interval estimation acknowledges the inherent uncertainty in making population inferences from a sample. By providing a range of values instead of a single point estimate, the marketing team of the company

- Range defined by two values within which the parameter is likely to fall gives a more comprehensive picture of the potential variability in the parameter being estimated. This approach helps decision-makers make informed choices and manage risks associated with their decisions, considering the potential fluctuations in the population characteristics.

- Point estimation risks inaccuracy due to the mismatch between a single sample and the entire population

Point estimation, while straightforward, comes with inherent limitations that can lead to estimation errors. For instance, when we use the sample mean x-bar to estimate the population mean μ (mu), it's important to recognize that the likelihood of obtaining an exact match between x-bar and μ (mu) is very low. This discrepancy between x-bar and μ (mu) is a consequence of drawing our estimation from a single sample, which is only a small subset of the entire population.

- Interval estimation gives a range of probable values for a parameter

In essence, interval estimation goes beyond single point estimates by furnishing a range of values within which we believe the true parameter lies. This range provides a degree of uncertainty around our point estimate and acknowledges the variability that arises from using limited sample data. By quantifying the error associated with the estimation, interval estimation offers a more comprehensive and informative perspective on the parameter being estimated. This approach is particularly valuable in decision-making scenarios where understanding the potential range of values adds depth to our analysis and aids in managing the inherent uncertainty of estimating population parameters from limited sample data.

3.3.8 Fisher's Properties of Estimators

Fisher's properties of estimators refer to a set of desirable qualities that characterise the performance of statistical estimators. These properties were introduced by the statistician and geneticist Ronald A. Fisher and are crucial in evaluating the effectiveness and reliability of different estimation techniques. The main properties include unbiasedness, consistency, efficiency, and sufficiency:

a. **Unbiasedness**: An estimator is unbiased if, on average, it yields an estimate that is equal to the true value of the parameter being estimated. In other words, the expected value of the estimator matches the population parameter it aims to estimate. An unbiased estimator minimises the systematic errors that may arise during the estimation process.

b. **Consistency**: An estimator is consistent if it converges in probability to the true parameter value as the sample size increases. In simpler terms, as more data are collected, the estimator becomes more accurate and approaches the true population parameter. Consistency ensures that the estimator becomes reliable when working with larger samples.

c. **Efficiency**: An estimator is efficient if it has a smaller variability (lower variance) than other estimators for the same parameter. An efficient estimator strikes a balance between providing accurate estimates and having minimal variability. In essence, it provides the most information about the parameter using the available data.

d. **Sufficiency**: An estimator is sufficient if it captures all relevant information about the parameter contained in the sample data. A sufficient estimator reduces the entire dataset down to a manageable summary, allowing for efficient estimation without unnecessary redundancy. This property is particularly important in cases where data are complex and extensive.

- Researchers choose the estimation method as the best parameter

By evaluating estimators based on these properties, researchers can assess the quality and performance of different estimation methods and choose the one that best suits their specific needs and objectives. It is important to note that no single estimator may possess all these properties simultaneously, and the choice of an estimator often involves a trade-off between these desirable qualities depending on the particular context and requirements of the statistical problem at hand.

Summarised Overview

The Z-distribution, also known as the standard normal distribution, is symmetric, bell-shaped, and characterized by a mean of 0 and a standard deviation of 1. The t-distribution is bell-shaped and symmetric, with its shape determined by the degrees of freedom, and approaches the standard normal distribution as the degrees of freedom increase. The chi-square distribution is right-skewed, its shape is determined by the degrees of freedom, and it only takes positive values. The F distribution is positively skewed, varies with the degrees of freedom of its numerator and denominator, and is used primarily in the context of comparing variances or ratios of sample variances. Point estimation involves providing a single value as an estimate for a population parameter, while interval estimation provides a range of values within which the population parameter is likely to lie along with a level of confidence.

Assignments

1. Write short note on the properties of the following
   a. Z distribution
   b. t distribution
   c. chi-square distribution
   d. F distribution.
2. Discuss in detail the Fisher's properties of estimators.

Suggested Reading

1. Anderson, D., D. Sweeney and T. Williams (2013): "*Statistics for Business and Economics*", Cengage Learning: New Delhi.
2. Goon, A.M., Gupta and Das Gupta B (2002): *Fundamentals of Statistics* (Vol I), World Press.

Reference

1. Mayes, Anne C., and David G. Mayes (1976). "*Introductory Economic Statistics*."
2. Beals, R. E. (1972). *Statistics for Economists: An Introduction*. Rand McNally College Publishing Company, U.S.A.
3. Hogg, R. V., McKean, J. W., & Craig, A. T. (2013). *Introduction to Mathematical Statistics*. Pearson Education India.
4. Webster, A. (2010). *Applied Statistics for Business and Economics*. Tata McGraw Hill Education Private Limited, New Delhi.
5. Ross, S. M. (2017). *Introductory Statistics*. Academic Press.

Space for Learner Engagement for Objective Questions

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UNIT 4
Methods of Estimation

Learning Outcomes

After completing this unit, learner will be able to:

- know the method of maximum likelihood to estimate parameters in statistical models
- calculate the method of least squares to estimate parameters in linear regression models
- compute a confidence interval for mean, proportion, and variance
- apply the various methods of estimation in real life data

Background

Confidence intervals provide a measure of uncertainty around estimated population parameters, aiding researchers and analysts in assessing the precision of their estimates. Confidence intervals for the mean, proportion, and variance offer a range of plausible values for these parameters along with a specified level of confidence, typically 95% in practice. Understanding how to construct and interpret confidence intervals is crucial for making informed decisions based on sample data and ensuring the reliability of statistical estimates. Together, the methods of least squares and maximum likelihood, coupled with confidence interval construction, form a robust framework for statistical inference, empowering practitioners across various domains to derive meaningful insights and draw reliable conclusions from data analysis.

Keywords

Confidence Interval, Method of Least Squares, Method of Maximum Likelihood

Discussion

3.4.1 Confidence Intervals for Mean

In statistical analysis, the computation of confidence intervals for mean, proportion, and variance is a crucial practice for providing accurate estimates while acknowledging the inherent uncertainty in sample data. This enables decision-makers to quantify the precision of their estimates and make informed judgments. In statistical analysis, estimation methods such as the method of least squares and the method of maximum likelihood play pivotal roles in extracting meaningful insights from real-life data scenarios.

- For providing accurate estimates

When calculating a confidence interval for the mean, the objective is to estimate the population mean using the mean of a sample. Essentially, if the population mean were already known, there would be no necessity for constructing a confidence interval. However, in order to illustrate the process of creating confidence intervals, we will take a reverse approach and start by assuming certain attributes of the population. From there, we will demonstrate how data from a sample can be utilised to establish a confidence interval.

In the case where the weights of 10-year-old children follow a normal distribution with a mean of 90 and a standard deviation of 36, the sampling distribution of the mean for a sample size of 9 can be understood as follows:

When you take multiple samples of size 9 from the population of 10-year-old children and calculate the mean weight for each sample, you will end up with a collection of sample means. The distribution of these sample means is what we refer to as the sampling distribution of the mean. This distribution will have certain characteristics:

Mean of Sampling Distribution: The mean of the sampling distribution of the mean will be the same as the population mean. In this case, it would be 90, which is the mean weight of 10-year-old children.

Standard Deviation of Sampling Distribution: The standard deviation of the sampling distribution of the mean, often called the standard error, is given by the population standard deviation divided by the square root of the sample size. So for a sample size of 9, the standard error would be

$$\frac{36}{\sqrt{9}} = 12$$

- Standard deviation of a sampling distribution is referred to as its standard error

It is important to note that the standard deviation of a sampling distribution is referred to as its standard error. The distribution depicting this concept is illustrated in Figure 3.4.1. In this diagram, the shaded region signifies the central 95% portion of the distribution and spans from 66.48 to 113.52. These boundaries were determined by subtracting and adding 1.96 standard deviations from the mean of 90, which can be computed as follows:

Lower Bound: 90 − (1.96) (12) = 66.48

Upper Bound: 90 + (1.96) (12) = 113.52

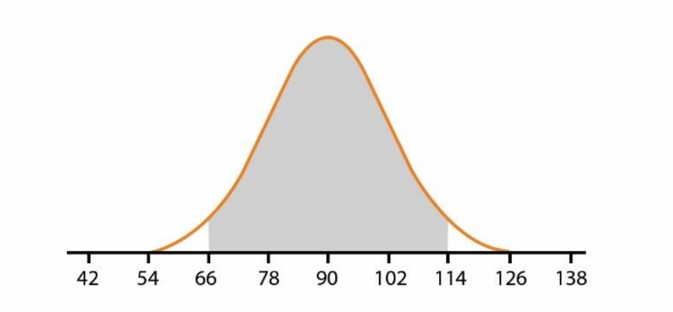

Fig 3.4.1 Standard Error

The value 1.96 is derived from the observation that 95% of the area under a normal distribution fall within 1.96 standard deviations of the mean, while 12 represents the standard error of the mean. This interval estimation technique provides a range within which we can be reasonably confident that the true population parameter lies.

3.4.1.1 Confidence Interval for Large Samples

If the statistic S is the sample mean $\bar{X}$, then 95% and 99% confidence limits for estimation of the population mean μ are given by $\bar{X} \pm 1.96_{\sigma_{\bar{x}}}$, and $\bar{X} \pm 2.58_{\sigma_{\bar{x}}}$, respectively. More generally, the confidence limits are given by $\bar{X} \pm z_c \sigma_{\bar{x}}$, where z_c, which depends on the particular level of confidence desired.

Using the values of $\boldsymbol{\sigma_{\bar{x}}}$, we see that the confidence limits for the population mean are given by

$$\boldsymbol{\bar{X} \pm z_c \frac{\sigma}{\sqrt{n}}} \text{ ----------------------- } (1)$$

in case sampling is from an infinite population or if sampling is with replacement from a finite population, and by

$$\boldsymbol{\bar{X} \pm z_c \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}} \text{ -------------------------- } (2)$$

if sampling is without replacement from a population of finite size *N*.

In general, the population standard deviation $\boldsymbol{\sigma}$ is unknown, so that to obtain the above confidence limits we use the estimator $\hat{S}$ or *S*.

A variable $Z = \frac{X-\mu_x}{\sigma_x}$

$$\underbrace{Z}_{std.normalvariate} = \left(\frac{\bar{X}-\mu_x}{\frac{\sigma_x}{\sqrt{n}}}\right) N(0,1)$$

If variable *X* has a mean $\boldsymbol{\mu_x}$ and variance $\boldsymbol{\sigma_x^2}$, then *Z* variable as defined above will have a zero mean and a variance of one. It is known as a unit or standardised variable.

Thus, any normally distributed random variable with a given mean and variance can be converted to a standard normal variable, which greatly simplifies our task of computing probabilities.

Example 1

It is given that *X*, the daily sale of bread in a bakery, follow *N* with $\boldsymbol{\mu}$= 70 and $\boldsymbol{\sigma^2}$= 9, *i.e., X* ~ (70, 9). What is the probability that on any given day the sale of bread is greater than 75 loaves?

Solution

Since X follows N, it follows that

$$\boldsymbol{Z = \frac{75-70}{3} \approx 1.67}$$

Follows standard normal distribution.

We want to find $P(\boldsymbol{Z} > \mathbf{1.67})[\boldsymbol{P}(\boldsymbol{X} > \mathbf{75})]$

Refer table of areas under the standardised normal distribution.

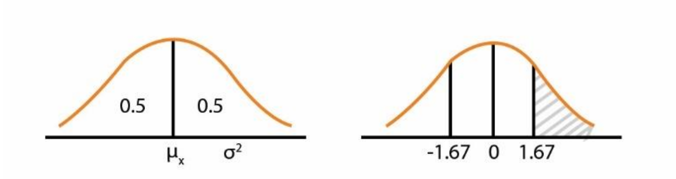

Fig 3.4.4 Sample Distribution

$$Z = \frac{X - \mu}{\sigma^2}$$

$$P = (Z > 1.67) = 0.5 - P(0 < Z < 1.96)$$

$$= 0.5 - 0.4525$$

$$= 0.0475$$

The same example on bakery can be extended to the interval estimation also.

Example 2

Find the probability that the daily sale of bread is between 65 and 75 loaves

Solution

$$Z_1 = \frac{65 - 70}{3} = -1.67$$

$$Z_2 = \frac{75 - 70}{3} \approx 1.67$$

$$P(-1.67 \leq Z \leq 1.67) = 0.4525 + 0.4525 = 0.9050$$

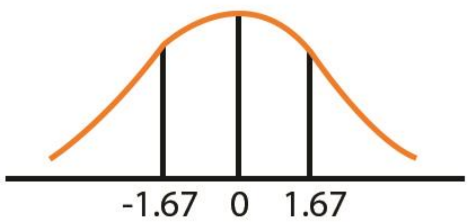

Fig 3.4.5 Sample Distribution

The larger the sample, the less the sampling error and the narrower the confidence limits. An explanation on setting the intervals is given below:

It is known from the study of normal distribution, that the area under the standard normal curve between $z = -1.96$ and $z = 1.96$ is 0.95

$$P(-1.96 \leq Z \leq 1.96) = 0.95$$

$$P(-1.96 \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq 1.96) = 0.95$$

$$P(-1.96 \times \frac{\sigma}{\sqrt{n}} \leq \bar{X} - \mu \leq 1.96 \times \sigma/\sqrt{n}) = 0.95$$

$$P(\mu - 1.96 \times \frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 1.96 \times \sigma/\sqrt{n}) = 0.95$$

i.e., at 95% confidence level (or 5% level of significance) the confidence interval for arithmetic mean is

$$(\mu - 1.96 \times \frac{\sigma}{\sqrt{n}}, \ \mu + 1.96 \times \frac{\sigma}{\sqrt{n}})$$

At 90% confidence level (or 1% level of significance) the confidence interval for arithmetic mean is

$$(\mu - 1.65 \times \frac{\sigma}{\sqrt{n}}, \ \mu + 1.65 \times \frac{\sigma}{\sqrt{n}})$$

3.4.1.2 Confidence Interval for Small Samples

It is to be noted that when the sample is small (i.e., n ≤ 30), the t test has to be conducted.

The confidence interval for the population mean is $\bar{x} \pm t_{\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$

Where $\bar{x}$ sample arithmetic mean

$t_{\frac{\alpha}{2}}$ – t- distribution table value with (n-1) degree of freedom

s- Standard deviation of sample

$n -$ sample size

Example

A sample of size 15 drawn from a normally distributed population has sample mean 35 and sample standard deviation 14. Construct a 95% confidence interval for the population mean

Solution

confidence interval 95% means 5% level of significance so

$\frac{\alpha}{2} = \frac{0.05}{2} = 0.025$

$n = 15$, there are $n - 1 = 51 - 1 = 14$ degree of freedom.

From t distribution table $t_{\frac{\alpha}{2}} = 2.145$

Confidence interval is

$\bar{x} \pm t_{\frac{\alpha}{2}} \times \frac{s}{\sqrt{n}}$ = (35-2.145× $\frac{14}{\sqrt{15}}$, 35+2.145× $\frac{14}{\sqrt{15}}$)

$= (35 - 7.754,\ 35 + 7.754)$

$= (27.246,\ 42.754)$

3.4.1.3 Interval Estimation for μ

Sample information:-

$$\boldsymbol{t = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}}$$

95% Confidence Interval
95 % Confidence level

$$P\left(-t_{n-1,\frac{\alpha}{2}} \le t \le t_{n-1,\frac{\alpha}{2}}\right) = 0.95$$

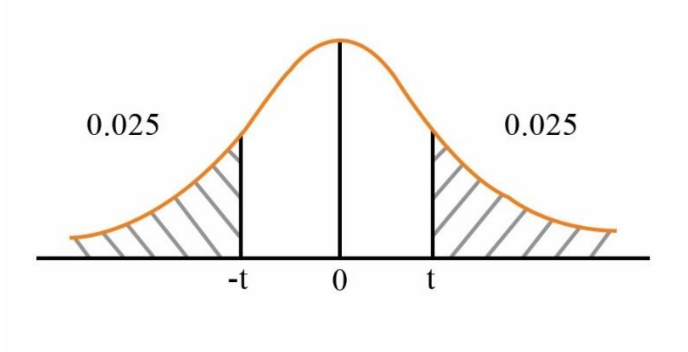

Fig 3.4.6 Sample Distribution

Critical values of *t* depends on $\boldsymbol{n - 1}$ degrees of freedom.
Example: If n = 28, d.f. = 27 and $t_{27,\frac{0.05}{2}} = 2.052$

For α = 0.01, $t_{27,\frac{0.01}{2}} = 2.771$

3.4.2 Confidence Interval for Proportions

- $P \pm z_c\sqrt{\frac{pq}{n}}$
  $= P \pm z_c\sqrt{\frac{p(1-p)}{n}}$

Suppose that the statistic S is the proportion of "successes" in a sample of size n ≧ 30 drawn from a binomial population in which p is the proportion of successes (i.e. the probability of success).

Then the confidence limits for p are given by P ± $\boldsymbol{z_c \sigma_P}$ where P denotes the proportion of successes in the sample of size n. Using the values of $\boldsymbol{\sigma_P}$ we see that the confidence limits for the population proportion are given by

$$P \pm z_c\sqrt{\frac{pq}{n}} = P \pm z_c\sqrt{\frac{p(1-p)}{n}}$$

in case sampling is from an infinite population or if sampling is with replacement from a finite population. Similarly, the confidence limits are

$$P \pm z_c\sqrt{\frac{pq}{n}}\sqrt{\frac{N-n}{N-1}}$$

if sampling is without replacement from a population of finite size N. Note that these results are obtained from (1) and (2) on replacing $\boldsymbol{\overline{X}}$ by P and $\boldsymbol{\sigma}$ by $\sqrt{\boldsymbol{pq}}$.

To compute the above confidence limits we use the sample estimate P for p.

Example 1

A sample poll of 100 voters chosen at random from all voters in a given district indicated that 55% of them were in favour of a particular candidate. Find (a) 95% (b) 99% and (c) 99.78% confidence limits for the proportion of all the voters in favour of this candidate.

Solution

a) The 95% confidence limits for the population p are

$$P \pm 1.96_{\sigma p} = P \pm 1.96\sqrt{\frac{p(1-p)}{n}}$$

$$0.55 \pm 1.96\sqrt{\frac{(0.55)(0.45)}{100}} = 0.55 \pm 0.10$$

where we have used the sample proportion 0.55 to estimate *p*.
b) The 99% confidence limits for *p* are

$$0.55 \pm 2.58\sqrt{(0.55)\,(0.45)/100} = 0.55 \pm 0.13.$$

c) The 99.73% confidence limits for *p* are

$$0.55 \pm 3\sqrt{(0.55)(0.45)/100} = 0.55 \pm 0.15.$$

3.4.3 Confidence Intervals for Variances

The fact that $\frac{\boldsymbol{nS^2}}{\boldsymbol{\sigma^2}} = \boldsymbol{(n-1)}\frac{\boldsymbol{s^2}}{\boldsymbol{\sigma^2}}$ has a chi-square distribution with *n* -1 degrees of freedom enables us to obtain confidence limits for $\boldsymbol{\sigma^2}$or $\boldsymbol{\sigma}$. For example, if $\boldsymbol{x^2_{.025}}$ and $\boldsymbol{x^2_{.975}}$ are the values of $\boldsymbol{x^2}$ for which 2.5% of the area lies in each "tail" of the distribution, then a 95% confidence interval is

$$x^2_{.025} \leqq \frac{nS^2}{\sigma^2} \leqq x^2_{.975}$$

or equivalently

$$x^2_{.025} \leqq \frac{(n-1)\hat{S}^2}{\sigma^2} \leqq x^2_{.975}$$

From these we see that can be estimated to lie in the interval

$$\frac{S\sqrt{n}}{x_{.975}} \leqq \sigma \leqq \frac{S\sqrt{n}}{x_{.025}}$$

or equivalently

$$\frac{\hat{S}\sqrt{n-1}}{x_{.975}} \leqq \sigma \leqq \frac{\hat{S}\sqrt{n-1}}{x_{.025}}$$

with 95% confidence. Similarly other confidence intervals can be found.

Example 1

The standard deviation of the lifetimes of a sample of 200 electric light bulbs was computed to be 100 hours. Find (a) 95% and (b) 99% confidence limits for the standard deviation of all such electric light bulbs.

In this case large sampling theory applies. Thus, confidence limits for the population standard deviation are given by $\boldsymbol{S} \pm \boldsymbol{z_c \, \sigma/\sqrt{2n}}$, where z_c indicates the level of confidence. We use the sample standard deviation to estimate $\boldsymbol{\sigma}$.

Solution

a) The 95% confidence limits are: $\mathbf{100 \pm 1.96\,(100)/\sqrt{400} = 100 \pm 9.8}$. Thus we can be 95% confident that the population standard deviation will lie between 90.2 and 109.8 hr.

b) The 99% confidence limits are: $\mathbf{100 \pm 2.58\,(100)/\sqrt{400} = 100 \pm 12.9}$. Thus we can be 99% confident that the population standard deviation will lie between 87.1 and 112.9 hr.

3.4.4 Methods of Estimation

The method of least squares and the method of maximum likelihood are integral to the estimation process. They offer distinct approaches to finding parameter estimates that best represent the data, with the former emphasising the minimisation of squared errors and the latter focusing on maximising the likelihood of the observed data under a given model.

3.3.4.1 Methods of least squares

In regression analysis, a comparable approach is employed where we determine the regression line that minimises the squared residuals.

$$\boldsymbol{e_i^2 = (y_i - \hat{y}_i)^2}$$

across all observations, resulting in the minimisation of the Sum of Squared Residuals

$$\sum e_i^2 = \Sigma(y_i - \hat{y}_i)^2$$

- Minimisation of squared errors

This process of identifying the estimated regression coefficients that achieve this minimisation is referred to as least squares estimation, offering a pragmatic approach to discovering the optimal estimated regression line among various alternatives.

Very often in practice a relationship is found to exist between two (or more) variables, and one wishes to express this relationship in mathematical form by determining an equation connecting the variables.

A first step is the collection of data showing corresponding values of the variables. For example, suppose x and y denote respectively the height and weight of an adult male Then a sample of x individuals would reveal the heights $x_1, x_2, \ldots, x_n$ and the corresponding weights $y_1, y_2, \ldots, y_n$.

A next step is to plot the points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ on a rectangular coordinate system. The resulting set of points is sometimes called a scatter diagram.

One of the main purposes of curve fitting is to estimate one of the variables (the dependent variable) from the other (the independent variable). The process of estimation often referred to as regression. If y is to be estimated from x by means of some equation we call the equation a regression equation of y on x and the corresponding curve a regression curve of y on x.

Generally, more than one curve of a given type will appear to fit a set of data. To avoid individual judgment in constructing lines, parabolas, or other approximating curves, it is necessary to agree on a definition of a "best-fitting line," "best-fitting parabola," etc.

3.4.5 Methods of Maximum Likelihood

Although confidence limits are valuable for estimating a population parameter it is still often convenient to have a single or point estimate. To obtain a "best" such estimate, we employ a technique known as the maximum likelihood estimate, due to Fisher.

To illustrate the method, we assume that the population has a density function which contains a population parameter, say $\boldsymbol{\theta}$, which is to be estimated by a certain statistic. Thus, the density function can be denoted by $\boldsymbol{f(x,\theta)}$. Assuming that there are n independent observations, $\boldsymbol{X_1}$, ..., $\boldsymbol{X_n}$, the joint density function for these observations is

$$L = f(x_1,\theta)f(x_2,\theta)\ldots.f(x_n,\theta)$$

- Maximizing the likelihood of the observed data

which is called the likelihood. The maximum likelihood can then be obtained by taking the derivative of L with respect to $\boldsymbol{\theta}$ and setting it equal to zero. For this purpose, it is convenient to first take logarithms and then take the derivative. In this way we find

$$\frac{1}{f(x_1,\theta)}\frac{\partial f(x_1,\theta)}{\partial \theta} + \cdots + \frac{1}{f(x_n,\theta)}\frac{\partial f(x_n,\theta)}{\partial \theta} = 0$$

From this we can obtain in terms of the $\boldsymbol{x_k}$.

The method is capable of generalisation. Thus, in case there are several parameters we take the partial derivatives with respect to each parameter, set them equal to zero and solve the resulting equations simultaneously.

Summarised Overview

The method of least squares seeks to find the parameters that minimise the sum of the squared differences between observed data points and the corresponding values predicted by a model. This method is particularly useful for linear regression, where the goal is to fit a straight line to the data that minimises the overall squared distances between the data points and the line. The method of maximum likelihood focuses on determining the parameter values that maximise the likelihood function, which measures the probability of observing the given data under a specific parameterised model.

Assignments

1. A random sample of 16 men in a village gives a mean height of 174 cms. Population σ =5 cms. What are the: (a) 95%, (b) 99% confidence limits for the mean height of the men?
2. In 40 tosses of a coin, 24 heads were obtained. Find (a) 95% and (b) 99.73% confidence limits for the proportion of heads which would be obtained in an unlimited number of tosses of the coin.
3. The standard deviation of the heights of 16 male students chosen at random in a school of 1000 male students is 6.10 centimeters. Find (a) 95% and (b) 99% confidence limits of the standard deviation for all male students at the school.

Suggested Reading

1. Anderson, D., D. Sweeney and T. Williams (2013): "*Statistics for Business and Economics*", Cengage Learning: New Delhi.
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Reference

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MASTER OF ARTS ECONOMICS

Hypothesis Testing

BLOCK 4

UNIT 1
Testing of Hypothesis

Learning Outcomes

After completing this unit, learner will be able to:

- understand the rationale behind hypothesis testing
- apply hypothesis testing in real-world scenarios
- formulate null and alternative hypotheses for different research scenarios
- differentiate between parametric and non-parametric tests of hypothesis

Background

Before discussing hypothesis testing, it is crucial to understand the fundamental concepts that form the backbone of this statistical process. A hypothesis is essentially a statement that researchers make about what they expect to find in their studies. There are two main types of hypothesis known as simple and composite hypothesis. A simple hypothesis predicts a specific outcome, while a composite hypothesis involves multiple possibilities or conditions. Within the component of hypothesis testing, we will also discuss the null hypothesis and the alternative hypothesis. The null hypothesis represents the status quo or a default assumption to be challenged, while the alternative hypothesis suggests a different outcome that the researcher aims to support.

In the statistical tests, there exists two broad categories. They are parametric and non-parametric tests. Parametric tests rely on assumptions about the underlying distribution of data, while non-parametric tests offer more flexibility, making fewer assumptions about the data's characteristics. Understanding these distinctions is vital for selecting the appropriate test based on the nature of the research and the data at hand.

Keywords

Simple Hypothesis, Composite Hypothesis, Null Hypothesis, Alternative Hypothesis, Parametric Tests, Non-Parametric Tests

Discussion

4.1.1 Hypothesis

A startup company introduces a new mobile app designed to improve financial literacy among young adults. The company claims that using its app for a specified duration helps users increase their savings by an average of ₹200 per month. To validate this claim, the company conducts a study where they randomly select a sample of app users and track their savings over a few months. After analysing the sample data, they calculated the sample mean increase in savings to be ₹220 per month.

In this scenario, the company wants to make a statistical inference about the population mean increase in savings for all users of the app. They can use the method of confidence intervals to estimate the true average increase in savings with a certain level of confidence. They might calculate a 95% confidence interval that ranges between ₹205 and ₹235, indicating that they are 95% confident that the true average increase in savings falls within this range.

Numerous real-world scenarios similar to this frequently arise in the realm of economic research.

- Hypothesis is an educated guess about a parameter

To understand the concept of hypothesis testing, it is essential to begin by grasping the concept of hypothesis. A hypothesis is an informed speculation about a parameter. With a hypothesis in place, the subsequent step involves gathering relevant data. Using the data, we need to arrive at a decision.

4.1.2 Simple and Composite Hypothesis

- Simple hypothesis focuses on a single distribution for X

Hypotheses can be categorised into two main types. They are simple and composite. A simple hypothesis confines itself to a singular distribution for *X*. In this scenario, the hypothesis defines one specific outcome or pattern that *X*'s distribution should adhere to.

- Composite hypothesis focuses on various distributions for X

On the other hand, a composite hypothesis entertains the possibility of multiple distributions for X. This means that the hypothesis encompasses a range of potential distributional patterns for X. The composite hypothesis acknowledges the variability in X's behaviour and contemplates various alternatives within its scope.

4.1.3 Null and Alternative Hypothesis

The initial phase of the testing process involves the formulation of two distinct hypotheses known as the null hypothesis and the alternative hypothesis. These hypotheses represent contrasting viewpoints and serve as the foundation for the subsequent analysis.

- Null hypothesis asserts the absence of distinctions between groups or relationships between variables

The null hypothesis is usually represented as H_0. It is a statement that signifies the absence of a significant difference or relationship between the variables under consideration. It is often regarded as the established state or the current situation, the null hypothesis necessitates action if it is not supported.

- Asserting the presence of a relationship

The alternative hypothesis is usually referred to as H_1 is an assertion that presents a proposition about the population that contradicts the null hypothesis, representing the researcher's intended outcome. It represents what is being sought to be proven through the analysis.

Given the contradictory nature of the null and alternative hypotheses, the examination of evidence becomes crucial in determining whether there is sufficient reason to reject the null hypothesis. The evidence in question is derived from the sample data.

A bottler of soft drinks may hypothesize that the mean fill is 16 ounces (μ = 16). This null hypothesis (H_0:) is tested against the alternative hypothesis (H_1:) stated to the contrary. In this case, the mean fill is not 16 ounces $\mu \neq 16$.

Thus, we would have

$H_0: \mu = 16$ $\qquad$ $H_1: \mu \neq 16$

Even if $\overline{X} = 16$, it does not prove that μ =16. It could be that μ is 16.15

A difference between the hypothesised population mean and the sample mean is small enough to attribute to sampling error.

If the difference between the hypothesised mean of 16 and the sample finding of 16.15 is insufficient to reject the null, the question then becomes just how large the difference must be to be statistically significant and to lead to a rejection of the null. Recall from our discussion of sampling distributions that we can transform any unit of measurement, such as the ounces for the bottler, to corresponding *Z*-values with the *Z*-formula:

$$Z = \frac{\overline{X} - \mu}{\sigma_{\overline{X}}} = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

If *s* is unknown, the sample standard deviation *s* is used.

The resulting normal distribution of Z-values has a mean of zero and a standard deviation of one. The empirical rule tells us 95 percent of the $\overline{\boldsymbol{X}}$'s in the sampling distribution is within 1.96 standard errors of the unknown population mean, this is shown in figure 1.1

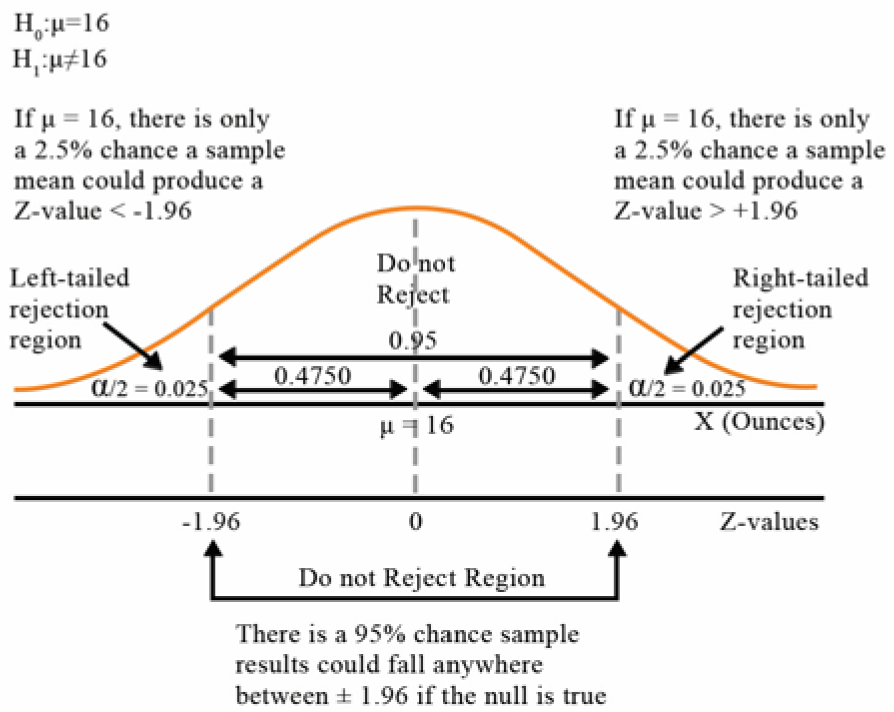

Figure 4.1.1 Critical Values for *Z* and Rejection Regions

These Z-values of ± 1.96 are the critical values indicating the rejection regions. To find them, divide the 95 percent by 2. From the Z-table, the area of $\frac{0.95}{2} = 0.4750$ produces a Z-value of 1.96. The remaining 5 percent is distributed in the two tails, with 2.5 percent in each rejection region.

4.1.4 Testing of Hypothesis

In Figure 4.1.1 note that if the bottler's hypothesis is correct and $\mu = 16$ ounces, it is unlikely (only a 5 percent chance) that any sample could produce a Z-value falling in either rejection region. Thus, if a Z-value greater than 1.96 or less than -1.96 does occur, it is unlikely that the distribution is centered at $\mu = 16$, and the null should be rejected.

These critical Z-values of ± 1.96 allow us to establish a decision rule that tells us whether to reject the null. The decision rule is: Do not reject the null if the Z-value is between ± 1.96. Reject if the Z-value is less than -1.96 or greater than +1.96.

The logic behind this decision rule, which is based simply on probabilities, should be clear. If the null is true, it is unlikely that a Z-value greater than 1.96 or less than -1.96 could possibly result. Only 5 percent of all the samples in the sampling distribution could produce a Z-value greater than 1.96 or less than -1.96. Therefore, if such a Z-value occurs, it is unlikely that $\mu = 16$, and the null should be rejected.

4.1.5 Parametric and Non-parametric Tests of Hypothesis

Many of the tests used for hypothesis testing assumed that the population was normally distributed or that variances fit particular patterns. When those assumptions cannot be made, or essential knowledge about the population is not attainable, we have to rely on nonparametric tests.

However, all these test situations exhibited one common characteristic: they required certain assumptions regarding the population. For example, t-tests and F-tests required the assumption that the population was normally distributed. Since such tests depend on postulates about the population and its parameters, they are called parametric tests. In practice, many situations arise in which it is simply not possible to safely make any assumptions about the value of a parameter or the shape of the population distribution. We must instead use other tests that do not depend on a single type of distribution or specific parametric values. These are called nonparametric (or distribution-free) tests.

Nonparametric Tests are the statistical procedures that can be used to test hypotheses when no assumptions regarding parameters or population distributions are possible.

Summarised Overview

Hypothesis testing is an essential statistical procedure used to draw conclusions about populations based on sample data. Parametric tests make assumptions about specific characteristics of the population distribution, while nonparametric tests offer flexibility and are suitable when distribution assumptions are unknown. The outcome of a hypothesis test leads to one of two conclusions: either rejecting the null hypothesis in favour of an alternative or failing to reject the null hypothesis. The null hypothesis is a statement that suggests no significant effect, no difference, or no change in the population parameter under investigation. It often represents the status quo or a baseline assumption. In mathematical terms, it is denoted as H_0. The alternative hypothesis is the statement that contradicts the null hypothesis. It posits a significant effect, a difference, or a change in the population parameter. The alternative hypothesis is what researchers aim to support with their data. In mathematical terms, it is denoted as H_1 or H_a.

Assignments

1. Distinguish between Null and Alternative Hypothesis
2. Distinguish between Parametric and non-parametric tests.

Suggested Reading

1. Anderson, D., D. Sweeney and T. Williams (2013): "*Statistics for Business and Economics*", Cengage Learning: New Delhi.
2. Goon, A.M., Gupta and Das Gupta B (2002): *Fundamentals of Statistics* (Vol I), World Press.

Reference

1. Webster, A. (2010). *Applied Statistics for Business and Economics*. Tata McGraw Hill Education Private Limited, New Delhi.
2. Ross, S. M. (2017). *Introductory Statistics*. Academic Press.
3. Mayes, Anne C., and David G. Mayes (1976). "*Introductory Economic Statistics*."

Space for Learner Engagement for Objective Questions

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UNIT 2
Concepts of Hypothesis Testing

Learning Outcomes

After completing this unit, learner will be able to:

- understand the hypothesis testing
- identify the Type I and Type II error
- determine the level of significance
- know the test procedure of the hypothesis

Background

Statistical hypothesis testing is a fundamental concept in the field of statistics. It offers a systematic framework for making informed inferences about population parameters. A statistical hypothesis is a statement or assumption concerning a population parameter, such as the mean or standard deviation. The process of hypothesis testing involves several key steps, starting with making assumptions about a population. This often involves statements about parameters like the population mean or standard deviation. After that, a statistical hypothesis is formulated, consisting of a null hypothesis representing a default assumption and an alternative hypothesis that contradicts the null. Next, sample data is collected from the population to provide a basis for analysis. Statistical methods are then applied to the sample data to assess whether the assumptions made about the population are substantiated by the observed data. This method is essential for making valid inferences about the broader population based on limited sample information.

The results of hypothesis testing enable researchers and statisticians to draw conclusions about population parameters. This contributes to evidence-based decision-making in various fields of study and application.

Keywords

Type I Error, Type II Error, Critical Region, Level of Significance, Power of a Test

Discussion

4.2.1 Type I and Type II Error

Hypothesis testing should be conducted with utmost care. This is due to its implications on decision-making and the potential consequences of drawing incorrect conclusions. Errors in hypothesis testing can lead to wrong decisions that impact real-world scenarios.

- Type I error occurs when you reject the null hypothesis

In testing a hypothesis, we can make two types of mistakes. A Type I error is rejecting a true hypothesis. In the example mentioned in the previous unit, if the bottler's hypothesis is true and $\mu = 16$, there is still a 5 percent chance that a sample mean could fall in either rejection region, causing us to incorrectly reject the null. Of all the sample means in the sampling distribution, 2.5 percent produces a Z-value > 1.96 in the right-tailed rejection region and 2.5 percent produces a Z-value < -1.96 in the left-tailed rejection region. This 5 percent is the level of significance, or α-value (α-value), and represents the probability of a Type I error.

- Type II error occurs when one accepts a false null hypothesis

A Type II error is not rejecting a false hypothesis. If the null hypothesis $\mu = 16$ is not correct, but our test fails to detect this, we commit a Type II error.

4.2.2 Critical Region

The critical region of a hypothesis test encompasses the values of the test statistic that would lead to the rejection of the null hypothesis. This region is also referred to as the rejection region because it defines the range of extreme or significant values that signal a departure from the null hypothesis.

4.2.3 Level of Significance

To determine the critical region, certain key points need to be considered:

Magnitude of the Test Statistic: For the null hypothesis to be rejected, the test statistic should exhibit extreme values, either very large or very small, depending on the specific test being conducted.

Level of Significance: When the significance level is set, it corresponds to the probability of committing a Type I error (rejecting a true null hypothesis). This level is typically chosen based on the desired level of confidence in the test results.

Coverage of the Critical Region: The critical region's extent is directly linked to the significance level α. For example, if the significance level is set at 0.05, the critical region should account for 5% of the probability distribution associated with the reference distribution (often the Standard Normal Distribution).

Interpretation: The critical region is particularly important for non-directional hypotheses. It identifies the range of values for the test statistic where the null hypothesis would be rejected. The Standard Normal Distribution characterizes the probabilities of obtaining specific values assuming the null hypothesis holds true.

Tests of significance are based on the fact that each null hypothesis can be tested with a particular type of statistical test. Every calculated statistic has a special distribution associated with it. The calculated value is then compared to the distribution to conclude whether the sample characteristics are different from what you would expect by chance.

Confidence intervals represent the best estimate of the range of the population value (or population parameter) based on the sample value (or sample statistic). A higher confidence interval (for example, a 99% confidence interval as compared with a 95% confidence interval) represents a greater degree of confidence, meaning that a wider range of values will be incorporated into the confidence interval.

4.2.4 Power of a Test

In cases where the sample size is small, even if there exists a noticeable difference between sample means, it might not be statistically significant. In such situations, making definitive conclusions about the population means becomes challenging. The lack of significance does not provide evidence in favour of the null hypothesis that the population means are equal, nor does it support the notion that the null hypothesis is false. Consequently, when an observed effect fails to attain significance, the outcome is inconclusive. In scenarios where resources are limited, you might prefer allocating funds to projects with a higher likelihood of yielding robust conclusions, allowing for more confident decision-making.

The power of a test is a crucial concept in statistical hypothesis testing, representing the probability of correctly identifying and rejecting a false null hypothesis.

4.2.5 Test Procedure

Applying a statistical test to any null hypothesis follows eight general steps:

1. Provide a statement of the null hypothesis.
2. Set the level of risk associated with the null hypothesis (significance level).
3. Select the appropriate test statistic.
4. Compute the test statistic value (also known as the obtained value).
5. Determine the value (the critical value) needed for rejection of the null hypothesis using the appropriate table of critical values for that particular statistic.
6. Compare the obtained value with the critical value.
7. If the obtained value is more extreme than the critical value, the null hypothesis must be rejected.

8. If the obtained value does not exceed the critical value, the null hypothesis cannot be rejected.

Summarised Overview

Hypothesis testing is a crucial and systematic process in statistics that involves making informed decisions about population parameters based on sample data. The key elements of hypothesis testing include the formulation of null and alternative hypotheses, the determination of a critical region, the choice of a significance level, and the consideration of both Type I and Type II errors. Hypothesis testing is a statistical tool used to draw conclusions about a population based on sample data. It involves formulating a null hypothesis, which states that there is no significant difference between the population parameter and the sample statistic, and an alternative hypothesis, which states that there is a significant difference between the population parameter and the sample statistic. There are two types of errors that can occur in hypothesis testing: Type I and Type II errors. A Type I error happens when a true null hypothesis is mistakenly rejected, leading to a false positive conclusion. On the other hand, a Type II error occurs when a false null hypothesis is not rejected, resulting in a false negative conclusion. To determine whether to reject or accept the null hypothesis, a critical region is established. This critical region is a range of values that, if observed in the sample data, would lead to the rejection of the null hypothesis. The level of significance, denoted by alpha (α), sets the threshold for rejecting the null hypothesis. It represents the probability of committing a Type I error. The power of a test is the probability of correctly rejecting a false null hypothesis. It is influenced by factors such as sample size and effect size. Therefore, it is essential to strike a balance between Type I and Type II errors, selecting an appropriate level of significance, and maximizing the power of a test to ensure the reliability and accuracy of conclusions drawn through hypothesis testing.

Assignments

1. Explain the key concepts of test hypothesis
2. Define the power of a statistical test and discuss its relationship with Type II errors.

Suggested Reading

1. Anderson, D., D. Sweeney and T. Williams (2013): "*Statistics for Business and Economics*", Cengage Learning: New Delhi.
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UNIT 3
Test of Significance

Learning Outcomes

After completing this unit, learner will be able to:

- understand the tools of hypothesis testing
- calculate the one-tailed and two-tailed test
- know the correlation coefficient and their properties
- recognise the chi-square test of goodness of fit

Background

Hypothesis testing is a fundamental statistical method used to make decisions and draw inferences about a population based on sample data. The process involves formulating and testing hypotheses regarding population parameters. Hypothesis testing is a helpful tool for making decisions based on empirical evidence. When making decisions that involve comparing two groups or populations, it is important to determine whether there is a significant difference in their means or proportions, depending on whether the data is quantitative or categorical. Statistical methods such as t-tests or chi-square tests are used to analyse sample data, and a test statistic is calculated to help make decisions.

Keywords

Significance Test, Chi-Square, Test of Goodness of Fit, Test for Independence of Attributes

Discussion

4.3.1 Test of Significance with Respect to Mean

- Method to uncover the unknown in various situations

Hypothesis testing is a method to uncover the unknown in various situations, encompassing different parameters such as mean, proportion, and variance. This process involves applying diverse tests to data, allowing for the acceptance or rejection of hypotheses in various contexts.

We usually conduct the test of significance with respect to mean when the sample consists of 'n' independent observations drawn from a normal distribution. The n independent random variables, follow a normal distribution with an unknown population mean (μ), and a known variance (σ^2).

Symbolically,

$X \sim N(\mu, \sigma)$

μ is unknown; σ^2 is known

The following are the key steps to conduct the test of significance with respect to mean.

Assertion $\mu = \mu_0$ (a value)

$$\overline{X} \sim \left(\mu, \frac{\sigma^2}{n}\right)$$

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

If the level of significance (LoS) $\alpha = 0.05$

$$|Z| = \frac{|\overline{X} - \mu_0|}{\frac{\sigma}{\sqrt{n}}}$$

If $|Z| > 1.96 \Rightarrow$ Reject the assertion.

If $|Z| < 1.96 \Rightarrow$ Do not reject the assertion.

i.e. Our sample information does not contradict the assertion.

$\alpha = 0.01$

$|Z| > 2.58 \Rightarrow$ Reject

$|Z| < 2.58 \Rightarrow$ Do not reject

The assertion that is made about the numerical value of the population parameter is called the H_0.

Alternative Hypothesis – negation of H_0

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0, \mu < \mu_0, \mu > \mu_0$ or $\mu = \mu_1$ (some other value)

In case null is rejected, H_1 is automatically accepted.

4.3.1.1 Two Tailed Test

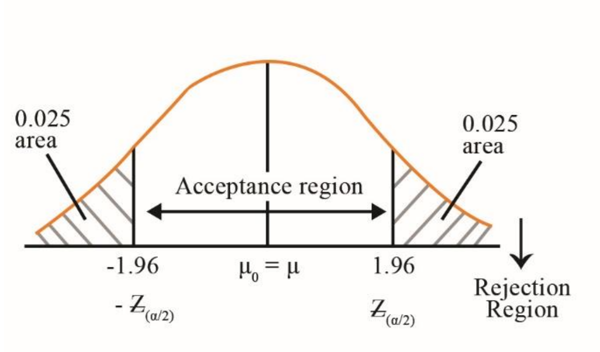


Fig 4.3.1 Two Tailed Test

In the above figure,

$\alpha = 0.05$ $\frac{\alpha}{2} = 0.025$

$|Z| > 1.96$ ($\leftarrow$ $critical\ value \rightarrow reject\ the\ null$)

Rejection region in two tails of tests is called two tailed test S.

Total Rejection region = α

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$, it permits μ to be > or < than μ_0.

Steps:

1. Put H_0 and H_1.
2. Calculate $|Z|$.
3. Compare it with critical value $Z_{\frac{\alpha}{2}}$

Example 1

A motor car company claims that their car average is 35 miles per gallon of Petrol. A random sample of 50 cars was tested and found to give an average of 32 miles per gallon. With a standard deviation of 1.2 gallons, test the null hypothesis μ = 35 against $H_1: \mu \neq 35$ at 5% level of significance.

Solution

$H_0: \mu = 35$

$H_1: \mu \neq 35$

$\overline{X} = 32$

$$|Z| = \frac{|\overline{X} - \mu_0|}{\frac{\sigma}{\sqrt{n}}} = \frac{|32 - 35|}{\frac{1.2}{\sqrt{50}}} = 17.75 > 1.96$$

Condition: We reject H_0, that the cars average 35 miles per gallon at both levels of significance 0.05

4.3.1.2 One Tailed Test

1. Left Tailed Test

$H_0: \mu = \mu_0$

$H_1: \mu < \mu_0$ → Left tailed test

α → Level of significance

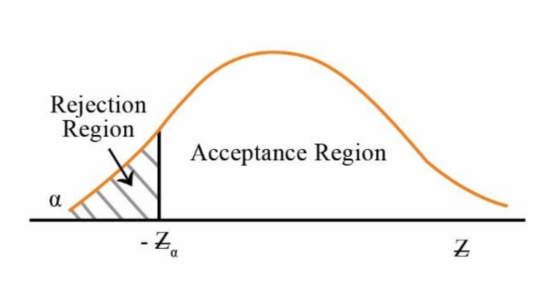


Fig 4.3.2 One Tailed Test

$\boldsymbol{Z_\alpha}$ is that critical value of Z which cuts off α% of area in the left tail of a standard normal curve.

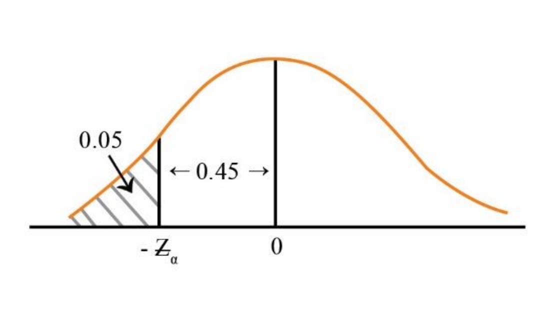


Fig 4.3.3 Left Tailed Test

$-1.65 \rightarrow 0.05$ level of significance

$-2.33 \rightarrow 0.01$ level of significance

If $Z < -1.65 \Longrightarrow$ Reject the H_0 at 5% LoS

$Z < -2.33 \Longrightarrow$ Reject the H_0 at 1% LoS

$Z < -Z_\alpha \Longrightarrow$ Reject the H_0

Otherwise, do not reject H_0

2. Right Tailed Test

$H_0 : \mu = \mu_0$

$H_1 : \mu < \mu_0 \rightarrow$ Left tailed test

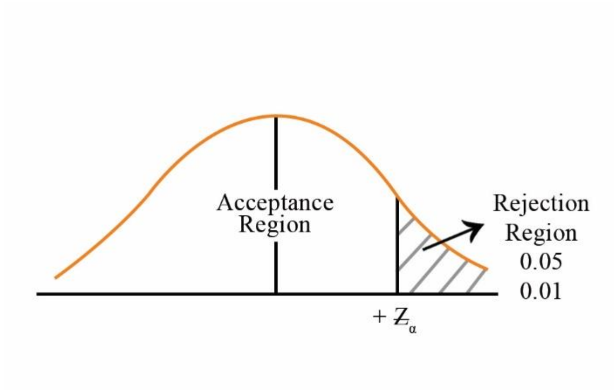


Fig 4.3.4 Right Tailed Test

5% + 1.65

1% + 2.33

$\boldsymbol{Z_\alpha}$ is that critical value of Z which cuts off α% of area in the right tail of a standard normal curve.

If $Z > 1.65 \Longrightarrow$ Reject the H_0 at 5% LoS

$Z > 2.33 \Longrightarrow$ Reject the H_0 at 1% LoS

Example 2

Suppose that a census of city dwellers reveals an average family size of 4.2 with a standard deviation of 0.5. A random sample of 100 city side families reveals a family size of 4.29. We wish to test whether the family size in the city side is the same as in the city.

Solution

$H_0: \mu = 4.2$

$H_1: \mu \neq 4.2$

$$|Z| = \frac{|\overline{X} - 4.2|}{\frac{0.5}{10}} = \frac{|4.29 - 4.2|}{0.05} = \frac{0.09}{0.05} = \frac{9}{5} = 1.8$$

$1.8 < 1.96 \Longrightarrow$ Do not reject H_0 at 5 % LoS

Conclusion: Family size in the city side is same as in the city.

At 1% LoS

$H_0: \mu = 4.2$

$H_1: \mu \neq 4.2$

$$|Z| = \frac{|4.29 - 4.2|}{\frac{0.5}{10}} = 1.8 < 1.65$$

$1.8 < 2.33 \rightarrow$ Do not reject H_0 *at* 1% LoS

Conclusion: Average family size in the city side is greater than the average family size in the city.

Example 3

A random sample of boots owned by 40 soldiers in a desert region showed an average life of 1.08 years with a standard deviation of 0.05 years. Under standard conditions, the boots are known to have an average life of 1.28 years. Is there a reason to assert at a level of significance of 0.05 that use in deserts causes the mean life of such boots to decrease.

Solution

$H_0: \mu = 1.28$

$H_1: \mu < 1.28$

$$|Z| = \frac{|\overline{X} - 1.28|}{\frac{0.05}{\sqrt{40}}} = \frac{|1.08 - 1.28|}{\frac{0.05}{6.32}} = \frac{-0.2}{0.008}$$

$$= -25 > 1.65$$

Reject H_0

Conclusion: Use in deserts causes the mean life of such boots to decrease.

Example 4

It is claimed that a random sample of 100 tyres with mean life of 15269 kms is drawn from a population of tyres which has a mean life of 15200 kms and standard deviation is 124.8 kms. Test the validity of the claim.

Solution

$H_0: \mu = 15200$

$H_1: \mu \neq 15200$

$$|Z| = \frac{|15269 - 15200|}{\frac{1248}{\sqrt{100}}} = \frac{69}{124.8} = 0.055 < 1.96 < 2.58$$

Do not reject H_0

At 1 % Los

$|Z| = 0.055 < 2.58$

Do not reject H_0

Conclusion: This sample is drawn from a population of tyres which has a mean life of 15200 kms.

Example 5

A census of retail establishments in a particular month revealed that the mean monthly turnover of suburban food stores was ₹2500. A random sample of 16 such stores taken in the following month from the normal population had a mean monthly turnover of ₹2660 and a standard deviation of turnover of ₹480. Could you conclude that the mean monthly turnover had changed since the census.

Solution

Please note that the number of sample S (n = 16) is less than 30, hence we conduct t test.

$n = 16, \bar{X} = 2660, s = 480, \mu = 2500$

$H_0: \mu = 2500$

$H_1: \mu \neq 2500$

$$t = \frac{|\bar{X} - \mu_0|}{\frac{s}{\sqrt{n}}} = \frac{160}{120} = 1.33$$

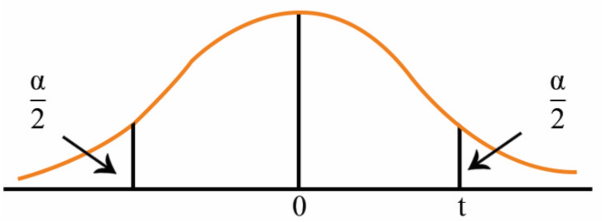


Fig 4.3.5 Critical Value for one Tail

Critical value for one tail is 0.025 and 15 *d.f.*
Critical Value of t: $|t_{15,0.05}|$ is 2.131

If $t > t_{n-1,\frac{\alpha}{2}}$ Reject H_0

If $t < t_{n-1,\frac{\alpha}{2}}$ Do not reject H_0

Here, $1.33 < 2.131$ so, do not reject H_0. There has not been a statistically significant change in mean monthly turnover of suburban food stores since the census.

4.3.1.3 Distribution of Difference in Means, Interval Estimation and Hypothesis Tests for Difference

There are situations where the distribution of difference in means. Then we need to adopt the hypothesis tests for difference.

Two Populations with	$\boldsymbol{\mu_1}$	$\boldsymbol{\mu_2}$
Population variance	$\boldsymbol{\sigma_1}$	$\boldsymbol{\sigma_2}$
Pick 2 samples of	$\boldsymbol{n_1}$	$\boldsymbol{n_2}$
Sample Mean	$\bar{X}_1$	$\bar{X}_2$

4.3.1.4 Sampling Distribution of $(\bar{X}_1 - \bar{X}_2)$ Difference in Means

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2)$$

$$= \mu_1 - \mu_2$$

$$V(\bar{X}_1 - \bar{X}_2) = V(\bar{X}_1) + V(\bar{X}_2)$$

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

$$(\bar{X}_1 - \bar{X}_2) \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

If the two samples are drawn from normal population if $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ exceeds 30.

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$ is the standard normal variable

Our concern is $\mu_1 - \mu_2$ (previously it was μ)

Point estimation is $\bar{X}_1 - \bar{X}_2$

Interval estimation: $P[-1.96 < Z < 1.96] = 0.95$

$$P\left[(\bar{X}_1 - \bar{X}_2) - Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\bar{X}_1 - \bar{X}_2) + Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right]$$

$$= 1 - \alpha \text{ or } (1 - \alpha) \times 100\%$$

Steps involved when $\boldsymbol{t}$-Distribution is using the *Difference in Means*: -

- σ is unknown
- We know s_1 and s_2
- Make an estimate s of the σ
- Pooled *s.d.*,

$S = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$ Replace s by σ

$$t = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}, t = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{S\sqrt{\frac{1}{N_1} + \frac{1}{N_2}}}$$

If $N_1 + N_2 - 2 < 30 \rightarrow t$-distribution.

$N_1 + N_2 - 2 \geq 30 \rightarrow$ normal distribution.

Example 6

Ten plots of land are treated with fertiliser $\boldsymbol{A}$ and 12 with fertiliser $\boldsymbol{B}$. The mean yield of the first is 6 bushels with a stand deviation of 0.03 bushels. The yields of second plots have 5.95 bushels with a stand deviation. of 0.04 bushels. At a **1**% LoS, is there a difference have a mean of the fertilisers?

Solution

$n_1 = 10, \bar{X}_1 = 6, s_1 = 0.03$

$n_2 = 12, \bar{X}_2 = 5.95, s_2 = 0.04$

Assumption: They have same σ, i.e. $\sigma_1 = \sigma_2 = \sigma$

S.D. of distribution of $\bar{X}_1 - \bar{X}_2$ is $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

On assumption, it is $\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

σ is unknown.

Replace σ by $s = \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$

$= \sqrt{\frac{9 \times (0.03)^2 + 11 \times (0.04)^2}{20}}$

$= \sqrt{0.001285}$

$= 0.036$

$H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

$\alpha = 0.05$

$d.f.\ n_1 + n_2 - 2 = 20$

$$t = \frac{|(\bar{X}_1 - \bar{X}_2) - 0|}{s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{0.05}{0.036\sqrt{\frac{1}{10}+\frac{1}{12}}} = \frac{0.05}{0.015} = 3.3$$

Table value t (20, $\frac{0.05}{2}$) = 2.086

Since, $3.3 > 2.086 \Rightarrow$ Reject H_0.

Conclusion: The two samples are unlikely to have come from the same population or $\bar{X}_1$ and $\bar{X}_2$ differ significantly. The sample results are evidence of a difference in effects of the two fertilisers.

4.3.2 Test of Significance with respect to Proportion

When a random sample of *N* observations are drawn from a dichotomous population which has a proportion of success of *p*. The formula for applying the test of significance with respect to proportion is discussed hereunder.

- $|Z| = \frac{|P-\pi|}{\sigma_p}$,

$H_0: \pi = \pi_0$ (Samples does come from such a population)

$H_1: \pi \neq \pi_0$

$|Z| = \frac{|P-\pi|}{\sigma_p}$, where $\sigma_p = \sqrt{\pi}$

Example 7

A sample of 400 electors selected at random gives a 51% majority to the party in office and 49% to the party in opposition. Could such a sample have been drawn from a population with a 50-50 division of political opinion?

Solution

N = 400, p = 0.51, $\pi = 0.5$, $\sigma_p = 0.025$,

$H_0: \pi = 0.5$,

$H_1: \pi \neq 0.5$

$$|Z| = \frac{|P - \pi_0|}{\sigma_p} = \frac{|0.51 - 0.5|}{0.025} = \frac{0.01}{0.025} = 0.4$$

$|Z| = 0.4 < 1.96 \Longrightarrow Do\ not\ reject$

Conclusion: This sample could have reasonably come from a population with a 50-50 division of political opinion.

4.3.3 Test of Significance in Respect of Variance

Under this section, we wish to test statistically whether the two samples have come from the same population with same mean, i.e. $\mu_1 = \mu_2$ and same variance σ^2.

In this case,

$H_0: \mu_1 - \mu_2 = 0(\text{ or } \mu_1 = \mu_2)$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$H_1: \mu_1 - \mu_2 \neq 0(\text{ or } \mu_1 \neq \mu_2)$

Sample Information

$$\frac{n_1}{X_1} \frac{n_2}{X_2}$$

$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \sim N(0,1)}$$

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - 0}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \text{(Same } \sigma \text{ under } H_0\text{)}$$

If $Z > 1.96 \Rightarrow$ Reject H_0 at 5% LoS.

If $Z < 1.96 \Rightarrow$ Do not reject H_0 at 5% LoS.

Example 8

A random sample of 40 male employees is taken at the end of a year and the mean number of hours of absenteeism for the year is found to be 63 hours. A similar sample of 50 female employees has a mean of 66 hours. Could these samples have been drawn from a population with the same mean and with $\sigma = 10hrs$.

Solution

Here, $n_1 = 40, n_2 = 50$

$\bar{X}_1 = 63, \bar{X}_2 = 66$

$\sigma = 10$

H_0 : samples drawn from same population with $\sigma = 10$ and $\mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

$\sigma = 5\%$ or 0.05

$$z = \frac{|\bar{X}_1 - \bar{X}_2|}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{|63 - 66|}{10\sqrt{\frac{1}{40} + \frac{1}{50}}} = \frac{63 - 66}{2.12} = \frac{3}{0.10 \times 0.212} = \frac{3}{2.12}$$

$= 1.42 < 1.96 \Rightarrow$ Do not reject H_0

Conclusion: There appears to be no difference in the incidence of absenteeism between the two groups of employees.

4.3.4 Correlation Coefficient and their Differences

- Focus is on understanding bivariate data

In various economic contexts, we often gather data on multiple variables for each individual or entity. For instance, we might collect information on income and expenditure, or unemployment rates and inflation rates, or GDP growth and trade balance. Instead of analysing these variables in isolation (univariate data), our focus is on understanding bivariate data, where we examine the relationship between two variables measured for each subject in our dataset. This involves investigating whether changes in one variable are associated with corresponding changes in the other variable. For instance, as GDP growth varies, are there corresponding changes in the trade balance?

In order to understand the association, we usually resort to correlation.

The formula for finding correlation coefficient is as follows:

$$r = \frac{\sum(xi - \bar{x}) - (yi - \bar{y})}{\sqrt{\sum(xi - \bar{x})^2 \sum(yi - \bar{y})^2}}$$

The properties of the correlation coefficient "r" are as follows:

- It always falls within the range of -1 and +1.
- The value of "r" remains consistent regardless of the units used for measuring the two variables
- Positive values of "r" indicate positive relationships between the variables.
- Negative values of "r" indicate negative relationships between the variables.

Example 9

Calculate the Pearson correlation coefficient for the following data, which represents the relationship between the number of hours studied (X) and the corresponding test scores (Y) of a group of students:

Table 4.3.1 Hours Studied and Test Scores

Hours Studies (X)	Test Score (Y)
4	85
6	89
7	91
9	97
10	99

Solution

Table 4.3.2 Solution Table- Example 9

	Hours Studies (x)	Test Score (y)	x- x̄	y - ȳ	(x- x̄)(y- ȳ)	$(x- \bar{x})^2$	$(y - \bar{y})^2$
	4	85	-3.2	-7.2	23.04	10.24	51.84
	6	89	-1.2	-3.2	3.84	1.44	10.24
	7	91	-0.2	-1.2	0.24	0.04	1.44
	9	97	1.8	4.8	8.64	3.24	23.04
	10	99	2.8	6.8	19.04	7.84	46.24
Total	**36**	**461**	**0**	**0**	**54.8**	**22.8**	**132.8**

$$\bar{X} = \frac{\sum X}{n} = \frac{36}{5} = 7.2$$

$$\bar{Y} = \frac{\sum Y}{n} = \frac{461}{5} = 92.2$$

$$r = \frac{\sum(xi - \bar{x}) - (yi - \bar{y})}{\sqrt{\sum(xi - \bar{x})^2 \sum(yi - \bar{y})^2}}$$

$$r = \frac{54.2}{\sqrt{22.8 \times 132.8}}$$

$$r = \frac{54.2}{55.3} = 0.99$$

The Pearson correlation coefficient (r) between the number of hours studied and the corresponding test scores is 0.99, indicating a strong positive linear relationship between the two variables. This means that as the number of hours studied increases, the test scores tend to increase.

4.3.5 Chi-square Test of Goodness of Fit

- Foundational for conducting tests of significance

The chi-square distribution is closely connected to the analysis of categorical data and tests of independence in statistics. In scenarios where there are two possible events, labelled as "success" and "failure," with respective probabilities p and $q = 1 - p$, the concept of the chi-square distribution emerges. This distribution is applied to random variables like $X = nP$, where n represents the sample size, P denotes the probability of the "success" event, and np signifies the expected frequency of "success" occurrences. This framework is foundational for conducting tests of significance and exploring associations in categorical data, where observed and expected frequencies are pivotal in evaluating the fit between observed data and theoretical expectations. The chi-square distribution provides a critical tool for statistical inference in such scenarios.

Example 10

Use the chi-square test to determine the goodness of fit of the data.

Table 4.3.3 Number of Heads

Number of Heads ***X***	***P*** **(*x* heads)**
0	0.0332
1	0.1619
2	0.3162
3	0.3087
4	0.1507
5	0.0294

Solution

Table 4.3.4 Solution Table- Example 10

Number of Heads ***x***	***P*** **(*x* heads)**	**Expected Frequency**	**Observed Frequency**
0	0.0332	33.2 or 33	38
1	0.1619	161.9 or 162	144
2	0.3162	316 or 316	342
3	0.3087	308.7 or 309	287
4	0.1507	150.7 or 151	164
5	0.0294	29.4 or 29	25

$$\chi^2 = \frac{(38-33.2)^2}{33.2} + \frac{(144-161.9)^2}{161.9} + \frac{(342-316.2)^2}{316.2} + \frac{(287-308.7)^2}{308.7} + \frac{(164-150.7)^2}{150.7} + \frac{(25-29.4)^2}{29.4}$$

$= 7.45$

Since the number of parameters used in estimating the expected frequencies is $m = 1$ (namely the parameter p of the binomial distribution),

$v = k - 1 - m = 6 - 1 - 1 = 4.$

For $v = 4, x^2_{.95} = 9.49$. Hence the fit of the data is good.

For $\boldsymbol{v = 4, x^2_{.05} = 0.711}$. Thus, since $\boldsymbol{x^2 = 7.54 > 0.711}$, the fit is not so good as to be incredible.

Example 11

A coin is tossed 100 times and we observed 58 heads and 42 tails. Is the result consistent with the hypothesis that the coin is unbiased?

Solution

Table 4.3.5 Solution Table- Example 11

	Event	**Observed frequency (O)**	**Probability**	**Calculate Expected frequency (E)**	$\boldsymbol{\frac{(O-E)^2}{E}}$
	Heads	58	1/2	50	$\frac{(58-50)^2}{50} = \frac{8^2}{50} = \frac{64}{50} = 1.28$
	Tails	42	1/2	50	$\frac{(42-50)^2}{50} = \frac{-8^2}{50} = \frac{64}{50} = 1.28$
Total		**100**	**1**	**100**	**2.56**

Next compute $\boldsymbol{\chi^2}$ statistic

$\boldsymbol{H_0}$: coin is unbiased

$\boldsymbol{H_A}$: Biased.

$\boldsymbol{\chi^2 = \sum_{i=1}^{k}}$

d.f. $\boldsymbol{k-1, k}$ = number of events

$\boldsymbol{\alpha = 0.05}$

χ^2 value is 2.56

χ^2 table value with degrees of freedom 1 is 3.841

Computed value is less than the table value so,

$\Rightarrow$ Do not reject $\boldsymbol{H_0}$ $\Rightarrow$ Coin is a fair coin

Our measure of the relative discrepancy between the observed and expected frequencies indicates a good fit.

4.3.6 Test for Independent of Attributes

Another test that come under the chi-square test is independent of attributes.

Example 12

A random sample of 400 dwellings classified according to area and nature of occupancy.

Table 4.3.6 Sample of Dwellings

Nature of Occupancy	**Area**			**Marginal Frequency**
	Metropolitan	**Urban Provincial**	**Rural**	**Total**
Owner occupied	102 (111.8)	41 (43.1)	100 (88.1)	243
Tenanted	82 (72.2)	30 (27.9)	45 (56.9)	157
Total Marginal Frequency	184	71	145	400

- It is called a contingency table.
- $\mathbf{3 \times 2}$ table, horizontal classification containing 3 categories, the vertical one two categories.
- Frequencies belonging to each cell are called cell frequencies.

→ Now, we wish to use above data to test the

H_0 : Two classifications are independent.
H_A : They are not.

→ We first need to find the expected frequency for each of the six cells.

→ Joint probability of two statistically independent variables is equal to the product of their marginal probabilities.

→ If H_0 is true joint probability P (metropolitan and owner occupied $= P$ (metropolitan$\times$ Powner occupied).

Product of Marginal probability $\frac{Ri}{N} \cdot \frac{Cj}{N}$

[$i = 1,2$ - rows; $j = 1,2,3$ - columns]

Expected frequency Probability $\times N = \frac{Ri \cdot Cj}{N}$

$$f_{11} = \frac{R_1 \cdot C_1}{N} = \frac{(243) \times 184}{400} = 111.8$$

$$f_{12} = \frac{R_1 \cdot C_2}{N} = \frac{243 \times 71}{400} = 43.1 \text{ and so on.....}$$

Now calculate χ^2 for the above table. (Right tailed test always)

$$\chi^2 = \sum \frac{(0i - Ei)^2}{Ei}$$

The χ^2 table value for degree of freedom 2 and level of significance 5% is 5.991.

Table 4.3.7 Solution Table – Example 12

O	**E**	**O – E**	$(\mathbf{O - E})^2$	$\mathbf{X^2 = \frac{(O - E)^2}{E}}$
102	111.8	-9.8	96.04	0.86
41	43.1	-2.1	4.41	0.1
100	88.1	11.9	141.61	1.61
82	72.2	9.8	96.04	1.33
30	27.9	2.1	4.41	0.16
45	56.9	-11.9	141.61	2.49
				6.55

Since, $6.55 > 5.991 \Rightarrow$ Reject H_0

Conclusion: The sample suggests that area and nature of occupancy of dwellings are related in some way.

Summarised Overview

Hypothesis testing is a method used to explore unknown aspects in various situations. It can be applied to both the mean and proportion, depending on the nature of the data. Testing the mean is specifically focused on situations where there is a sample of 'n' independent observations drawn from a normal distribution. The sample follows a normal distribution with an unknown population mean (μ) and a known variance (σ^2). In cases with two possible outcomes (dichotomous population), the hypothesis test is conducted on the proportion of success (p). When exploring relationships between pairs of economic variables like income and expenditure, unemployment rates and inflation rates, or GDP growth and trade balance, the correlation coefficient is indeed a useful statistical measure.

The correlation coefficient quantifies the strength and direction of the relationship between two variables. If the correlation coefficient is close to +1, it indicates a strong positive correlation, while a value close to -1 suggests a strong negative correlation. A correlation coefficient near 0 implies a weak or no linear correlation. When testing associations in categorical data, particularly when dealing with events labelled as "success" and "failure," the chi-square distribution is commonly used. The chi-square test is a statistical test that assesses the independence or association between two categorical variables.

Assignments

1. It has been found by experience that the mean breaking strength of a particular brand of thread is 10 units with a standard deviation of 1.5 units. A random sample of 49 pieces of thread showed a mean breaking strength of 9.1 units. Can we conclude at a LoS of 0.05 that the thread has become inferior.
2. A national survey of 1200 adults found that 450 of those surveyed were pro-choice on the abortion issue. Find a 95% confidence interval for p, the proportion of all adults who are pro choice on the abortion issue.

3. Calculate the linear correlation coefficient for the following data. X = 4, 8, 12, 16 and Y = 5, 10, 15, 20.

4. Number of orders received by a manufacturer in a weeks time is given below. Conduct the Chi-square Test.

Days of Week	Number of Orders
Monday	7
Tuesday	12
Wednesday	15
Thursday	11
Friday	15
	60

Suggested Reading

1. Anderson, D., D. Sweeney and T. Williams (2013): "*Statistics for Business and Economics*", Cengage Learning: New Delhi.
2. Goon, A.M., Gupta and Das Gupta B (2002): *Fundamentals of Statistics* (Vol I), World Press.

Reference

1. Webster, A. (2010). *Applied Statistics for Business and Economics*. Tata McGraw Hill Education Private Limited, New Delhi.
2. Ross, S. M. (2017). *Introductory Statistics*. Academic Press.
3. Mayes, Anne C., and David G. Mayes (1976). "*Introductory Economic statistics*."

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

UNIT 4
Non- Parametric Tests

Learning Outcomes

After completing this unit, learner will be able to:

- understand nonparametric tests for hypothesis testing
- apply non parametric tests to different scenarios
- know about the concepts of analysis of variance

Background

Parametric tests, such as t-tests and ANOVA, are valuable tools in statistical analysis when specific assumptions about the distribution and characteristics of the data can be met. However, real-world data often deviate from these assumptions, challenging the toughness of parametric methods. In such scenarios, nonparametric tests emerge as a powerful alternative. Nonparametric tests are useful in statistics, especially when there are uncertainties about the data's distribution or when dealing with small sample sizes. They provide a strong and adaptable alternative in situations where parametric assumptions cannot be confidently made. However, when working with known distributions or larger sample sizes, they may be less efficient than parametric tests. The major non-parametric tests are the Sign Test, Wilcoxon-Mann Whitney U Test, Signed Rank Test, Kruskal Wallis Test, Wald-Wolfowitz Test, and Analysis of Variance.

Keywords

Sign Test, Wilcoxon- Mann Whitney U Test, Signed Rank Test, Kruskal Wallis Test, Wald-Wolfowitz Test, Analysis of Variance

Discussion

4.4.1 Non Parametric Tests

- For making meaningful statistical inferences

Nonparametric tests become essential tools for making meaningful statistical inferences when the distribution is unknown. These tests allow economists to assess relationships, differences, or trends within the data without relying on assumptions about the underlying distribution or specific measurement scales. By providing valid alternatives to parametric tests, nonparametric methods enable economists to derive valuable insights and draw conclusions from diverse economic contexts where traditional assumptions might not hold.

4.4.2 Sign Test

- Sign test is the counterpart to the t-test for matched pairs

A nonparametric test commonly used to make business decisions is the sign test. It is most often used to test the hypothesis comparing two population distributions, and often involves the use of matched pairs.

Example 1

A market analyst wants to measure the effectiveness of a promotional game of a firm's product. Prior to the promotional game, he selects 12 retail outlets and records sales for the month, rounded to the nearest ₹100. During the second month, the promotional game is implemented and again record sales. Table 1 displays these sales levels, along with the algebraic sign that results when sales in the second month are subtracted from those in the first month. A plus sign recorded in the last column means that sales went down during the second month.

Table 4.4.1 Sales for Twelve Retail Stores

Store	Before the Game (₹)	During the Game (₹)	Sign
1	42	40	+
2	57	60	-
3	38	38	0
4	49	47	+
5	63	65	-
6	36	39	-
7	48	49	-

8	58	50	+
9	47	47	0
10	51	52	-
11	83	72	+
12	27	33	-

The analyst wants to test at the 5 percent level the hypothesis that the promotion increased sales. If sales went up in the second month when the promotion was in effect, subtracting those sales from sales in the first month would produce minus signs. You would then expect the number of minus signs, *m*, to exceed the number of plus signs, *p*. That is, *m* > *p*. This statement does not contain an equal sign and is, therefore, the alternative hypothesis, producing a right-tailed test:

$H_0: m \leq p$

$H_1: m > p$

Table 1 shows 6 minus signs and 4 plus signs for a total of *n* = 10 signs. Values resulting in a zero difference are ignored. Observations 3 and 9 are therefore dropped from consideration. We must then determine the probability of six or more minus signs or four or fewer plus signs if the probabilities of both are $\boldsymbol{\pi} = 0.50$. If this probability is less than the chosen α-value, the sample results are significant and the null hypothesis is rejected.

However, if the probability of the sample results is greater than α, the results can be attributed to sampling error; do not reject the null. That is, if the sample results actually observed are likely to occur, they are not interpreted as being a significant finding and the null cannot be rejected.

From the table, the probability of six or more minus signs is

$P(m \geq 6, n = 10, \pi = 0.5) = 1 - P(X \leq 5)$

$= 1 - 0.6230$

$= 0.3770$

Of course, if you obtained six or more minus signs, you must have obtained four or fewer plus signs. Therefore, the probability of four or fewer plus signs is also 0.3770:

$P(p \leq 4, n = 10, \pi = 0.5) = 0.3770$

This value of 0.3770 is the probability of obtaining six or more minus signs (or four or fewer plus signs) if π, the probability of the occurrence of either sign on any trial, is 0.50. We noted that if the number of minus signs was unusually large, it would refute the null. However, 6 is not an unusually large number. The probability of getting six or more signs is quite high at 37.7 percent. Since the probability of their occurrence is greater than 5 percent, the event of six minus signs is not considered large, and the null that $H_0: m \geq p$ is not rejected.

If the promotion were effective, there would be a large number of minus signs, and the null that $\boldsymbol{m \leq p}$ would be rejected. But as we have seen, six minus signs is not an unusually large number, and you cannot consider the promotion successful.

4.4.3 Wilcoxon-Mann Whitney U Test

- Wilcoxon-Mann Whitney U Test is the nonparametric counterpart to the t-test for independent

The Wilcoxon-Mann-Whitney *U* test (or simply the *U* test also known as Wilcoxon rank sum test) tests the equality of two population distributions. It is based on the assumption that two random samples are independently drawn from continuous variables. In its broadest sense, the null hypothesis states that the distributions of two populations are identical. However, the test can be tailored to examine the equality of two population means or medians. To test the equality of means, we must assume that the populations are symmetrical and have the same variance. Under these conditions the Mann-Whitney U test serves as the nonparametric alternative to the *t*-test, except it does not require the assumption of normality. If the assumption of symmetry is dropped, the median replaces the mean as the test statistic.

The data are ordered or ranked from lowest to highest. There is no effort to match pairs, as we have often done when two samples were taken.

Example 2

A pottery factory wants to compare the time it takes for clay pieces to cool after being "fired" in the oven by two different firing methods. Potters fire 12 pieces using method 1, and 10 using method 2. The number of minutes required for each piece to cool is as follows:

Table 4.4.2 Time Taken

Method 1	Method 2
27*	34
31	24*
28	38
29	28
39	30
40	34
35	37
33	42
32	41
36	44
37	
43	

The observations are then ordered and ranked from lowest to highest as shown in Table 2. The value 24 in method 2 is the lowest of all 22 observations and is given the rank of 1, and 27 in method 1 has a rank of 2. Ties, such as 28, are averaged over the appropriate ranks. The value 28 is the third lowest observation, and both values of 28 receive a ranking of 3.5. There is no rank of 4, since two observations have the rank of 3.5. The rankings are then summed, yielding $\sum R_1$ and $\sum R_2$.

Table 4.4.3 Ranking Cooling Times

Method 1	Rank	Method 2	Rank
		24	1
27	2		
28	3.5	28	3.5
29	5		
		30	6
31	7		
32	8		

33	9		
		34	10.5
		34	10.5
35	12		
36	13		
37	14.5	37	14.5
		38	16
39	17		
40	18		
		41	19
		42	20
43	21		
		44	22
$\sum \boldsymbol{R_1} = \mathbf{130}$		$\sum \boldsymbol{R_2} = \mathbf{123}$	

We calculate the Mann-Whitney *U*-statistic for each sample from the equations as

Mann-Whitney U-statistic for first sample,

$$U_1 = n_1 n_2 + \frac{n_1(n_1+1)}{2} - \sum R_1$$

Mann-Whitney U-statistic for second sample,

$$U_2 = n_1 n_2 + \frac{n_2(n_2+1)}{2} - \sum R_2$$

$$U_1 = (12)(10) + \frac{12(12+1)}{2} - 130$$

$= 68$

and $U_2 = (12)(10) + \frac{10(10+1)}{2} - 123$

$= 52$

Notice that $U_1 + U_2 = n_1 n_2$ provides a quick check of your arithmetic.

If n_1 and n_2 are both at least 10, the mean and standard deviation of the sampling distribution for the *U*-statistic are

Mean of the sampling distribution for Mann-Whitney *U* test,

$$\mu_u = \frac{n_1 n_2}{2}$$

and Standard deviation of the sampling distribution for the Mann-Whitney *U* test,

$$\sigma_u = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}}$$

In the present case, we find

$$\mu_u = \frac{(12)(10)}{2} = 60$$

and

$$\sigma_u = \sqrt{\frac{(12)(10)(12 + 10 + 1)}{12}}$$

$$= 15.17$$

The distribution of the *U*-statistic can then be normalised by the formula.

Z-value to normalise the Mann-Whitney U test, $Z = \frac{U_i - \mu_u}{\sigma_u}$ where U_i is the appropriate *U*-value, either U_1 or U_2, depending on the nature of the test.

Let us now determine which *U*-value is appropriate.

In the example, the pottery factory may want to test the hypothesis that the mean cooling times of method 1 and method 2 are the same. This requires a two-tailed test with hypotheses

$H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

In a two-tailed test, either U_1 or U_2 can be used in Formula [14.12]. Thus, arbitrarily using U_2, we find

$$Z = \frac{52 - 60}{15.17}$$

-0.53

If α = 10 percent, the decision rule, as reflected in Figure is

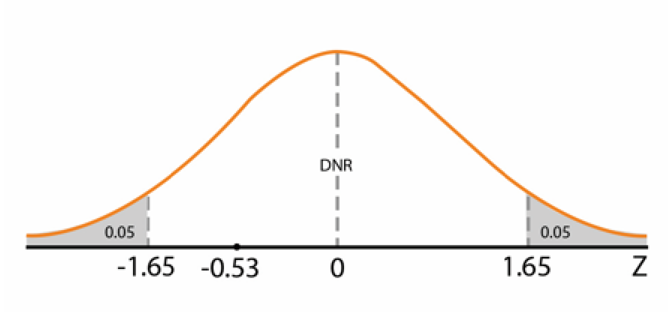


Fig 4.4.1 Two- Tailed Test of Mean Cooling Times

Decision Rule Do not reject if $-1.65 \leq Z \leq 1.65$. Reject if $Z < -1.65$ or $Z > 1.65$.

4.4.4 Signed Rank Test

- Operates based on the relative magnitudes of data pairs

The signed rank test operates based on the relative magnitudes of data pairs rather than their actual values. It's important to note that this test does not require precise data values, only information about which value is larger in each pair. While the signed rank test is straightforward to use, it may not be the most efficient method for testing the null hypothesis that the population distributions are identical. If this null hypothesis holds true, the distribution of paired differences not only has a median of zero but also exhibits a stronger property of symmetry around zero.

Specifically, for any given number x, the likelihood of the first value in the pair being larger than the second by an amount x is the same as the likelihood of the second value being larger than the first by the same amount. The signed rank test focuses solely on verifying whether the median value of the distribution of differences is zero, rather than assessing the symmetry of the distribution.

Example 3

A psychology instructor wanted to see if students would perform equally well on two different examinations. He selected 12 students, who all agreed to take part in the experiment. Six of the students were given examination A. and the other six examination B. On the next day the students were tested on the examination they had not yet taken. Thus, each of the 12 students took both examinations. The following pairs of scores were obtained by the students on the two examinations:

Table 4.4.4 Scores Obtained

Examination	A	B
1	763	797
2	419	404
3	586	576
4	920	855
5	881	762
6	758	707
7	262	195
8	332	341
9	717	728
10	909	817
11	940	947
12	835	849

Thus, for instance, student 3 scored 586 on examination A and 576 on test B. The paired differences are as follows:

−34, 15, 10, 65, 119, 51, 67, −9, −11, 92, −7, −14

Ordering these in increasing order of their absolute values gives

−7, −9, 10, −11, −14, 15, −34, 51, 65, 67, 92, 119

Since the differences in positions 1,2, 4, 5, and 7 are negative, the values of the test statistic is

$TS = 1 + 2 + 4 + 5 + 7 = 19$

To obtain the p value, which computes the *p* value for the signed-rank test that a population distribution is symmetric about 0. Our sample size is 12 and our observed value of the sum of signed ranks is 19. The *p* value is 0.1293945.

Thus, the *p* value is 0.129, and so the null hypothesis that the distributions of student scores on the two examinations are identical cannot be rejected at the 10 percent level of significance.

4.4.5 Kruskal Wallis Test

- Compares three or more populations to determine whether a difference exists in the distribution of the populations

If we need to compare more than two populations, the Kruskal-Wallis test will be used. The test is applied to hypotheses regarding the distribution of three or more populations. In this capacity, the Kruskal-Wallis test functions as the nonparametric counterpart to the completely randomized design used in ANOVA tests.

The null hypothesis states that no difference exists in the distribution of the *k* populations under comparison. The hypotheses are thus

H_0: All *k* populations have the same distribution.

H_A: Not all *k* populations have the same distribution.

The Kruskal-Wallis statistic is

$$K = \frac{12}{n(n+1)}\left[\sum \frac{R_i^2}{n_i}\right] - 3(n+1)$$

where n_i is the number of observations in the i^{th} sample
n is the total number of observations in all samples
R_i is the sum of the ranks of the i^{th} sample

Example 4

Consider the three customers whose time it takes to settle their accounts with a company.

Table 4.4.5 The Days Taken and Rankings

Customer 1			Customer 2			Customer 3	
Days	Rank		Days	Rank		Days	Rank
			11	1			
13	2						
			14	3			
19	4						
			20	5			
21	6.5		21	6.5			
22	8.5		22	8.5			
			26	10.5		26	10.5
28	13						
28	13					28	13

29	15						
						31	16
						35	17
						37	18
$\sum R_1 = \overline{62}$			$\sum R_2 = \overline{34.5}$			$\sum R_3 = \overline{74.5}$	

K is then found, using Formula, as

$$K = \frac{12}{18(18+1)}\left[\frac{(62)^2}{7} + \frac{(34.5)^2}{6} + \frac{(74.5)^2}{5}\right] - 3(18+1)$$

= 8.18

It is now left to compare *K* with a critical value. The distribution of *K* is approximated by a chi-square distribution with $k-1$ degrees of freedom. If *K* exceeds the critical value for chi-square, the null is rejected. We choose an α-value of 5 percent, the critical chi-square value, given 3 − 1 = 2 degrees of freedom, becomes $X^2_{0.05,2} = 5.99$.

Decision Rule Do not reject if $k \leq 5.99$. Reject if $k > 5.99$.

Since $k = 8.18 > 5.99$, we reject the null that there is no difference in the time it takes these three customers to settle their accounts with the company.
In the event that the null hypothesis is rejected, the next logical step is to determine which differences are statistically significant and which are due to sampling error.
We must first compute the average rank for each sample $\overline{R}_i$ by dividing the sum of the sample's rank by the number of observations in that sample. For the first sample this is

$$\overline{R}_i = \frac{62}{7} = 8.86$$

Similarity, $\overline{R}_2 = \frac{34.5}{6} = 5.75$ and $\overline{R}_3 = \frac{74.5}{5} = 14.9$. The absolute differences are found as

$$|\overline{R}_1 - \overline{R}_2| = |8.86 - 5.75| = 3.11$$

$$|\overline{R}_1 - \overline{R}_3| = |8.86 - 14.9| = 6.04$$

$$\left|\overline{R}_2 - \overline{R}_3\right| = |5.75 - 14.9| = 9.15$$

These absolute values are then compared to a critical value to determine whether they differ significantly. This critical value $\boldsymbol{C_k}$ is calculated as

Critical value for the Kruskal-Wallis test,

$$C_k = \sqrt{X^2_{\alpha,k-1}\left[\frac{n(n+1)}{12}\right]\left[\frac{1}{n_i} + \frac{1}{n_j}\right]}$$

where $\boldsymbol{X^2_{\alpha,k-1}}$ is the chi-square value used to test the original hypothesis, $\boldsymbol{n_i}$ and $\boldsymbol{n_j}$ are the sizes of the two samples under comparison.

If the actual difference between the average ranks of the two samples is greater than the critical difference, it is considered a significant difference and the two populations are found to be different.

If we have a balanced design with equal size samples, $\boldsymbol{C_k}$ will be the same for all pair- wise comparisons. If the design is unbalanced as in this case, a different value for $\boldsymbol{C_k}$ must be computed for each comparison. To compare customer 1 and customer 2 for Pox, $\boldsymbol{C_k}$ is

$$C_k = \sqrt{5.99\left[\frac{(18)(19)}{12}\right]\left[\frac{1}{7} + \frac{1}{6}\right]}$$

$$= 7.27$$

Since $\left|\overline{R}_1 - \overline{R}_2\right| = |8.86 - 5.75| = 3.11$ is less than 7.27, populations 1 and 2 do not differ. In a similar manner, C_k for populations 1 and 3 is 7.65. Since the observed absolute difference between 1 and 3 is 6.04 < 7.65, these two populations are not different. Finally, C_k for populations 2 and 3 is 7.91 The actual difference between 2 and 3 is 9.15 > 7.91. and populations 2 and 3 are said to be different.

Common underscoring can be used to summarize based on the average ranks.

$\overline{R}_1$	$\overline{R}_2$	$\overline{R}_3$
5.75	8.86	14.9

4.4.6 Wald-Wolfowitz Test

- Used to asses the distribution of the data is unknown

The Wald-Wolfowitz run test is a nonparametric method utilized when parametric tests are not applicable. This test involves obtaining two distinct random samples from populations with different continuous cumulative distribution functions. The null hypothesis assumes no significant statistical distinction between the two cumulative distribution functions, implying that the populations are identical. The observations from the samples are ranked and coded, forming runs that are summed up to create the test statistic. Small values suggest different populations, while large values indicate identical populations.

Example 5

Data on value of imports of selected agricultural production inputs by a country in million rupees during recent 12 years is given below. Is the sequence random?

Table 4.4.6 Value of Imports

5.2	
5.5	+
3.8	-
2.5	-
8.3	+
2.1	-
1.7	-
10	+
10	0
6.9	-
7.5	+
10.6	+

The hypothesis are

H_0 : the Sequence is random

H_1 : the Sequence is non random

Here n = 11, the number of runs r = 7, Critical value for α = 0.05 from the table are r_{cl} (lower value) = 4 and r_{cu} (upper value) = 10.

Here $r_{cl} \leq r \leq r_{cu}$. i.e., observed r lies between 4 and 10. So H_0 is accepted and conclude that the sequence is random

4.4.7 Analysis of Variance (ANOVA)

- methodology for comparing data involving multiple means among distinct groups

Analysis of variance (ANOVA) is a methodology for comparing data involving multiple means among distinct groups. It provides insights into patterns and trends within intricate and diverse datasets. We assumed that the two populations from which the samples were drawn had the same variance. In many situations there is a need to test the significance of differences among three or more sampling means, or equivalently to test the null hypothesis that the sample means are all equal.

The steps to perform the one way ANOVA test are given below:

- Calculate the mean for each group.
- Calculate the average mean. This is done by adding all the means and ividing it by the total number of means.
- Calculate the SSB.
- Calculate the between groups degrees of freedom.
- Calculate the SSE.
- Calculate the degrees of freedom of errors.
- Determine the MSB and the MSE.
- Find the f test statistic.
- Using the f table for the specified level of significance, α, find the critical value. This is given by F(α, df_1. df_2).
- If f > F then reject the null hypothesis.

Example 6

Using the following data, perform a one way analysis of variance using α = .05.

Group1	Group2	Group3
51	**23**	**56**
45	**43**	**76**
33	**23**	**74**
45	**43**	**87**
67	**45**	**56**

Solution

Table 4.4.7 Group I

Value	Mean	Deviations	Sq. deviation
51	48.2	2.8	7.84
45	48.2	-3.2	10.24
33	48.2	-15.2	231.04
45	48.2	-3.2	10.24
67	48.2	18.8	353.44
			612.8

Table 4.4.8 Group II

Value	Mean	Deviations	Sq. deviation
23	35.4	-12.4	153.76
43	35.4	7.6	57.76
23	35.4	-12.4	153.76
43	35.4	7.6	57.76
45	35.4	9.6	92.16
			515.2

Table 4.4.9 Group III

Value	Mean	Deviations	Sq. deviation
56	69.8	-13.8	190.44
76	69.8	6.2	38.44
74	69.8	4.2	17.64
87	69.8	17.2	295.84
56	69.8	-13.8	190.44
			732.8

$H_0: \mu_1 = \mu_2$

$H_0: \mu_1 \neq \mu_2$

Sample means (x) for the groups:

48.2, 35.4, 69.8

Total Mean = 153.4

Average Mean $= \frac{153.4}{3} = 51.13$

$$\text{SSB} = \sum n_j(\bar{X} - \bar{x})^2 = 5 \times (48.2 - 51.13)^2 + 5 \times (35.4 - 51.13)^2 + 5 \times (69.8 - 51.13)^2$$

$$= 3022.9335$$

$$\text{SSE} = \sum\sum(X - \bar{X})^2 = 612.8 + 515.2 + 732.8 = 1860.8$$

$$MSB = \frac{SSB}{k-1} = \frac{3022.9335}{3-1} = 1511.47$$

$$MSE = \frac{SSE}{k-1} = \frac{1860.8}{15-3} = 155$$

$$F = \frac{MSB}{MSE} = \frac{1511.47}{155} = 9.75$$

Table 4.4.10 Anova Table

Source of Variation	**SS**	**Df**	**MS**	**F**
Group	3022.9335	2	1511.47	9.75
Error	1860.8	12	155	
Total				

$F_{\text{critical}}\,(2{,}12) = 3.89$

$F_{\text{critical}}\,(2{,}12) < F$

Reject H_0

Summarised Overview

Nonparametric tests are recognized as essential and valuable tools in statistics, particularly when dealing with unknown or non-normally distributed data. Major nonparametric tests include the Sign test, Wilcoxon-Mann-Whitney U Test, Signed Rank test, Kruskal-Walli's test, and Wald-Wolfowitz test. In statistical analysis, various nonparametric tests offer valuable alternatives to traditional parametric methods, especially in scenarios where data may not meet stringent distributional assumptions. The sign test, a straightforward yet powerful tool, is commonly employed in business contexts to evaluate significant differences between two population distributions. Similarly, the Wilcoxon-Mann-Whitney U test focuses on testing the equality of two populations, offering robustness when parametric assumptions are not applicable. The signed rank test steps in by assessing the relative magnitudes of data pairs, providing insights into whether one value in a pair is consistently larger than the other.

Moving beyond pairwise comparisons, the Kruskal-Wallis test becomes essential when dealing with three or more populations, offering a nonparametric alternative for assessing differences among multiple groups. Its versatility makes it particularly suitable for complex scenarios where parametric ANOVA tests might fall short. Analysis of Variance (ANOVA), a well-established methodology, proves useful for comparing data involving multiple means across distinct groups. It serves as a powerful tool in situations demanding an exploration of variations and differences among various groups, providing insights into the underlying patterns and trends within diverse datasets.

Assignments

1. A type company tested the wear resistance of two types of tyre tread on its motorbike. Ten bikes were randomly chosen. Mechanics mounted tyres with one type of tread on the front, and the other tread on the rear. After driving the bikes a specified number of miles under set conditions, they gave a wear rating between 0 and 40 to each tyre. A higher rating indicated a better tyre. The results are shown here. Honda's research analysts want to test the hypothesis that there is no difference in wear ratings at the 10 percent level.

Tyres	Wear Rating	
	Tread Type I	Tread Type II
1	32	37
2	27	25
3	21	21
4	13	17
5	25	29
6	38	39
7	17	23
8	29	33
9	32	34
10	34	37

2. An education researcher is comparing four different algebra curricula. Eighth grade students are randomly assigned to one of the four groups. Their state achievement test scores are compared at the end of the year. Use the appropriate statistical procedure to determine whether the curricula differ with respect to math achievement. An alpha criterion of 0.05 should be used for the test.

Curriculum	Sample Size	Sample Mean	Sample Standard Deviation
1	50	170.5	14.5
2	50	158.3	12.8
3	65	167.6	17.7
4	50	162.8	16.8

Suggested Reading

1. Anderson, D., D. Sweeney and T. Williams (2013): "*Statistics for Business and Economics*", Cengage Learning: New Delhi.
2. Goon, A.M., Gupta and Das Gupta B (2002): *Fundamentals of Statistics* (Vol I), World Press.

Reference

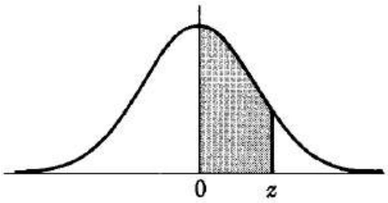
1. Webster, A. (2010). *Applied Statistics for Business and Economics*. Tata McGraw Hill Education Private Limited, New Delhi.
2. Ross, S. M. (2017). *Introductory Statistics*. Academic Press.
3. Mayes, Anne C., and David G. Mayes (1976). "*Introductory Economic Statistics*."

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

Appendix I

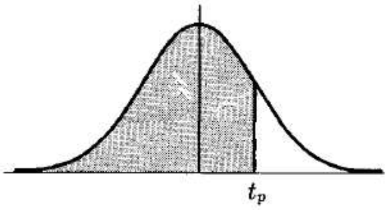
Areas Under the Standard Normal Curve from 0 to *z*



z	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

Appendix II

Percentile Values (t_p)
for
Student's t Distribution
with ν Degrees of Freedom
(shaded area = p)

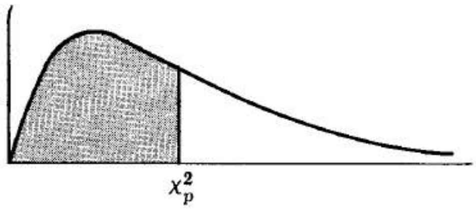


ν	$t_{.995}$	$t_{.99}$	$t_{.975}$	$t_{.95}$	$t_{.90}$	$t_{.80}$	$t_{.75}$	$t_{.70}$	$t_{.60}$	$t_{.55}$
1	63.66	31.82	12.71	6.31	3.08	1.376	1.000	.727	.325	.158
2	9.92	6.96	4.30	2.92	1.89	1.061	.816	.617	.289	.142
3	5.84	4.54	3.18	2.35	1.64	.978	.765	.584	.277	.137
4	4.60	3.75	2.78	2.13	1.53	.941	.741	.569	.271	.134
5	4.03	3.36	2.57	2.02	1.48	.920	.727	.559	.267	.132
6	3.71	3.14	2.45	1.94	1.44	.906	.718	.553	.265	.131
7	3.50	3.00	2.36	1.90	1.42	.896	.711	.549	.263	.130
8	3.36	2.90	2.31	1.86	1.40	.889	.706	.546	.262	.130
9	3.25	2.82	2.26	1.83	1.38	.883	.703	.543	.261	.129
10	3.17	2.76	2.23	1.81	1.37	.879	.700	.542	.260	.129
11	3.11	2.72	2.20	1.80	1.36	.876	.697	.540	.260	.129
12	3.06	2.68	2.18	1.78	1.36	.873	.695	.539	.259	.128
13	3.01	2.65	2.16	1.77	1.35	.870	.694	.538	.259	.128
14	2.98	2.62	2.14	1.76	1.34	.868	.692	.537	.258	.128
15	2.95	2.60	2.13	1.75	1.34	.866	.691	.536	.258	.128
16	2.92	2.58	2.12	1.75	1.34	.865	.690	.535	.258	.128
17	2.90	2.57	2.11	1.74	1.33	.863	.689	.534	.257	.128
18	2.88	2.55	2.10	1.73	1.33	.862	.688	.534	.257	.127
19	2.86	2.54	2.09	1.73	1.33	.861	.688	.533	.257	.127
20	2.84	2.53	2.09	1.72	1.32	.860	.687	.533	.257	.127
21	2.83	2.52	2.08	1.72	1.32	.859	.686	.532	.257	.127
22	2.82	2.51	2.07	1.72	1.32	.858	.686	.532	.256	.127
23	2.81	2.50	2.07	1.71	1.32	.858	.685	.532	.256	.127
24	2.80	2.49	2.06	1.71	1.32	.857	.685	.531	.256	.127
25	2.79	2.48	2.06	1.71	1.32	.856	.684	.531	.256	.127
26	2.78	2.48	2.06	1.71	1.32	.856	.684	.531	.256	.127
27	2.77	2.47	2.05	1.70	1.31	.855	.684	.531	.256	.127
28	2.76	2.47	2.05	1.70	1.31	.855	.683	.530	.256	.127
29	2.76	2.46	2.04	1.70	1.31	.854	.683	.530	.256	.127
30	2.75	2.46	2.04	1.70	1.31	.854	.683	.530	.256	.127
40	2.70	2.42	2.02	1.68	1.30	.851	.681	.529	.255	.126
60	2.66	2.39	2.00	1.67	1.30	.848	.679	.527	.254	.126
120	2.62	2.36	1.98	1.66	1.29	.845	.677	.526	.254	.126
∞	2.58	2.33	1.96	1.645	1.28	.842	.674	.524	.253	.126

Source: R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research* (5th edition), Table III, Oliver and Boyd Ltd., Edinburgh,

Appendix III

Percentile Values (χ_p^2)
for
the Chi-Square Distribution
with ν Degrees of Freedom
(shaded area = p)



ν	$\chi^2_{.995}$	$\chi^2_{.99}$	$\chi^2_{.975}$	$\chi^2_{.95}$	$\chi^2_{.90}$	$\chi^2_{.75}$	$\chi^2_{.50}$	$\chi^2_{.25}$	$\chi^2_{.10}$	$\chi^2_{.05}$	$\chi^2_{.025}$	$\chi^2_{.01}$	$\chi^2_{.005}$
1	7.88	6.63	5.02	3.84	2.71	1.32	.455	.102	.0158	.0039	.0010	.0002	.0000
2	10.6	9.21	7.38	5.99	4.61	2.77	1.39	.575	.211	.103	.0506	.0201	.0100
3	12.8	11.3	9.35	7.81	6.25	4.11	2.37	1.21	.584	.352	.216	.115	.072
4	14.9	13.3	11.1	9.49	7.78	5.39	3.36	1.92	1.06	.711	.484	.297	.207
5	16.7	15.1	12.8	11.1	9.24	6.63	4.35	2.67	1.61	1.15	.831	.554	.412
6	18.5	16.8	14.4	12.6	10.6	7.84	5.35	3.45	2.20	1.64	1.24	.872	.676
7	20.3	18.5	16.0	14.1	12.0	9.04	6.35	4.25	2.83	2.17	1.69	1.24	.989
8	22.0	20.1	17.5	15.5	13.4	10.2	7.34	5.07	3.49	2.73	2.18	1.65	1.34
9	23.6	21.7	19.0	16.9	14.7	11.4	8.34	5.90	4.17	3.33	2.70	2.09	1.73
10	25.2	23.2	20.5	18.3	16.0	12.5	9.34	6.74	4.87	3.94	3.25	2.56	2.16
11	26.8	24.7	21.9	19.7	17.3	13.7	10.3	7.58	5.58	4.57	3.82	3.05	2.60
12	28.3	26.2	23.3	21.0	18.5	14.8	11.3	8.44	6.30	5.23	4.40	3.57	3.07
13	29.8	27.7	24.7	22.4	19.8	16.0	12.3	9.30	7.04	5.89	5.01	4.11	3.57
14	31.3	29.1	26.1	23.7	21.1	17.1	13.3	10.2	7.79	6.57	5.63	4.66	4.07
15	32.8	30.6	27.5	25.0	22.3	18.2	14.3	11.0	8.55	7.26	6.26	5.23	4.60
16	34.3	32.0	28.8	26.3	23.5	19.4	15.3	11.9	9.31	7.96	6.91	5.81	5.14
17	35.7	33.4	30.2	27.6	24.8	20.5	16.3	12.8	10.1	8.67	7.56	6.41	5.70
18	37.2	34.8	31.5	28.9	26.0	21.6	17.3	13.7	10.9	9.39	8.23	7.01	6.26
19	38.6	36.2	32.9	30.1	27.2	22.7	18.3	14.6	11.7	10.1	8.91	7.63	6.84
20	40.0	37.6	34.2	31.4	28.4	23.8	19.3	15.5	12.4	10.9	9.59	8.26	7.43
21	41.4	38.9	35.5	32.7	29.6	24.9	20.3	16.3	13.2	11.6	10.3	8.90	8.03
22	42.8	40.3	36.8	33.9	30.8	26.0	21.3	17.2	14.0	12.3	11.0	9.54	8.64
23	44.2	41.6	38.1	35.2	32.0	27.1	22.3	18.1	14.8	13.1	11.7	10.2	9.26
24	45.6	43.0	39.4	36.4	33.2	28.2	23.3	19.0	15.7	13.8	12.4	10.9	9.89
25	46.9	44.3	40.6	37.7	34.4	29.3	24.3	19.9	16.5	14.6	13.1	11.5	10.5
26	48.3	45.6	41.9	38.9	35.6	30.4	25.3	20.8	17.3	15.4	13.8	12.2	11.2
27	49.6	47.0	43.2	40.1	36.7	31.5	26.3	21.7	18.1	16.2	14.6	12.9	11.8
28	51.0	48.3	44.5	41.3	37.9	32.6	27.3	22.7	18.9	16.9	15.3	13.6	12.5
29	52.3	49.6	45.7	42.6	39.1	33.7	28.3	23.6	19.8	17.7	16.0	14.3	13.1
30	53.7	50.9	47.0	43.8	40.3	34.8	29.3	24.5	20.6	18.5	16.8	15.0	13.8
40	66.8	63.7	59.3	55.8	51.8	46.6	39.3	33.7	29.1	26.5	24.4	22.2	20.7
50	79.5	76.2	71.4	67.5	63.2	56.3	49.3	42.9	37.7	34.8	32.4	29.7	28.0
60	92.0	88.4	83.3	79.1	74.4	67.0	59.3	52.3	46.5	43.2	40.5	37.5	35.5
70	104.2	100.4	95.0	90.5	85.5	77.6	69.3	61.7	55.3	51.7	48.8	45.4	43.3
80	116.3	112.3	106.6	101.9	96.6	88.1	79.3	71.1	64.3	60.4	57.2	53.5	51.2
90	128.3	124.1	118.1	113.1	107.6	98.6	89.3	80.6	73.3	69.1	65.6	61.8	59.2
100	140.2	135.8	129.6	124.3	118.5	109.1	99.3	90.1	82.4	77.9	74.2	70.1	67.3

Source: Catherine M. Thompson, *Table of percentage points of the χ^2 distribution, Biometrika, Vol. 32 (1941)*

MODEL QUESTION PAPER I

SREENARAYANAGURU OPEN UNIVERSITY

QP CODE: ……… **Reg. No : ………...............**

Name : ………………...

FIRST SEMESTER - MA ECONOMICS EXAMINATION

DISCIPLINE CORE – 04 - M23EC04DC

QUANTITATIVE METHODS FOR ECONOMICS I

2023-24 - Admission Onwards

Time: 3 Hours **Max. Marks: 70**

Section A - Objective Type Questions

Answer any 10 questions. Each question carries 1 mark (10 X 1=10 Marks)

1. Define the term hypothesis.
2. What is the primary criterion for a well-defined set?
3. Determine the first derivative of $f(y) = 4\sqrt[5]{y^3} - \frac{1}{8y^2} - \sqrt{y}$
4. What type of function does not express the dependent variable explicitly in terms of the independent variable?
5. Solve $\int (x^e + e^x + e^e)\, dx$
6. Define is a critical region.
7. Who is the Italian mathematician credited with the development of the Lagrange Multipliers method?
8. What is sample variance?
9. Optimise $f(t) = t^3 - 8t^2 + 12$
10. What principle in economics involves individuals optimising their well-being within the confines of limited resources?
11. What is the equation for the variance of the sampling distribution of means, denoted by $(\sigma_{\bar{x}}^2)$, if a population is infinite or if sampling is with replacement?
12. Define optimisation.
13. What is Z distribution?

14. Express the mathematical form for measuring Pearson's correlation coefficient.
15. What is objective function in an LPP?

Section B - Very Short Questions

Answer any 5 questions. Each question carries 2 marks (5X2=10 Marks)

16. Evaluate $\lim_{x\to 8} \frac{2x^2-17x+8}{8-z}$
17. Explain the uses of F test.
18. What information do higher-order derivatives provide about the behaviour of economic models?
19. What is the primary goal of optimisation in economics?
20. The demand equation for a certain product is $p = 6 - \frac{1}{2}x$ rupees. Find the level of production that results in maximum revenue.
21. The cost function of a company is $C = x^2 - 8x + 56$, where *C* is cost per unit and *x*, the number of units produced. Find the minimum value of the cost and the corresponding number of units to be produced.
22. Given $Q = 3000 - (3 - y)^2$. When is Q maximum? What is the maximum value of Q?
23. Find $\int \frac{8}{x} + \frac{5}{x^2} + \frac{6}{x^3} dx$
24. What is the rejection region?
25. Find the second order derivative of $f(y) = (4y^2 - y)(y^2 + 3y^2 + 2)$

Section C - Short Answer

Answer any 5 questions. Each question carries 4 marks. (5X4=20 Marks)

26. Explain the significance of marginal concepts in economics. Provide examples of important marginal concepts related to demand and supply.
27. Distinguish between null and Alternative hypothesis.
28. Solve the following
    a) $\int \sqrt{y} + \frac{1}{3\sqrt{y}}\ dy$
    b) $\int 9e^{\frac{x}{4}} dx$
    c) $\int (2x^2 - 1)^2 dx$
    d) $\int 12y^{\frac{3}{4}} - 9y^{\frac{5}{3}} dy$

29. Explain the central limit theorem.
30. Explain the process of optimisation. Explain revenue maximisation, cost minimisation and profit maximisation?
31. An economist is interested in studying the incomes of consumers in a particular country. The population standard deviation is known to be Rs.6,000. A random sample of 50 individuals resulted in a mean income of Rs. 5,0000. What is the width of the
    a. 90% confidence interval b. 95% confidence interval
32. Find the maximum and minimum values of $f(x,y) = 81x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$.
33. The following table gives the classification of 100 workers according to gender and nature of work. Test whether nature of work is independence of the gender of the worker.

	Skilled	Unskilled
Male	**40**	**20**
Female	**10**	**30**

Section D - Long Answer/Essay Question

Answer any 3 questions. Each question carries 10 marks. (3X10=30 Marks)

34. Explain in detail about Kruskal Wallis test and ANOVA.
35. Explain the difference between relation and function in mathematics. Illustrate each concept using examples.
36. Explain properties of different Sampling Distributions (Chi-square, F, t, Z).
37. The mean strength of a certain cutting tool is 41.5 hours with the standard deviation of 2.5 hrs. A random sample of size 50 drawn from this population. what is the probability that
    a) Mean of the sample is between 40.5 hours and 42 hours
    b) Mean of the sample is less than 40.5 hours
    c) Mean of the sample is greater than 42 hours
38. ABC Corporation produces a product with a total weight requirement of 150 kg. The manufacturing process involves two primary raw materials, X and Y, priced at ₹2 and ₹8 per unit, respectively. The production must include a minimum of 14 units of Y and a maximum of 20 units of X. Each unit of X weighs 5 kg, and each unit of Y weighs 10 kg. The objective is to determine the optimal allocation of X and Y for each unit of the final product to minimise costs. Utilise the simplex method to solve this linear programming problem.

39. Three different kinds of food are tested on three groups of rats for 5 weeks. The objective is to check the difference in mean weight (in grams) of the rats per week. Apply one-way ANOVA using a 5% significance level to the following data:

Food I	**Food II**	**Food III**
8	4	11
12	5	8
19	4	7
8	6	13
6	9	7
11	7	9

MODEL QUESTION PAPER II

SREENARAYANAGURU OPEN UNIVERSITY

QP CODE: ………

Reg. No : ……….............

Name : ………………...

FIRST SEMESTER - MA ECONOMICS EXAMINATION

DISCIPLINE CORE – 04 - M23EC04DC – QUANTITATIVE METHODS FOR ECONOMICS I

2023-24 - Admission Onwards

Time: 3 Hours **Max. Marks: 70**

Section A - Objective Type Questions

Answer any 10 questions. Each question carries 1 mark (10 X 1=10 Marks)

1. Define ANOVA.
2. What is the term used to describe the set operation that consists of all elements belonging to either of two sets or both?
3. What is standard Normal Distribution?
4. If A = (1,2,3) and B = (a, b). Find A × B.
5. What is standard error?
6. Solve $\int 4x\, e^{2x}\, dx$.
7. Define one tailed test.
8. Given the demand function $P = 25 - Q$, find the marginal revenue function.
9. What is the primary objective in the context of utility, profit, and revenue when using optimisation techniques?
10. What is sign test?
11. Find the marginal cost, if the total cost function is $C = 20 + 12q - 8q^2 + 2q^3$.
12. Give the mathematical form for the standard error of the sample mean ($\sigma\bar{x}$).
13. What is the formula for calculating the Z-test statistic in hypothesis testing?

14. Define null hypothesis.
15. What is the power of test?

Section B - Very Short Questions

Answer any 5 questions. Each question carries 2 marks (5X2=10 Marks)

16. Distinguish between sample and population.
17. The total revenue function of a firm is given by $R = 21 - q^2$ where q is the output. Find the output at which the TR is maximum.
18. What is the Chi-square distribution?
19. Find the first order derivative of $y = x^3 logx$.
20. Distinguish between simple and composite hypothesis.
21. Which are the properties of the correlation coefficient "r"?
22. A company is conducting a survey to assess employee satisfaction with a new work policy. They randomly selected 80 employees and asked them about their satisfaction. If 35 employees express satisfaction with the new policy, estimate the proportion of all employees who are satisfied.
23. A smartphone manufacturer claims that the average battery life of their latest model is 12 hours. To test this claim, a random sample of 50 smartphones was taken, and the average battery life was found to be 11.5 hours with a standard deviation of 0.8 hours. Test the null hypothesis μ=12 against the alternative hypothesis $H1$: $\mu \neq 12$ at a 5% level of significance.
24. Write about Wilcoxon-Mann Whitney U Test.
25. Define Law of Large numbers.

Section C - Short Answer

Answer any 5 questions. Each question carries 4 marks. (5X4=20 Marks)

26. Solve the following
    a. $\int x\sqrt{x+1}\ dx$
    b. $\int \frac{x^3}{(x^2+2)^2}dx$
27. The mean breaking strength of the cables supplied by a manufacturer is 1800 with a standard deviation of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable was increased. To test the claim, a sample of 50 cables are tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1% level of significance?
28. Describe the simplex method for solving Linear Programming Problems (LPP). Explain its key characteristics.

29. An economist wishes to estimate the average family income in a certain population. The population standard deviation is known to be Rs.4,500 and the economist uses the random sample of size $n = 225$. What is the probability that the sample mean will fail within Rs.800 of the population mean?

30. What are the Fisher's properties of estimators?
31. The cost function of a company is $C = x^2 - 8x + 56$, where C is cost per unit and x, the number of units produced. Find the minimum value of the cost and the corresponding number of units to be produced.
32. Explain in detail about Concavity and Convexity.
33. Distinguish between type I and type II error.

Section D- Long Answer/Essay Question

Answer any 3 questions. Each question carries 10 marks. (3X10=30 Marks)

34. Use the simplex method to solve following LP problem.
$Max\ Z = 6x + 17y + 10z$
Subject to $x + y + z \leq 2000$
$2x + y + z \leq 3600$
$x + 2y + 2z \leq 2400$
$x \leq 30$ and
$x, y, z \geq 0$
35. Find Karl Pearson's coefficient of correlation between sales and expenses of the following ten firms:

Firm	1	2	3	4	5	6	7	8	9	10
Sales ('000 units)	50	50	55	60	65	65	65	65	60	50
Expenses ('000 rupees)	11	13	14	16	16	15	15	14	13	13

36. Explain in detail about optimisation of single variable and multi variable function.
37. Discuss in detail about various applications of derivatives.
38. Answer both the questions:
    a. In 16 one-hour test runs, the gasoline consumption of an engine averaged 16.4 gallons with a standard deviation of 2.1 gallons. Test the claim that the average gasoline consumption of this engine is 12.0 gallons per hour.
    b. A soft-drink vending machine is set-so that the amount of drink dispensed is a random variable with a mean of 200 milliliters and a standard deviation of 15 milliliters. What is the probability that the average (mean) amount dispensed in a random sample of size 36 is at least 204 milliliters?
39. Elaborate on different types of functions.

സർവ്വകലാശാലാഗീതം

വിദ്യയാൽ സ്വതന്ത്രരാകണം
വിശ്വപൗരരായി മാറണം
ഗ്രഹപ്രസാദമായ് വിളങ്ങണം
ഗുരുപ്രകാശമേ നയിക്കണേ

കൂരിരുട്ടിൽ നിന്നു ഞങ്ങളെ
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നീതിവൈജയന്തി പാറണം

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ബോധരശ്മിയിൽ തിളങ്ങുവാൻ
ജ്ഞാനകേന്ദ്രമേ ജ്വലിക്കണേ

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SREENARAYANAGURU OPEN UNIVERSITY

Regional Centres

Kozhikode
Govt. Arts and Science College
Meenchantha, Kozhikode,
Kerala, Pin: 673002
Ph: 04952920228
email: rckdirector@sgou.ac.in

Thalassery
Govt. Brennen College
Dharmadam, Thalassery,
Kannur, Pin: 670106
Ph: 04902990494
email: rctdirector@sgou.ac.in

Tripunithura
Govt. College
Tripunithura, Ernakulam,
Kerala, Pin: 682301
Ph: 04842927436
email: rcedirector@sgou.ac.in

Pattambi
Sree Neelakanta Govt. Sanskrit College
Pattambi, Palakkad,
Kerala, Pin: 679303
Ph: 04662912009
email: rcpdirector@sgou.ac.in

QUANTITATIVE METHODS FOR ECONOMICS I

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Sreenarayanaguru Open University

Kollam, Kerala Pin- 691601, email: info@sgou.ac.in, www.sgou.ac.in Ph: +91 474 2966841