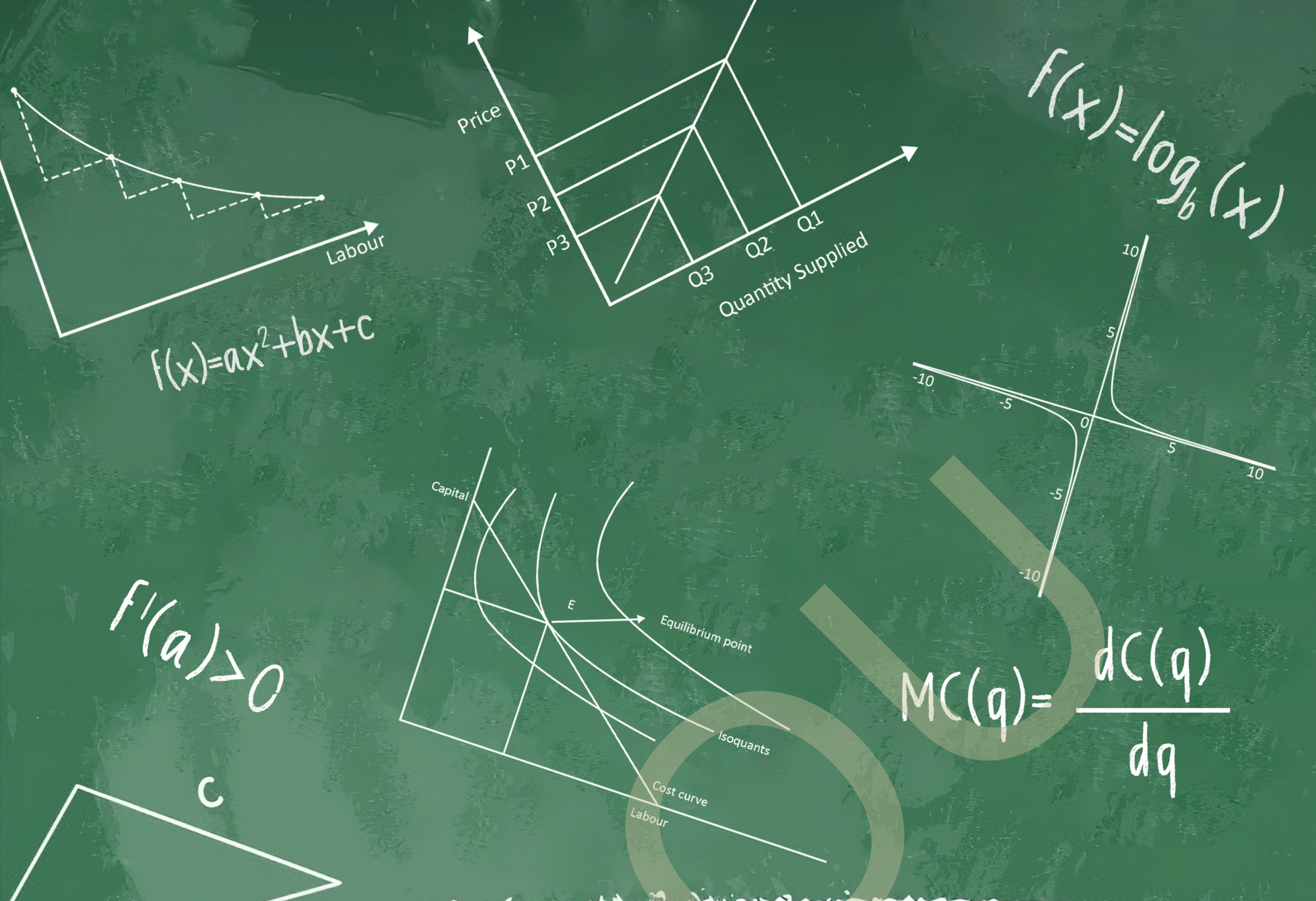


Mathematical Economics

COURSE CODE: B21EC01DE

Undergraduate Programme in Economics

Discipline Specific Elective Course

Self Learning Material

$$Ed = \frac{\Delta Q}{Q} \div \frac{\Delta P}{P}$$

SREENARAYANAGURU OPEN UNIVERSITY

SREENARAYANAGURU OPEN UNIVERSITY

The State University for Education, Training and Research in Blended Format, Kerala

SREENARAYANAGURU OPEN UNIVERSITY

Vision

To increase access of potential learners of all categories to higher education, research and training, and ensure equity through delivery of high quality processes and outcomes fostering inclusive educational empowerment for social advancement.

Mission

To be benchmarked as a model for conservation and dissemination of knowledge and skill on blended and virtual mode in education, training and research for normal, continuing, and adult learners.

Pathway

Access and Quality define Equity.

Mathematical Economics

Course Code: B21EC01DE

Semester - IV

Discipline Specific Elective Course
Undergraduate Programme in Economics
Self Learning Material

SREENARAYANAGURU OPEN UNIVERSITY

SREENARAYANAGURU OPEN UNIVERSITY

The State University for Education, Training and Research in Blended Format, Kerala

SREENARAYANAGURU
OPEN UNIVERSITY

MATHEMATICAL ECONOMICS

Course Code: B21EC01DE

Semester- IV

Discipline Specific Elective Course

Undergraduate Programme in Economics

Academic Committee

Dr. Sanathanan Velluva
Dr. Priyesh C. A.
Dr. Jisha K. K.
Dr. Muneer Babu M.
Dr. Suchithra Devi S.
Dr. Ratheesh C.
Dr. Resmi C. Panicker
Dr. Suprabha L.
Dr. Rajeev S.R.

Development of the Content

Yedu T. Dharan
Dr. Sindhu M.
Dr. Lakshmi Ravi

Review and Edit

Dr. K.N. Kamalasanan
Dr. M. Jayaprakas

Linguistics

Dr. K.N. Kamalasanan

Scrutiny

Dr. Suchithra K.R.
Yedu T. Dharan
Soumya V.D.
Muneer K.
Dr. Smitha K.

Design Control

Azeem Babu T.A.

Cover Design

Jobin J.

Co-ordination

Director, MDDC :
Dr. I.G. Shibi
Asst. Director, MDDC :
Dr. Sajeevkumar G.
Coordinator, Development:
Dr. Anfal M.
Coordinator, Distribution:
Dr. Sanitha K.K.

Scan this QR Code for reading the SLM on a digital device.

Edition
January 2025

ISBN 978-81-984025-8-5

MESSAGE FROM VICE CHANCELLOR

Dear learner,

I extend my heartfelt greetings and profound enthusiasm as I warmly welcome you to Sreenarayanaguru Open University. Established in September 2020 as a state-led endeavour to promote higher education through open and distance learning modes, our institution was shaped by the guiding principle that access and quality are the cornerstones of equity. We have firmly resolved to uphold the highest standards of education, setting the benchmark and charting the course.

The courses offered by the Sreenarayanaguru Open University aim to strike a quality balance, ensuring students are equipped for both personal growth and professional excellence. The University embraces the widely acclaimed "blended format," a practical framework that harmoniously integrates Self-Learning Materials, Classroom Counseling, and Virtual modes, fostering a dynamic and enriching experience for both learners and instructors.

The university aims to offer you an engaging and thought-provoking educational journey. The undergraduate programme in Economics is designed to be on par with the high-quality academic programmes offered at state universities throughout the country. The curriculum incorporates the latest methodologies for presenting economic ideas and concepts. It stimulates students' interest in developing a deeper comprehension of the discipline. The curriculum encompasses both theoretical concepts and historical evidence. Suitable emphasis is placed on India's experiences with economic transformation. This would aid learners in preparing for competitive examinations, should they choose to take them. Upon successfully completing the programme, we anticipate that students will be well-equipped to handle key areas within the economics discipline. The Self-Learning Material has been meticulously crafted, incorporating relevant examples to facilitate better comprehension.

Rest assured, the university's student support services will be at your disposal throughout your academic journey, readily available to address any concerns or grievances you may encounter. We encourage you to reach out to us freely regarding any matter about your academic programme. It is our sincere wish that you achieve the utmost success.

Regards,
Dr. Jagathy Raj V. P.

01-01-2025

Contents

BLOCK 1

Introduction to Mathematical Economics

UNIT 1

Nature and Scope of Mathematical Economics

Learning Outcomes

After completing this unit, the learner will be able to:

- understand the concept and scope of mathematical economics
- identify the role of mathematical representation in economic models
- distinguish between variables, constants, and parameters in economic equations

Prerequisites

One day, a farmer named Ravi decided to expand his farm. He wanted to figure out how much land he could cultivate with the resources he had like seeds, water, labour, and money. But soon, Ravi realised he could not just guess the outcomes. He needed to calculate how different factors like weather, costs, and the market demand would affect his farming decisions. This is where mathematics became useful for him. By using simple equations and patterns, Ravi started to predict the results of his efforts, making his decisions smarter and more effective.

Much like Ravi's experience, economics often requires a structured way of thinking, where we analyse the world through numbers, patterns, and relationships. Think about how businesses plan their budgets, governments decide on policies, or individuals manage their expenses. All of these involve choices, trade-offs, and predictions that can be better understood with the help of mathematics.

Keywords

Mathematical Economics, Economic Models, Variables, Constants, Parameters

Discussion

1.1.1 Mathematical Economics

Mathematical economics integrates mathematical tools and principles into the study of economics, for the development of theories and the analysis of complex economic problems. By employing mathematics, economists can create well-defined models, derive conclusions through logical reasoning, and validate these findings with statistical data. This approach enables the formulation of accurate predictions about future economic behaviour. The combination of mathematics, statistical methods, and economic principles has given rise to econometrics, a field that grows on advancements in computing power, big data analytics, and sophisticated mathematical techniques. Mathematical economics emphasises the representation of all relevant assumptions, conditions, and causal relationships in economic theories through mathematical expressions. This approach offers two key advantages, firstly mathematical tools like algebra and calculus help economists describe economic phenomena with clarity and draw specific inferences from foundational assumptions and definitions. Secondly, by translating theories into mathematical terms, economists can test them using quantitative data. When validated, these theories provide reliable quantitative predictions that are valuable for businesses, investors, and policymakers.

1.1.1.1 Nature of Mathematical Economics

Mathematical economics is an interdisciplinary field that uses mathematical tools and methods to model, analyse, and solve economic problems. By incorporating precise, structured, and quantifiable techniques, it allows economists to develop theories that are both logically accurate and empirically testable. Understanding its nature requires looking at several defining characteristics:

1. **Quantitative Representation:** Mathematical economics transforms economic theories into mathematical models by using equations, functions, and graphs. These models express relationships between different economic variables, such as supply and demand, price and quantity, or income and consumption. This quantitative approach helps make abstract economic concepts more real and measurable, allowing economists to study their effects under various conditions.

2. **Abstract and Simplified Framework:** Real-world economic phenomena

can be highly complex, with numerous variables and unpredictable outcomes. Mathematical economics simplifies this complexity by making assumptions that focus only on the most essential factors. For example, a model might assume perfect competition or rational behaviour to isolate specific economic relationships. These assumptions allow for clearer analysis but may not always reflect the full range of real-world complexities.

3. **Logical Consistency:** Mathematical economics demands that economic analysis follow a logical, step-by-step methodology. This ensures that assumptions, theorems, and conclusions are derived systematically and consistently. The use of mathematics guarantees a high degree of precision and minimises the potential for uncertainty or logical errors, thus increasing the reliability of economic models.

4. **Predictive Power:** One of the most powerful aspects of mathematical economics is its predictive capability. By analysing mathematical models, economists can make predictions about future economic behaviour and trends. For instance, a model that describes the relationship between inflation and unemployment can predict future economic conditions under various policy scenarios. This predictive power is key for decision-making in business, government, and international finance.

5. **Interdisciplinary Approach:** Mathematical economics does not function in isolation. It borrows concepts and methods from other disciplines such as mathematics, statistics, and computer science. Techniques such as calculus, linear algebra, optimisation, and differential equations are frequently used to solve economic problems. This interdisciplinary nature increases the flexibility and applicability of mathematical economics in addressing a wide range of economic issues.

6. **Theoretical and Applied Dimensions:** Mathematical economics serves both theoretical and practical purposes. On the theoretical side, it helps economists formalise and refine abstract concepts, offering a clearer understanding of economic principles. On the applied side, mathematical economics plays a key role in real-world decision-making, including business strategy, policy formulation, and resource allocation.

The nature of mathematical economics is defined by its use of quantitative models, logical accuracy, predictive capabilities, and interdisciplinary foundations. By simplifying complex economic phenomena and offering precise, testable predictions, it plays a major role in both the development of economic theory and the practical application of economic principles.

1.1.1.2 Scope of Mathematical Economics

Mathematical economics has a broad and diverse scope, which spans various domains of economic theory and its applications. The ability to model complex systems mathematically has led to advancements in many areas of economics, ranging from theoretical foundations to practical decision-making processes. The following are some of the key areas where mathematical economics has proven very important.

1. **Development of Economic Theory:** Mathematical economics provides a framework for the formalisation and development of economic theories. It is used to explain key concepts in microeconomics, such as consumer behaviour, production theory, and market equilibrium. Mathematical tools enable economists to rigorously analyse and express these theories, leading to more accurate predictions about how individuals and firms make decisions.

2. **Optimisation Problems:** A major application of mathematical economics is in optimisation. Economists frequently solve optimisation problems where the goal is to maximise or minimise some economic objective, such as profit, utility, or cost. Techniques like linear programming, constrained optimisation, and calculus are used to solve these problems, which are critical in both business and policy contexts.

3. **Market Analysis:** Mathematical models help explain how markets function, how prices are determined, and how resources are allocated. For example, mathematical economics can model the interaction of supply and demand, price elasticity, and market equilibrium. These models are key tools for understanding and predicting market outcomes in various economic environments.

4. **Macroeconomic Models:** Mathematics plays a crucial role in constructing models of national economies. These models include the determination of national income, the effects of fiscal and monetary policies, and long-term economic growth. Mathematical methods analyse complex macroeconomic relationships, such as the interaction between investment, consumption, and government spending.

5. **Game Theory and Strategic Behaviour:** Game theory focuses on the strategic interaction between different agents, such as firms, governments, and consumers. By using mathematical models to analyse competitive situations, game theory explain decision-making in scenarios where the outcome depends on the choices of others, such as in oligopolies or international trade negotiations.

6. **Welfare Economics:** Mathematical tools are also used in welfare economics to assess the efficiency and equity of resource allocation. By evaluating social welfare functions, income distribution, and the implications of different policies, mathematical economics helps policymakers design systems that promote economic well-being.

7. **Empirical Testing and Econometrics:** Mathematical economics is closely tied to econometrics, which uses statistical methods to test economic theories and validate models using real-world data. This empirical testing is essential for evidence-based policy-making and for refining economic theories based on observed behaviour.

8. **International Trade and Finance:** In the global context, mathematical economics helps analyse trade theories, exchange rate models, and the balance of payments. Mathematical models are used to understand how international markets operate and how factors like trade barriers, capital flows, and exchange rate fluctuations impact global economies.

The scope of mathematical economics is extensive, covering a wide range of theoretical and practical applications. Its interdisciplinary nature and predictive power make it an essential tool for economists, businesses, and policymakers alike.

1.1.2 Economic Models and their Mathematical Representation

Economic models are abstract representations of the real-world economic processes designed to analyse and understand how various economic factors interact. These models can take various forms, with mathematical models being one of the most accurate and widely used approaches. By using mathematical methods, economists can formulate hypotheses, test assumptions, and derive predictions about the behaviour of different economic agents. An economic model provides a theoretical framework for analysing economic phenomena. Mathematical models offer a structured approach to express the relationships between economic variables. A mathematical economic model typically consists of a set of equations that capture the core assumptions and relationships among the variables within the model. These equations allow economists to mathematically represent how different factors such as price, demand, and supply are related. By applying mathematical operations (e.g., algebraic operations, differentiation, or integration), economists can derive logical conclusions from these equations.

1.1.2.1 Variables, Constants, and Parameters

Variables: A variable in an economic model is a quantity that can change and take on different values depending on the circumstances of the model. Variables are central to economic models because they represent the key factors whose relationships and interactions are being studied. Examples of commonly used economic variables include price (P), quantity of goods (Q), national income (Y), consumption (C), investment (I), exports (X), and imports (M). The variability of these factors is what makes them crucial in modelling, as they can assume different values under different conditions. For instance, the price of a good (P) can change depending on supply and demand conditions, or national income (Y) may fluctuate due to economic policy changes or external factors.

In a model, variables can be classified into two types:

- **Endogenous Variables**: These are the variables whose values are determined within the model itself. They depend on the relationships specified in the model and are the quantities that we seek to solve for. For instance, in a supply and demand model, the equilibrium price (P) and quantity (Q) would be endogenous variables.

- **Exogenous Variables**: These variables are determined outside the model and are treated as given or fixed. Exogenous variables are not explained by the model but influence the endogenous variables. For example, in the same supply and demand model, factors like government policy or external shocks (e.g., global events) may be considered exogenous.

It is important to note that what is endogenous in one model might be exogenous in another. For example, the price of wheat (P) might be endogenous in a market-clearing model, but in a consumer behaviour model, it could be an exogenous variable for the individual consumer.

Constants: A constant in an economic model refers to a value that remains fixed and does not change. Constants are often used to simplify the equations and capture unchanging factors in the model. For example, the rate of interest or technological coefficients in a production function might be represented as constants.

Constants often appear in mathematical equations alongside variables, and they help describe the strength or nature of the relationship between variables. For example, in a linear demand equation like: $Q = a - bp$, here, a and b are constants. The constant a represents the intercept, while b determines the slope of the demand curve.

Parameters: A parameter is a type of constant that characterises the behaviour of a particular model or a set of models. Parameters help define the specific form or structure of the model. For instance, in a production function like $Q = f(L,K)$ where L is labour and K is capital, the function (f) may include parameters that dictate the relationship between these inputs and output (e.g., technological progress or returns to scale). Parameters can also be used to modify the assumptions or relationships in a model to make it more accurate or applicable to specific conditions. For example, in the Solow growth model, the savings rate and depreciation rate are parameters that influence the model's predictions regarding economic growth.

In mathematical economics, understanding the role of variables, constants, and parameters is key to interpreting and solving economic models. Variables represent the dynamic elements whose values are influenced by the model's internal relationships, while constants and parameters help to define the structural properties and assumptions of the model. The use of mathematical tools helps economists create clearer, more structured analyses of economic systems and theories.

Recap

- Mathematical economics uses mathematical tools to develop theories
- It analyse economic problems, allowing for accurate predictions about economic behaviour
- It integrates algebra, calculus, and econometrics
- It represents economic assumptions, conditions, and relationships through mathematical expressions
- Mathematical models represent relationships between economic variables through equations
- Variables in economic models change depending on circumstances
- It represent factors like price, quantity, income, consumption, and investment
- Endogenous variables are determined within the model
- Exogenous variables are fixed outside the model but influence the endogenous variables
- Constants in economic models remain fixed and simplify equations
- Some examples are interest rates or technological coefficients in production functions
- Parameters are constants that define the behaviour or structure of a model
- It may modify assumptions to make the model more applicable or accurate

Objective Questions

1. What is the field that integrates mathematical tools and principles to study economics?
2. What is the interdisciplinary field that combines mathematical tools, statistics, and economics to analyse economic problems?
3. What mathematical method is used to model relationships between economic variables?

4. Which method is frequently used to solve optimisation problems in mathematical economics?
5. What is the type of variable whose value is determined outside the economic model?
6. What type of model explains the strategic interactions between different agents in economics?

Answers

1. Mathematical Economics
2. Econometrics
3. Algebra
4. Calculus
5. Exogenous Variable
6. Game Theory

Assignments

1. Explain the nature of mathematical economics.
2. Describe the scope of mathematical economics.
3. How is game theory applied in strategic decision-making in economics?
4. Discuss the role of parameters in determining the output.
5. Explain in detail about constants, variables and parameters.

References

1. Agarwal, M. (2015). *Mathematical Methods in Economics* (2nd ed.). Vikas Publishing House.

2. Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4th ed.). McGraw-Hill.

3. Saluja, M., & Sharma, R. (2013). *Mathematical Economics: An Introduction*. Pearson Education India.

4. Sundaram, K. P. M. (2010). *Mathematics For Economics and Finance*. Tata McGraw-Hill Education.

Suggested Readings

1. Spiegel, M. R. (2008). *Theory And Problems of Mathematical Methods for Economists*. McGraw-Hill.

2. Weitzman, M. L. (2003). *Economic Models and Mathematical Representation*. The MIT Press.

3. Mathur, G. (2006). *Mathematical Methods for Economics* (3rd ed.). Cambridge University Press.

2

UNIT

Functions and Economic Applications

Learning Outcomes

After learning this unit, the learner will be able to:

- understand various types of functions used in economics
- differentiate between algebraic and non-algebraic functions
- understand the concept of explicit and implicit functions
- discuss economic functions such as demand, supply, cost, and revenue

Prerequisites

Imagine you are running a small food stall at a market. Every day, you notice different patterns in your business. Some days, the crowd is overwhelming, and you sell out your items quickly. Other days, even though the market is busy, fewer people seem to stop by your stall. You begin to wonder why this happens. Is it because of the weather? The pricing? Or maybe the availability of similar food at a nearby stall? You decide to keep track of your sales and observe the behaviour of your customers. Soon, you notice certain patterns: when prices are low, sales increase; when prices rise, sales drop. You also realise that the costs of raw materials directly affect the profit you make each day. These patterns and relationships start to give you insights into how your stall functions in the market.

Now, imagine scaling this concept to larger businesses, industries, or even entire economies. The relationships between variables such as price, quantity, cost, and revenue become even more critical to understand and predict. By studying these relationships, individuals and businesses can make informed decisions that lead to success.

Keywords

Functions, Explicit Functions, Implicit Functions, Algebraic Functions, Non-Algebraic Functions, Demand, Supply, Cost, Revenue

Discussion

1.2.1 Functions

A function, denoted as y = f(x), is essentially a rule that associates a given value of x with a corresponding value of y. For instance, the rule y = f(x) = 2x + 1 defines a function. More generally, any equation of the form y = mx + b, where m and b are constants, is known as a linear function. Functions can be expressed in a variety of forms. They may be defined through an algebraic formula (or a set of formulas), graphically, or by a table of values derived from experiments. In the latter case, we often have a series of points, which we can connect with a smooth curve to represent the function. For any value of x, a function can produce at most one corresponding value of y.

In functions, the variable x is called the *independent variable*, while y is the *dependent variable*, as its value depends on the value of x. In applied problems, determining the appropriate variables and their meanings is a crucial step.

Functions can also be combined through various operations. If f and g are two functions, we can create new functions by adding, subtracting, multiplying, or dividing them.

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ with the restriction that $g(x) \neq 0$

Another way to combine two functions f and g is through *composition*. The composition of f and g, denoted $f \circ g$, is defined as:

$$(f \circ g)(x) = f(g(x))$$

The domain of the composition $f \circ g$ consists of all x values in the domain of g such that $g(x)$ is also in the domain of f.

1.2.1.1 Types of Functions

Functions come in various forms, each with its unique characteristics. Functions can be broadly classified into Algebraic Functions and Non-Algebraic Functions, based on whether they are defined using basic algebraic operations or not.

Algebraic Functions

An algebraic function is a type of mathematical function that can be expressed using a finite combination of algebraic operations such as addition, subtraction, multiplication, division, and root extraction (e.g., square roots, cube roots). The defining feature of algebraic functions is that they satisfy polynomial equations with coefficients that are constants or functions of the independent variable.

An algebraic function f(x) satisfies an equation of the form, $P(f(x), x) = 0$, where P is a polynomial in f(x) and x.

The list of algebraic functions are as follows :

1. Linear Functions: A linear function is a function of the form, $f(x) = mx + b$, where m is the slope (rate of change), and b is the y-intercept (the value of y when x = 0).

Example:

Consider the function:

$$f(x) = 2x + 3$$

Here:

- m = 2, which means that for every unit increase in x, y increases by 2 times.
- b = 3, which means the function crosses the y-axis at $y = 3$.

The graph of this function is a straight line with slope 2, and it intersects the y-axis at $(0,3)$

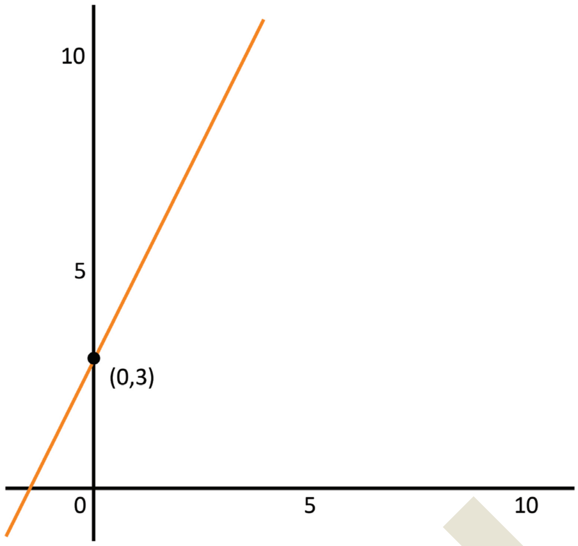


Fig 1.2.1 Linear Function

2. Quadratic Functions: A quadratic function is a polynomial function of degree 2, which has the general form, $f(x) = ax^2 + bx + c$, where a, b, and c are constants. The graph of a quadratic function is a parabola.

Example:

Consider the function:

$$f(x) = x^2 - 4x + 3$$

The graph of this function is a parabola that opens upward (because the coefficient of x^2, a = 1, is positive). The vertex of the parabola represents the minimum point of the graph. To find the vertex, we can use the formula:

$$x = -\frac{b}{2a}$$

Here, $a = 1$ and $b = -4$, so:

$$x = -\frac{-4}{2(1)} = \frac{4}{2} = 2$$

Substitute $x = 2$ into the equation to find the corresponding y-value:

$$f(2) = 2^2 - 4(2) + 3 = 4 - 8 + 3 = -1$$

Thus, the vertex is (2,−1), and the function has a minimum value at this point.

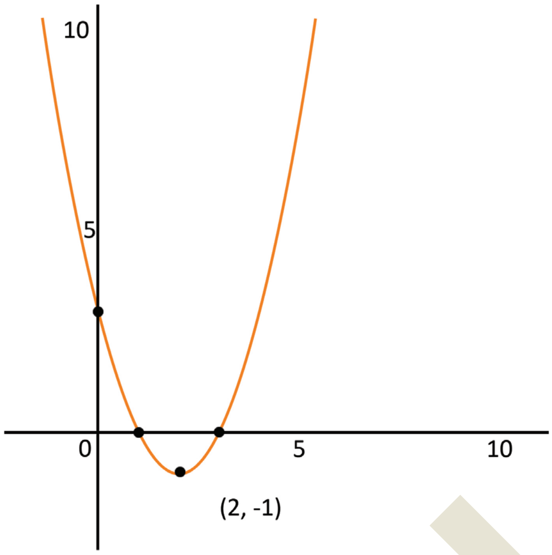


Fig 1.2.2 Quadratic Function

3. Polynomial Functions: A polynomial function is a function that involves terms with non-negative integer exponents of x, such as:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots a_0$ are constants and n is a non-negative integer.

Example:

Consider the function:

$$f(x) = 4x^3 - 2x^2 + x - 7$$

This is a cubic polynomial function with a degree of 3.

4. Rational Functions: A rational function is the ratio of two polynomials, which has the general form, $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, and $q(x) \neq 0$.

Example:

Consider the function:

$$f(x) = \frac{1}{x}$$

This is a basic rational function, also known as the reciprocal function. The graph of this function has two distinct parts (one in the first quadrant and one in the third quadrant), with an asymptote at x = 0 since division by zero is undefined) and another asymptote at y = 0.

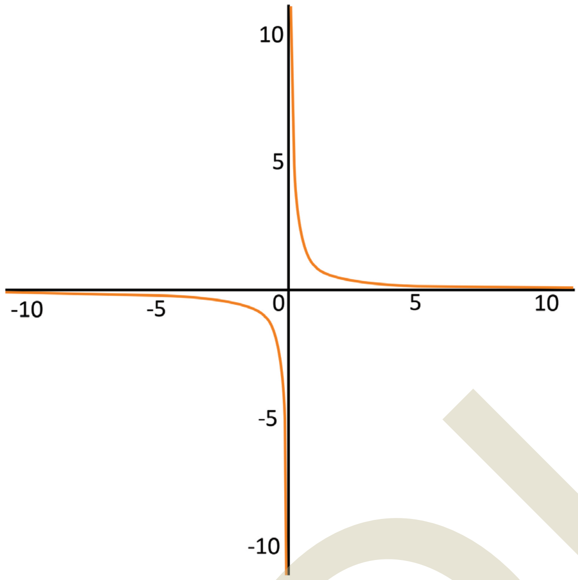


Fig 1.2.3 Rational Function

Non-Algebraic Functions

Non-algebraic functions are functions that cannot be expressed using a finite combination of algebraic operations such as addition, subtraction, multiplication, division, and raising to a power. They typically involve operations such as exponentiation, logarithms, trigonometry, or other complex relationships.

The list of few non- algebraic functions are as follows :

1. Exponential Functions: An exponential function is a function of the form, $f(x) = a.b^x$, where a is a constant and b is the base, which is a positive real number not equal to 1. The graph of an exponential function shows rapid rise or fall.

Example:

Consider the function:

$$f(x) = 2^x$$

The graph of this function rises exponentially as x increases. It passes through the point (0,1) because 2^0- 1, and as $x \to -\infty$, the function approaches 0 but never touches the x-axis. This function represents exponential growth.

If the base is between 0 and 1 (e.g., $f(x) = \left(\frac{1}{2}\right)^x$, the function will fall exponentially instead of growing.

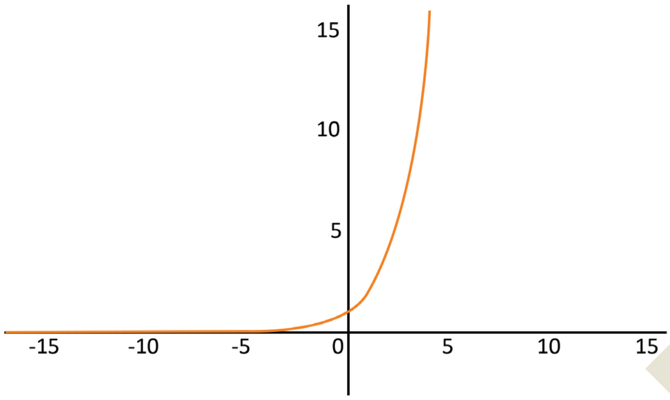


Fig 1.2.4 ***Exponential Function***

2. Logarithmic Functions: A logarithmic function is the inverse of an exponential function and has the general form, $f(x) = log_b(x)$, where b is the base of the logarithm and x > 0.

Example:

Consider the function:

$$f(x) = log_2(x)$$

The graph of this function is the inverse of the graph of $f(x) = 2^x$. It passes through the point (1,0) because $\log_2(1) = 0$, and as $x \to 0^+$, the function approaches negative infinity. It has a vertical asymptote at x= 0.

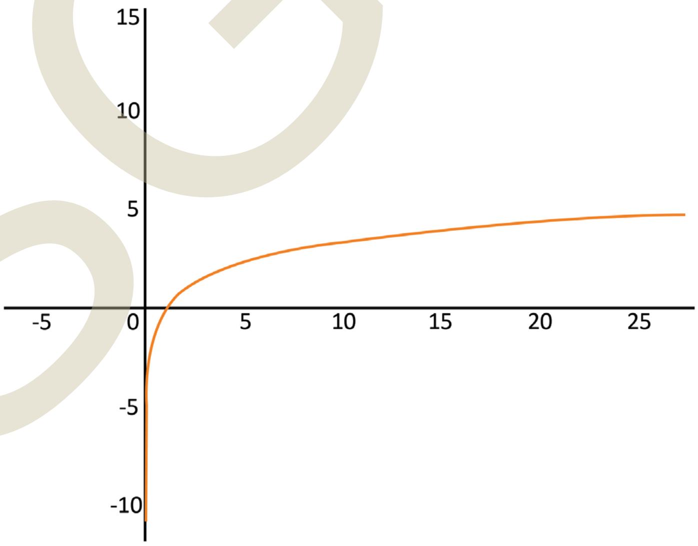


Fig 1.2.5 Logarithmic Function

3. Trigonometric Functions: Trigonometric functions such as sine, cosine, and tangent are based on the ratios of sides in a right triangle and are periodic functions.

The general forms of some common trigonometric functions are:

$f(x) = sin(x)$

$f(x) = cos(x)$

$f(x) = tan(x)$

Example:

Consider the function:

$f(x) = sin(x)$

This is a periodic function with a period of 2π and oscillates between -1 and 1. It has an amplitude of 1 and is symmetric about the origin.

1.2.1.2 Implicit and Explicit Functions

Functions play a key role in mathematics by describing relationships between variables. These relationships can be represented in two distinct forms: explicit and implicit functions.

1. Explicit Functions: An explicit function is a function where the dependent variable (y) is explicitly written in terms of the independent variable (x). In other words, the formula clearly shows how y depends on x.

General Form: $y = f(x)$

Examples:

$y = 2x + 3$ (Linear function)

$y = \sqrt{x + 1}$ (Square - root function)

In these examples, the value of y can be directly calculated for a given x, making the relationship straightforward.

2. Implicit Functions: An implicit function is a function where the dependent variable (y) and the independent variable (x) are interrelated through an equation but are not explicitly solved for one variable in terms of the other. In this case, finding y may require solving an equation.

General Form: $F(x, y) = 0$

Examples:

$x^2 + y^2 - 1 = 0$ (Equation of a circle)

$xy - 5 = 0$ (Hyperbola)

For these functions:

$x^2 + y^2 - 1 = 0$ represents a circle with radius 1.

$xy - 5 = 0$ can be rewritten explicitly as $y = \frac{5}{x}$, but its implicit form is useful in some contexts.

Implicit functions can often be rewritten as explicit ones when the equation allows solving for y in terms of x, but this is not always possible.

1.2.2 Economic Functions

Economic functions are mathematical representations that describe the relationship between key economic variables. They serve as the base for analysing and predicting market behaviour, production efficiency, and overall economic performance. The most common economic functions include demand, supply, cost, revenue, and profit functions. Each plays a vital role in microeconomic and macroeconomic analysis, helping businesses and policymakers in making informed decisions. Economic functions helps to understand how different factors interact within an economy. By establishing relationships between variables, these functions allow for modelling and forecasting.

Major economic functions are as follows:

1. Demand Function

The demand function expresses the quantity of a good or service that consumers are willing and able to purchase at various price levels. It typically has a downward-sloping curve, showing an inverse relationship between price and quantity demanded. External factors such as income levels, preferences, and the prices of substitutes and complements also influence demand.

2. Supply Function

The supply function represents the relationship between the quantity of a good or service that producers are willing to offer and the market price. Generally, the supply curve slopes upward, explaining the direct relationship between price and the quantity supplied. Factors like production costs, technological advancements, and government policies impact supply.

3. Cost Function

The cost function shows the relationship between the cost of production and the quantity of output. It can include fixed costs (unaffected by output level) and variable costs (dependent on output). Businesses use this function to analyse economies of scale and production efficiency.

4. Revenue Function

The revenue function indicates the total income a business earns by selling a certain quantity of goods or services at a given price. It is calculated as the product of price and quantity sold. Understanding the revenue function is key for identifying the optimal price and output level.

5. Profit Function

The profit function is derived by subtracting total cost from total revenue. It explains the relationship between profit and various factors, such as output level, production costs, and market conditions. Businesses aim to maximise profit by identifying the output level where marginal cost equals marginal revenue.

Economic functions are generally represented graphically, as it makes it easier to interpret relationships between variables. Economic functions provide a systematic way to study and predict economic behaviour. They form the base of decision-making for businesses and policymakers, helping to balance resources, costs, and consumer preferences.

1.2.2.1 Demand Function

The demand function describes the relationship between the quantity of a good or service demanded by consumers at a given price. The demand function helps in understanding consumer behaviour and market dynamics, forming the basis for analysing pricing strategies, market equilibrium, and the effects of external economic shocks.

A demand function is a mathematical expression of the form:

$$Q_d = f(P, I, T, P_r, E)$$

Where:

Q_d: Quantity demanded

P: Price of the good

I: Consumer income

T: Consumer tastes and preferences

P_r: Prices of related goods (substitutes or complements)

E: Other external factors (e.g., taxes, expectations)

In its simplest form, the demand function may be written as:

$$Q_d = a - bP$$

Where a and b are constants, P is the price of the good, and Q_d is the quantity demanded.

Other things are considered constant for the purpose of analysis. The demand curve is plotted with price (P) on the vertical axis and quantity demanded (Q_d) on the horizontal axis. The curve slopes downward, showing the inverse relationship between price and quantity demanded.

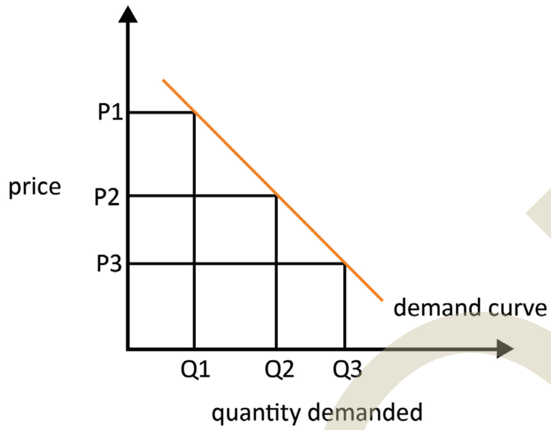


Fig 1.2.6 Demand Function

The demand function is a key economic theory offering a structured way to analyse how consumers respond to changes in price, income, and other variables.

1.2.2.2 Supply Function

The supply function is a mathematical representation that describes the relationship between the quantity of a good or service that producers are willing to supply at a given price. It is useful in explaining producer behaviour and market dynamics.

The supply function is typically written as:

$$Q_s = f(P, T, C, G, N)$$

Where:

Q_s: Quantity supplied

P: Price of the good or service

T: Technology level

C: Cost of production

G: Government policies (e.g., taxes, subsidies)

N: Number of sellers in the market

The supply curve represents supply function in a graph. Other things are considered constant for the purpose of analysis. The x-axis represents the quantity supplied (Q_s). The y-axis represents the price (P). The curve slopes upwards, demonstrating the positive relationship between price and quantity supplied.

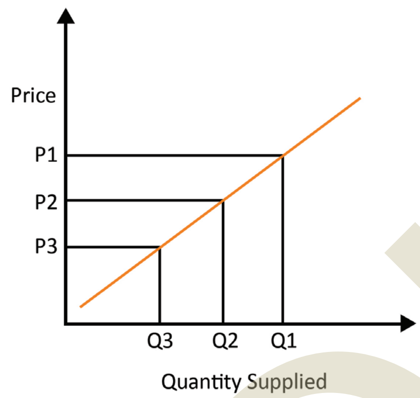


Fig.1.2.7 Supply Function

The supply function helps in understanding how producers respond to market conditions and external factors.

1.2.2.3 Cost Function

The cost function is a mathematical representation that describes the relationship between the cost incurred by a firm and the level of output it produces. It helps firms analyse their production costs, optimise resource allocation, and determine the most cost-effective production levels.

The cost function, denoted as $C(q)$, represents the total cost (C) of producing a certain quantity (q) of goods or services. It accounts for all costs, including both fixed and variable components, and can be expressed as:

$$C(q) = F + V(q)$$

Where:

F: Fixed costs, which remain constant regardless of output level.

V(q): Variable costs, which change depending on the quantity produced.

Components of the Cost Function

1. **Fixed Costs (F)**: Fixed costs are costs that remain unchanged irrespective of production levels in the short run but may vary in the long run due to capacity expansion. Examples include rent, salaries of permanent staff, and insurance. Even if no goods are produced, fixed costs must still be paid.

2. **Variable Costs ($V(q)$):** These costs depend directly on the quantity of output. Examples include raw materials, energy usage, and wages for temporary labour. As production increases, variable costs also rise.

3. **Total Cost ($C(q)$) :** This is the sum of fixed and variable costs for a given output level.

4. **Average Cost (AC)**: The cost per unit of output, calculated as: $AC(q) = \frac{C(q)}{q}$

5. **Marginal Cost (MC)**: The additional cost incurred for producing one more unit of output, calculated as: $MC(q) = \frac{dC(q)}{dq}$

The Fixed Cost Curve is a horizontal line, reflecting that fixed costs remain constant regardless of output. The Total Cost Curve starts at the level of fixed costs and slopes upward as output increases, explaining the growing total cost with higher production levels. The Marginal Cost Curve has a U-shape, explains initially decreasing costs followed by increasing costs due to diminishing returns. Similarly, the Average Cost Curve is also U-shaped, with its lowest point representing the most efficient production level, where average costs are minimised.

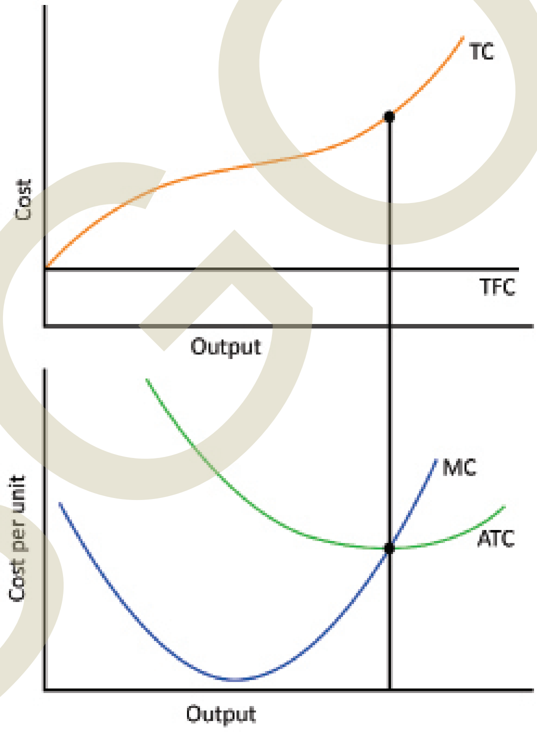


Fig 1.2.8 Cost Curves

The cost function helps in economic and managerial decision-making.

1.2.2.4 Revenue Function

The revenue shows the total revenue a firm generates from selling a given quantity of goods or services. It explains the relationship between the quantity sold, price, and total income generated by a firm.

Revenue is the total income a firm earns from selling its products or services. It is expressed as:

$$Revenue\ (R) = Price\ (P) \times Quantity\ (Q)$$

Here:

- Price (P): The selling price of one unit of the good or service.
- Quantity (Q): The number of units sold.

Components of the Revenue Function

1. **Total Revenue (TR):** Total Revenue is the total income received from sales. It is calculated as: $TR = P \times Q$, Where the price P can be constant (in the case of perfect competition) or vary with the quantity sold (in imperfect competition).

2. **Average Revenue (AR):** Average Revenue is the revenue earned per unit of output sold. It is given by: $AR = \frac{TR}{Q}$. Under perfect competition, AR equals P, as all units are sold at the same price.

3. **Marginal Revenue (MR):** Marginal Revenue is the additional revenue earned by selling one more unit of output. It is calculated as: $MR = \frac{\Delta TR}{\Delta Q}$. Under perfect competition, $MR=P$. In imperfect competition, MR diminishes as output increases due to the price drop required to sell additional units.

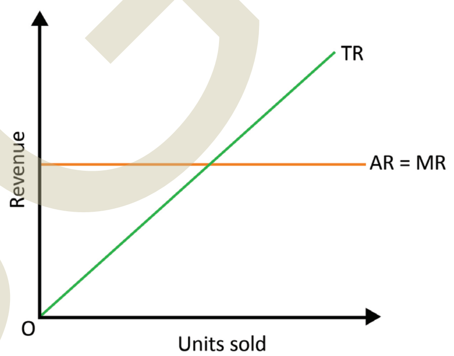


Fig 1.2.9 Revenue Curves under Perfect Competition

In the graphical representation of the revenue function, the behaviour of Total Revenue (TR), Average Revenue (AR), and Marginal Revenue (MR) differs under perfect and imperfect competition. Under perfect competition, Total Revenue increases linearly because the firm can sell any quantity at a constant price. Here, both Average Revenue and Marginal Revenue are horizontal lines, indicating that $AR=MR=P$, as all units are sold at the same price. In contrast, under imperfect competition, Total Revenue curve has a parabolic shape: it rises initially, reaches a maximum point, and then declines as the firm reduces prices to sell additional units. Average Revenue slopes downward, reflecting the decreasing price required to sell more units. Marginal Revenue lies below the AR curve and can become negative when Total Revenue starts decreasing beyond a certain output level.

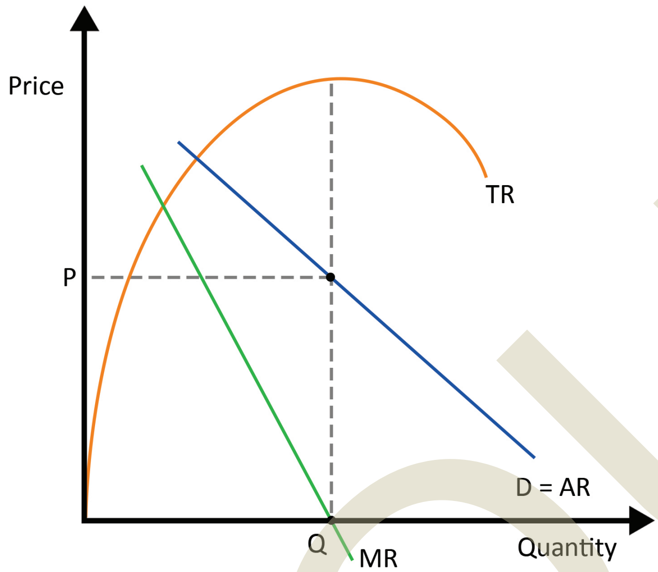


Fig 1.2.10 Revenue Curves under Imperfect Competition

The revenue function helps to analyse a firm's financial performance and generate market strategy.

1.2.2.5 Profit Function

The profit function is a mathematical representation of a firm's profit, which is the difference between its total revenue (TR) and total cost (TC). It helps in understanding a firm's financial performance and is a cornerstone of decision-making in economics and business management.

Profit is the financial gain achieved when a firm's total revenue exceeds its total cost. It is expressed as:

$$Profit\ (\pi) = Total\ Revenue\ (TR) - Total\ Cost\ (TC)$$

Where:

- **Total Revenue (TR):** The income generated from selling goods or services ($TR=P\times Q$).
- **Total Cost (TC):** The total expenses incurred in the production of goods or services. This includes fixed costs (FC) and variable costs (VC).

Types of Profit

Economic Profit: Economic profit considers both explicit costs (actual monetary expenses) and implicit costs (opportunity costs of using resources). It is expressed as:

$$Economic\ Profit = TR - (Explicit\ Costs + Implicit\ Costs)$$

Accounting Profit: Accounting profit only accounts for explicit costs and is calculated as:

$$Accounting\ Profit = TR - Explicit\ Costs$$

Normal Profit: Normal profit occurs when total revenue equals total costs (including opportunity costs), representing a breakeven point. It is the minimum profit required to keep resources employed in their current use.

Example

Suppose a firm faces a demand function P=100-2Q and a cost function TC=20+10Q. Find Profit maximisation level of output and maximum profit.

1. Total Revenue (TR):

$$TR = P \times Q = (100 - 2Q)Q = 100Q - 2Q^2$$

2. Total Cost (TC):

$$TC = 20 + 10Q$$

3. Profit Function (π):

$$\pi(Q) = TR - TC = (100Q - 2Q^2) - (20 + 10Q)$$

4. Profit Maximisation: To maximise profit, set *MR*=*MC*:

$$MR = \frac{d(TR)}{dQ} = 100 - 4Q$$

$$MC = \frac{d(TC)}{dQ} = 10$$

$$100 - 4Q = 10 \implies Q = 22.5$$

At $Q = 22.5$:

$$TR = 100(22.5) - 2(22.5)^2 = 2250 - 1012.5 = 1237.5$$

$$TC = 20 + 10(22.5) = 245$$

$$\pi = TR - TC = 1237.5 - 245 = 992.5$$

Profit maximisation level of output is at 22.5 units and maximum profit is 992.5.

The profit function helps in analysing and optimising a firm's financial performance. By understanding the revenue, cost, and output, firms can make informed decisions on pricing, production, and resource allocation.

Recap

- A function associates a value of x with a corresponding value of y, and for any x, there can be at most one y
- Functions can be combined by addition, subtraction, multiplication, or division; composition is another method
- Algebraic functions can be expressed using basic algebraic operations, such as addition, subtraction, and root extraction
- Linear functions have the form $f(x) = mx + b$, with a straight-line graph, slope m, and y-intercept b
- Quadratic functions have the form $f(x) = ax^2 + bx + c$, with a parabolic graph and a vertex representing the minimum or maximum point
- Rational functions are ratios of polynomials, and the denominator cannot be zero
- Non-algebraic functions cannot be expressed using algebraic operations and include exponential, logarithmic, and trigonometric functions
- Exponential functions show rapid growth or fall
- Logarithmic functions are the inverse of exponential functions, with the general form $f(x) = log_b(x)$
- An explicit function clearly shows the dependent variable (y) in terms of the independent variable (x)
- General form of an explicit function: $y = f(x)$
- Implicit functions do not explicitly solve for one variable in terms of the other

- General form of an implicit function: $F(x, y) = 0$
- Economic functions describe the relationship between economic variables
- It helps in analyse and predict market behaviour, production efficiency, and economic performance
- Demand function shows the relationship between the quantity of a good demanded and its price, with external factors influencing demand
- Supply function explains the relationship between the quantity supplied and price, impacted by production costs, technology, and government policies
- Cost function relates production costs to output levels, analysing fixed and variable costs to determine production efficiency
- Revenue function calculates the total income from sales, helping firms identify optimal pricing and output levels
- Profit function represents the difference between total revenue and total cost, helping firms maximise profit by optimising output levels

Objective Questions

1. What is the general form of a linear function?
2. What is the function form that represents a quadratic function?
3. What operation is represented by (f + g)(x)?
4. What is the base of the logarithmic function $f(x) = log_b(x)$?
5. What is the general form of a demand function?
6. What type of function has the form $f(x) = ax^2 + bx + c$?
7. What is the shape of the graph of a quadratic function?
8. What is the general form of a rational function?
9. What is the inverse of an exponential function called?

10. What is the general form of an explicit function?

11. What does the demand function relate?

12. What does the supply function describe?

13. What does the cost function describe?

14. How do we find the profit?

15. What does the revenue function represent?

Answers

1. $y = mx + b$
2. $f(x) = ax^2 + bx + c$
3. $f(x) + g(x)$
4. b
5. $Q_d = f(P, I, T, P\_r, E)$
6. Quadratic function
7. Parabola
8. $f(x) = \frac{p(x)}{q(x)}$
9. Logarithmic function
10. $y = f(x)$
11. Relationship between Quantity demanded and price
12. Relationship between Quantity supplied and price
13. Relationship between cost and output
14. Profit = Revenue−Cost
15. Total income from selling goods or services

Assignments

1. Define a function and provide examples of different types of functions.
2. Explain the difference between a rational function and a polynomial function.
3. Discuss on exponential functions.
4. Given the linear function $f(x) = 3x + 2$, calculate the value of y when $x = 4$.
5. For the quadratic function $f(x) = x^2 - 2x + 5$, find the vertex of the parabola.
6. Consider the polynomial function $f(x) = 5x^3 - 3x^2 + 7x - 4$. What is the degree of the function?
7. Define an explicit function and give an example of one.
8. Derive the profit function if the revenue function is $R(x) = 5x$ and the cost function is $C(x) = 3x + 10$.

References

1. Agarwal, A. (2018). *Mathematics for Economics* (2nd ed.). Vikas Publishing House.
2. Jha, R. (2015). *Advanced Microeconomics: Theory and Applications* (4th ed.). Himalaya Publishing House.
3. Varian, H. R. (2010). *Intermediate Microeconomics: A Modern Approach* (8th ed.). W.W. Norton & Company.

Suggested Readings

1. Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4th ed.). McGraw-Hill Education.

2. Mankiw, N. G. (2021). *Principles of Economics* (9th ed.). Cengage Learning.

2 BLOCK

Microprinciples:- Marginal Utility and Elasticity of Demand

UNIT 1

Marginal Concepts and Their Applications

Learning Outcomes

Upon completion of this unit, the learner will be able to

- understand marginal utility, marginal product, marginal cost, and marginal revenue
- discuss the concepts of MRS and MRTS
- define the relationship between average and marginal values in economic analysis

Prerequisites

Imagine you are running a small café in your neighbourhood. Every day, you bake fresh bread and brew coffee, hoping to attract more customers. On a busy morning, you notice that when you bake one extra loaf of bread, the number of customers who stop by increases. The smell of fresh bread brings in more people, and you sell more than usual. Encouraged by this success, you bake another loaf the next day, and then another the day after that. At first, the extra loaves seem to bring in more customers, and your sales go up. However, as you continue baking more and more loaves, something interesting happens. The new customers do not come in as often as they did at the start, and your bakery begins to feel crowded. The extra loaves of bread do not seem to attract as many customers anymore. You also notice that, while the extra bread is still selling, it is not making as much money as it did when you baked just a few extra loaves. This situation reflects a universal pattern: At first, the benefits of doing more are clear, but after a certain point, the additional effort yields smaller returns. It is not just about baking more bread—it is about finding the balance between what is added and what is gained. The challenge is to figure out how much is enough, without overdoing it. In the world of economics, these small but important changes are studied closely. They help us understand how people, businesses, and even entire economies make decisions about how much to produce, how to use resources, and when to stop.

Keywords

Marginal Utility, Marginal Product, Marginal Cost, Marginal Revenue, Marginal Rate of Substitution (MRS), Marginal Rate of Technical Substitution (MRTS)

Discussion

2.1.1 Marginal Concepts in Economics

Marginal concepts in economics are foundational principles used to analyse decision-making and resource allocation. These concepts focus on the incremental changes in costs, benefits, outputs, or inputs due to changes in economic activity. They are key in microeconomic analysis and are extensively used in studying production, consumption, and pricing. Below are some key marginal concepts:

2.1.1.1 Marginal Utility

Marginal utility represents the additional satisfaction or benefit a consumer derives when he/she consumes one more unit of a good or service. This explains that the utility derived from each additional unit of a product decreases as the consumer already possesses more of it. In other words, marginal utility is inversely related to the quantity of the good the consumer already owns. This is theoretically explained as theory of diminishing marginal utility.

Consider a family of 5 members with seven slices of bread. If they are offered an additional slice, the marginal utility of that slice would be significant because it directly impacts their satisfaction by reducing their hunger. The difference between having seven slices and eight slices is proportionally meaningful. Now imagine the same family has 30 slices of bread. If they receive one more slice, the marginal utility of that extra slice would be much lower. The difference between 30 and 31 slices is relatively insignificant, as their hunger is already satisfied by the earlier slices. Eventually, as the family gets more slices than they can consume, the marginal utility of additional slices diminishes and could reach zero when the need is completely satisfied. At this point, the family derives no benefit from acquiring further units of bread. Marginal utility can be calculated using the formula:

$$Marginal\ Utility\ (MU) = \frac{Change\ in\ Total\ Utility}{Change\ \ in\ Quantity\ Consumed}$$

$$MU = \frac{\Delta TU}{\Delta Q}$$

Where:

- MU: Marginal Utility
- ΔTU: Change in Total Utility
- ΔQ: Change in Quantity

To find ΔTU, subtract the total utility from the previous level (TU_{n-1}) from the current level of total utility (TU_n):

$$\Delta TU = TU_n - TU_{n-1}$$

Similarly, calculate ΔQ by subtracting the previous quantity of units (Q_{n-1}) from the current quantity (Q_n):

$$\Delta Q = Q_n - Q_{n-1}$$

Using this formula allows us to quantify how much additional satisfaction a consumer gains from consuming one more unit of a product.

Example:

A consumer derives utility from consuming chocolate bars. The total utility (TU) he/she derives from consuming different quantities of chocolate bars is as follows:

Table 2.1.1 Quantities of Chocolate

Number of Chocolate Bars (Q)	Total Utility (TU)
1	10
2	18
3	24
4	28
5	30

Calculate Marginal Utility.

Let us use the formula:

$$MU = \frac{\Delta TU}{\Delta Q}$$

1. For the 2nd chocolate bar:

$$MU = \frac{TU_2 - TU_1}{Q_2 - Q_1} = \frac{18 - 10}{2 - 1} = 8$$

2. For the 3rd chocolate bar:

$$MU = \frac{TU_3 - TU_2}{Q_3 - Q_2} = \frac{24 - 18}{3 - 2} = 6$$

3. For the 4th chocolate bar:

$$MU = \frac{TU_4 - TU_3}{Q_4 - Q_3} = \frac{28 - 24}{4 - 3} = 4$$

4. For the 5th chocolate bar:

$$MU = \frac{TU_5 - TU_4}{Q_5 - Q_4} = \frac{30 - 28}{5 - 4} = 2$$

Table 2.1.2 Quantities of Chocolate with Marginal Utility

Number of Chocolate Bars (Q)	Total Utility (TU)	Marginal Utility (MU)
1	10	-
2	18	8
3	24	6
4	28	4
5	30	2

The marginal utility decreases as more chocolate bars are consumed. The first chocolate bar provides the most satisfaction (10 utility points). However, as consumption increases, the additional satisfaction from each subsequent bar declines. This example explains the law of diminishing marginal utility, which states that the added utility derived from consuming additional units of a good decreases as consumption increases.

2.1.1.2 Marginal Product

Marginal product refers to the additional output generated when an extra unit of input is used, while all other production factors, remain constant. It measures the change in production output resulting from a change in the input. This concept is critical in short-run production analysis, particularly in understanding how inputs like labour contribute to output levels while other factors remain constant.

The marginal product can be expressed mathematically as:

$$Marginal\ Product\ (MP) = \frac{Change\ in\ Output\ Produced}{Change\ in\ Input\ Used}$$

Alternatively,

$$MP = \frac{\Delta TP}{\Delta X}$$

Or in expanded form:

$$MP = \frac{Q_n - Q_{n-1}}{X_n - X_{n-1}}$$

Where:

- Q_n: Total production at the current level of input (n)
- Q_{n-1}: Total production at the previous level of input (n−1)
- X_n: Number of input units at the current level
- X_{n-1}: Number of input units at the previous level

Marginal product assesses how output changes when additional labour is employed. By understanding this relationship, firms can determine the optimal workforce size to achieve maximum productivity and revenue. This makes marginal product analysis an essential component of decision-making in production planning and economic modelling.

Example

A company produces widgets using labour as the primary input, keeping other inputs constant (e.g., machinery and land). The production output is measured as the number of widgets produced, depending on the number of workers employed.

The table below shows the total production (TP) as the company increases the number of workers (L):

Table 2.1.3 Total Production (TP) And Number of Workers

Number of Workers (L)	Total Production (TP) (Widgets)
1	20

2	45
3	65
4	80
5	90

Calculate the Marginal Productivity.

Using the formula:

$$MP = \frac{\Delta TP}{\Delta X}$$

Compute the marginal product for each additional worker:

1. From 1 to 2 workers:

$$MP = \frac{Q_n - Q_{n-1}}{X_n - X_{n-1}} = \frac{45 - 20}{2 - 1} = 25$$

2. From 2 to 3 workers:

$$MP = \frac{65 - 45}{3 - 2} = 20$$

3. From 3 to 4 workers:

$$MP = \frac{80 - 65}{4 - 3} = 15$$

4. From 4 to 5 workers:

$$MP = \frac{90 - 80}{5 - 4} = 10$$

Table 2.1.4 Total Production (TP) And Number of Workers with MP

Number of Workers (L)	Total Production (TP)	Marginal Product (MP)
1	20	-
2	45	25

3	65	20
4	80	15
5	90	10

The marginal product decreases as more workers are employed. This reflects the law of diminishing marginal returns, where adding additional units of input (labour) leads to smaller increases in output because other factors of production remain fixed. The company can use this data to decide the optimal number of workers to employ. For instance, hiring beyond four workers results in diminishing returns that may not justify the cost of additional labour.

2.1.1.3 Marginal Cost

Marginal cost represents the additional cost a business incurs when producing one more unit of a good or service. It is calculated by dividing the total change in production costs by the change in the quantity of goods produced.

The marginal cost can be expressed as:

$$Marginal\ Cost = \frac{Change\ in\ Total\ Cost}{Change\ in\ Quantity\ Produced}$$

$$MC = \frac{\Delta TC}{\Delta Q}$$

or

$$MC = \frac{TC_n - TC_{n-1}}{Q_n - Q_{n-1}}$$

Where:

- TC_n: Total cost at the current period (n)
- TC_{n-1}: Total cost at the previous period (n−1)
- Q_n: Number of quantities at the current period
- Q_{n-1}: Number of quantities at the previous period

Example:

Suppose a company incurs ₹1,000,000 in production costs to manufacture 24 units of heavy machinery. If increasing production to 25 units raises costs to ₹1,050,000. Calculate the marginal cost.

The change in total expenses is ₹1,050,000 - ₹1,000,000 = ₹50,000 , and

the change in quantity is 25 – 24 = 1

Thus, the marginal cost for the additional unit is:

$$Marginal\ Cost = \frac{1{,}050{,}000\ -\ ₹1{,}000{,}000}{25 - 24} = \frac{50{,}000}{1} = 50{,}000$$

Thus the MC for producing the 25th unit is ₹50,000

This formula is versatile and can be applied to cases where multiple units are produced. However, businesses must consider variations in marginal costs across different production levels due to factors like changes in input costs or production capacities.

2.1.1.4 Marginal Revenue

Marginal revenue refers to the additional income a company earns from the sale of one more unit of output. While marginal revenue can remain constant over a certain range of production, it typically diminishes as output increases, in line with the law of diminishing returns. In the context of perfectly competitive markets, firms continue to produce as long as marginal revenue equals marginal cost.

To calculate marginal revenue, the change in total revenue is divided by the change in quantity of goods sold:

$$Marginal\ Revenue = \frac{Change\ in\ Total\ Revenue}{Change\ in\ Quantity\ Consumed}$$

$$MR = \frac{\Delta TR}{\Delta Q}$$

or

$$MR = \frac{TR_n - TR_{n-1}}{Q_n - Q_{n-1}}$$

Where:

TR_n: Total Revenue at the current period (n)

TR_{n-1}: Total Revenue at the previous period (n−1)

Q_n: Number of quantities at the current period

Q_{n-1}: Number of quantities at the previous period

Example

Imagine a company sells 100 units of a product in one week, generating total revenue of ₹1,000. If the company then sells 115 units the following week for a total revenue of ₹1,100, the marginal revenue for the additional 15 units sold is ₹100. Calculate marginal revenue.

Using the formula - $MR = \frac{\Delta TR}{\Delta Q}$

$$Marginal\ Revenue = \frac{1100 - 1000}{115 - 100} = \frac{1000}{15} = 6.67$$

Thus, the marginal revenue per unit is ₹6.67 for units 101 to 115.

Marginal revenue helps companies understand the financial effects of selling additional units, set appropriate pricing strategies, and plan production schedules more effectively.

2.1.1.5 Marginal Rate of Substitution (MRS)

The Marginal Rate of Substitution (MRS) represents the amount of one good that a consumer is willing to give up to acquire an additional unit of another good, while maintaining the same level of utility. It quantifies the relative marginal utility between two goods. MRS is most commonly analysed using indifference curves, which depict combinations of two goods that yield the same level of satisfaction or utility for the consumer. The consumer is indifferent between these combinations, meaning that they would derive equal satisfaction from either combination. At equilibrium, the MRS between two goods is constant, but it may vary along different points of the curve as the relative quantities of the goods change. The MRS expresses the consumer's preference in substituting one good for another. It is determined by the slope of the indifference curve. Essentially, the MRS measures how much of one good a consumer is prepared to sacrifice for the benefit of an additional unit of the other good, while keeping their overall utility unchanged. This means that the marginal utility of the two goods being compared is equal, even though they may be of different types.

$$MRS = \frac{\Delta Y}{\Delta X} = \frac{MU_X}{MU_Y}$$

where MU_Y and MU_X are the marginal utilities of goods Y and X, respectively. It quantifies how much of one good (Y) a consumer is willing to give up in exchange for an additional unit of another good (X).

The indifference curve serves as a graphical tool to show the trade-offs between these two goods. Each curve represents a different level of utility, with the slope indicating the rate of substitution between the goods. As we move along the indifference curve, the MRS changes, reflecting diminishing marginal utility. For instance, when a consumer consumes more of one good, he/she are generally willing to give up less of the other good, demonstrating the law of diminishing marginal rate of substitution. This principle suggests that as one good becomes more abundant, its marginal utility diminishes, and thus the consumer is less willing to substitute it for another good. The shape of indifference curve changes according to the type of goods.

1. Normal Goods: Typically, indifference curves are convex to the origin, meaning that the MRS diminishes as a consumer moves along the curve. This reflects the decreasing willingness to substitute one good for another as the quantity of the first good increases.

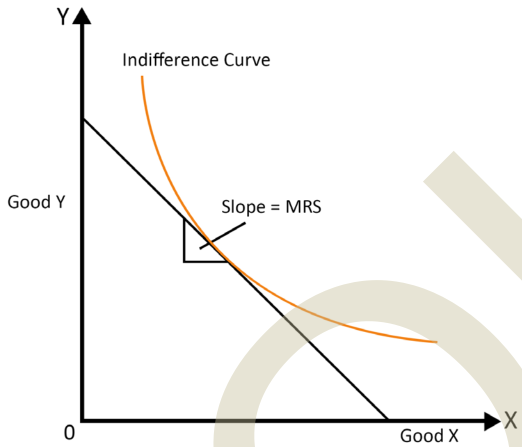


Fig 2.1.1 Normal Goods

2. Perfect Substitutes: When the two goods are perfect substitutes, the MRS is constant. The indifference curve is a straight line, with a constant slope (often 45 degrees). For perfect substitutes, the consumer is willing to trade one good for another at a fixed ratio.

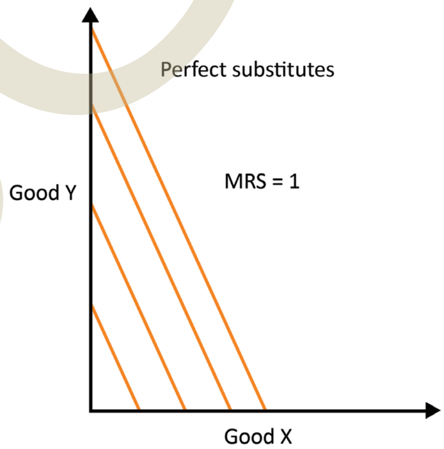


Fig 2.1.2 Perfect Substitutes

3. Perfect Complements: When two goods are perfect complements, the indifference curve forms a right angle. In this case, the consumer requires the goods in fixed proportions, and the MRS is zero when one good is in excess of the other.

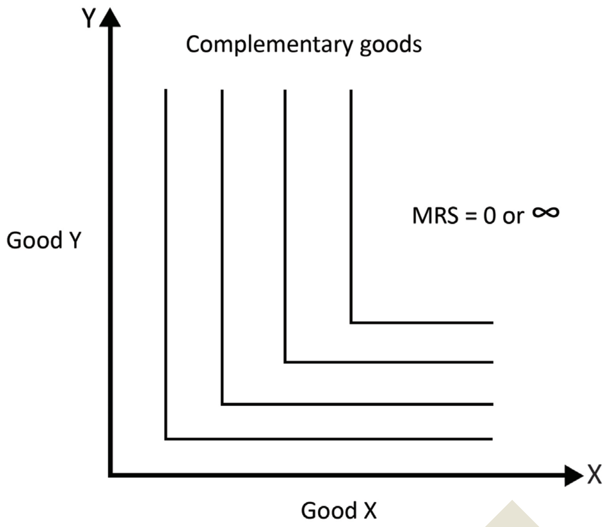


Fig 2.1.3 Perfect Complements

The Law of Diminishing Marginal Rate of Substitution:

The law of diminishing marginal rate of substitution states that as a consumer substitutes one good for another, the willingness to give up one good in exchange for an additional unit of the other decreases. This is reflected by the convex shape of the indifference curve, where the slope becomes flatter as one moves down the curve. The marginal rate of substitution (MRS) is a foundational concept in understanding how consumers make choices between different goods while maintaining the same level of satisfaction or utility.

2.1.1.6 Marginal Rate of Technical Substitution (MRTS) in Economics

The marginal rate of technical substitution (MRTS) refers to the rate at which one input factor, like labour, can be substituted for another input, such as capital, while keeping the output level unchanged. In essence, it measures the trade-off between two inputs that enables a firm to maintain its production without affecting the overall output. MRTS is distinct from the Marginal Rate of Substitution (MRS), which relates to consumer behaviour, showing how a consumer is willing to exchange one good for another while maintaining a constant level of utility. On the other hand, MRTS is focused on the producer's decision-making, specifically the substitution of factors of production. The formula for the marginal rate of technical substitution can be written as:

$$MRTS(L,K) = -\frac{\Delta K}{\Delta L} = \frac{MP_L}{MP_K}$$

where:

K represents capital,

L represents labour,

MP_L and MP_K are the marginal products of labour and capital, respectively.

The MRTS is a ratio of the marginal products of labour and capital. The MRTS is the rate at which one input can be substituted for another to maintain constant output, demonstrating diminishing returns to substitution as resources become constrained.

An isoquant is a graphical representation showing all the combinations of labour and capital that gives the same level of output. The slope of the isoquant indicates the MRTS. For instance, it tells us how much capital can be substituted for a unit of labour (or vice versa) without affecting the output. As the firm adjusts its input mix, the MRTS shows the amount of one input that can be replaced by another to maintain the same output level. If labour increases, capital can be reduced by the value indicated by the MRTS, keeping the production level constant.

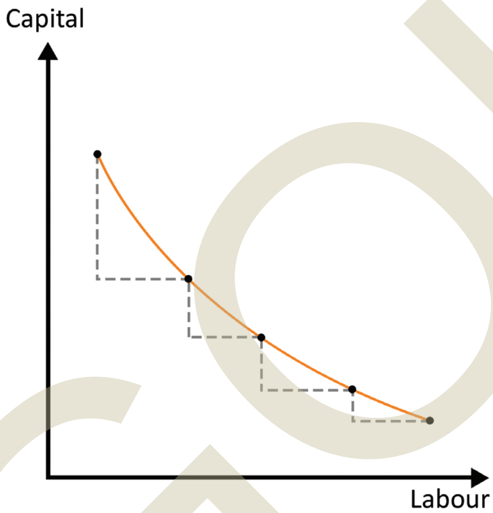


Fig 2.1.4 MRTS

The more of one input the firm substitutes for another, the less efficient that substitution becomes, as the marginal product of the substituting input begins to decline. For example, as labour increases in place of capital, the amount of capital that can be reduced to maintain the same output decreases. Initially, one additional unit of labour might replace several units of capital, but over time, each additional unit of labour replaces fewer units of capital.

The MRTS can be calculated by examining the slope of the isoquant at a given point. The slope at any point on the isoquant curve represents the MRTS between labour and capital. In graphical terms, this slope is $\frac{dL}{dK}$, indicating how much labour can be substituted for capital without affecting output.

MRTS applies to production and measures the trade-off between two factors of production to maintain a constant output level. MRTS relates to producer equilibrium, while MRS pertains to consumer equilibrium. MRTS shows how producers can optimise the combination of inputs - capital and labour - to achieve a fixed level of output. By examining MRTS, firms can make informed decisions on resource allocation, ensuring that they can meet production targets with the most efficient use of resources.

2.1.2 Relationship between Average and Marginal Concepts

The relationship between average and marginal concepts is a key concept in economics that helps to understand how incremental changes impact overall averages. The marginal concept refers to the additional amount resulting from a one-unit change in a variable, while the average concept represents the total amount divided by the number of units.

2.1.2.1 Relationship between Total Product, Average Product and Marginal Product

The relationship between Total Product (TP), Average Product (AP), and Marginal Product (MP) is a fundamental concept in the theory of production. Total Product represents the total quantity of goods or services produced with given inputs. Average Product is calculated by dividing the Total Product by the number of inputs used, offering a per-unit measure of productivity. Marginal Product, on the other hand, refers to the additional output generated by employing one more unit of input.

The Relationship between TP and MP

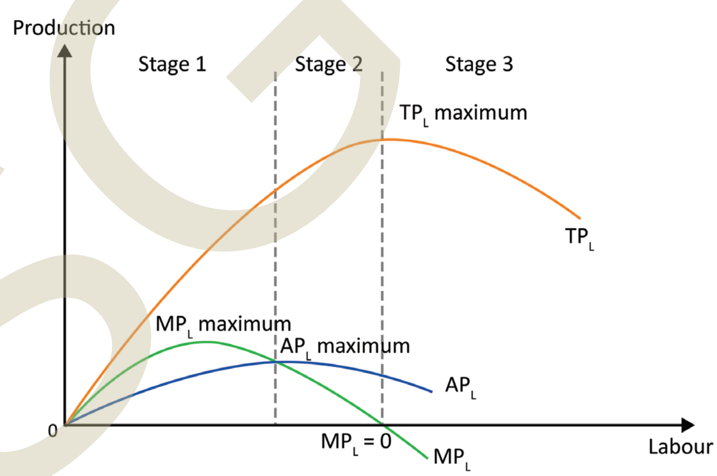


Fig 2.1.5 TP, AP, & MP Curves

The interaction between Total Product (TP) and Marginal Product (MP) shows how changes in inputs affect production output. They happen in these stages:

1. Increasing Marginal Product

When TP increases at an increasing rate, MP also rises. At this stage, the efficiency of additional units of the variable input is high. This is often due to better utilisation of fixed factors as more variable inputs are employed. When the first few workers are added to a production process, they complement existing machinery and space, leading to significant output gains.

2. Decreasing Marginal Product

When TP increases at a diminishing rate, MP starts to decline. This reflects the Law of Diminishing Marginal Returns, which states that as more units of a variable input are added to a fixed input, the additional output per unit of input decreases. Adding a fourth or fifth worker might lead to crowding, where each worker's productivity is lower due to limited equipment or workspace.

3. Maximum Total Product

When TP reaches its peak, MP falls to zero. At this point, adding more units of input no longer increases total output because the fixed inputs cannot accommodate any additional productivity from the variable inputs. A factory might reach full capacity where additional workers cannot contribute effectively, resulting in no further output increase.

4. Negative Marginal Product

When TP begins to decline, MP becomes negative. Beyond the optimal level of input, adding more units of a variable factor leads to inefficiencies that reduce overall production. Overemployment in a small workshop can lead to issues, such as workers interfering with each other, resulting in a decline in total output.

The Relationship between AP and MP

The relationship between Average Product (AP) and Marginal Product (MP) explains how output per unit of input behaves relative to the contribution of additional inputs:

1. Rising AP (MP > AP)

When **MP is greater than AP**, the average product increases. Each additional unit of input contributes more than the current average, which pulls the average up. If the current average output per worker is 5 units, but an additional worker produces 7 units, the average output increases.

2. Maximum AP (MP = AP)

When **MP equals AP**, AP reaches its peak. At this point, the additional output from one more input unit matches the average output, indicating the optimal allocation of inputs. If every worker contributes exactly the same as the average, the average output reaches its maximum and remains unchanged until MP falls below AP.

3. Falling AP (MP < AP)

When **MP is less than AP**, the average product starts to decline. Each additional unit of input contributes less than the average, pulling the overall average down. If the average output is 6 units per worker but an additional worker contributes only 4 units, the overall average decreases.

Even when MP becomes negative, AP stays positive as long as the total product is greater than zero. AP is derived from total output and does not account for declines in MP until total output itself begins to fall. MP typically declines faster than AP because MP reflects changes at the margin (additional units), while AP represents an average over all units.

2.1.2.2 Relationship between Average Cost and Marginal Cost

The relationship between Average Cost (AC) and Marginal Cost (MC) is a fundamental concept in production economics. It describes how the cost of producing additional units of output impacts the overall cost structure of a firm. Average cost is the total cost of production (fixed and variable costs) divided by the number of units produced. It reflects the per-unit cost of production. Marginal cost is the additional cost incurred from producing one more unit of output. It reflects how total cost changes as output changes.

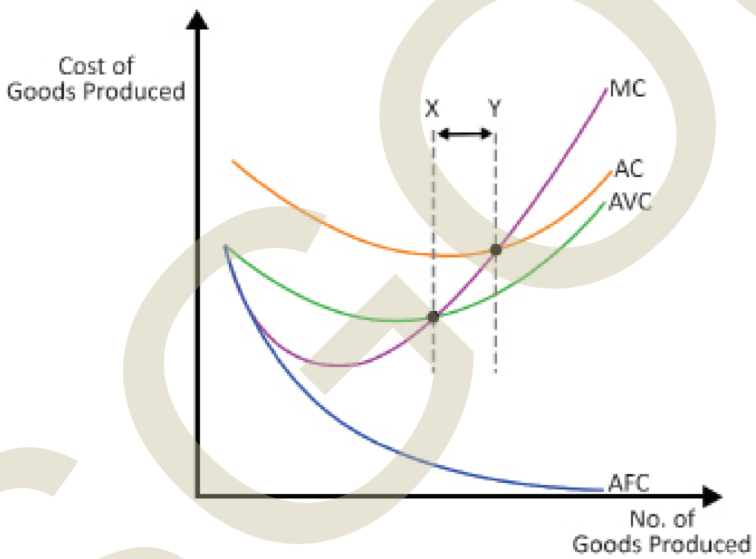


Fig 2.1.6 Cost Curves

The relationship between AC and MC can be understood by analysing how the addition of each unit of output influences the average cost.

a. When Marginal Cost is Less Than Average Cost (MC < AC):

When the marginal cost of producing an additional unit is lower than the average cost, the average cost decreases. This occurs because the additional unit produced is less costly than the existing average, thus pulling the overall average down. Imagine the scenario where your current exam average is 85%, and then you score 90% on the next exam. Your new average will increase (move closer to 90%). The average cost curve slopes downwards as production increases because each additional unit is cheaper than the current average cost. This is typically the phase where a firm benefits from economies of scale, meaning that it becomes more efficient in production as output increases.

b. When Marginal Cost Equals Average Cost (MC = AC):

When the marginal cost is exactly equal to the average cost, it indicates the minimum point of the average cost curve. This is the point of optimal efficiency, where the firm's average cost is the lowest it can be for the level of output produced. This is similar to a scenario in which your exam score exactly matches your average score, meaning your average does not change. At this point, the firm is producing at the most efficient level where the cost per unit of output is minimised. This is also the turning point, beyond which the firm may face diseconomies of scale, meaning increasing output leads to higher costs.

c. When Marginal Cost is Greater Than Average Cost (MC > AC):

When the marginal cost is higher than the average cost, the average cost begins to increase. This is similar to a situation where your exam score falls below your average, thus dragging your overall average down. In production, if the firm produces an additional unit at a higher cost than the average, the overall cost per unit will rise. The average cost curve slopes upwards as production increases. This typically happens after the firm has reached the optimal scale of production and is now experiencing diseconomies of scale, where producing more output leads to inefficiencies and higher per-unit costs. The AC curve is typically U-shaped because it initially decreases as the firm benefits from economies of scale (increased production efficiency) but later increases due to the law of diminishing returns (where additional units become more costly to produce). The MC curve typically slopes upwards after a certain point because, as production expands, it becomes more expensive to produce additional units due to diminishing returns. The intersection of the MC curve and the AC curve marks the minimum point of the AC curve, which is the point of optimal efficiency for the firm.

The Marginal Cost (MC) curve plays a key role in determining the most cost-effective level of output. When MC is below AC, increasing production can lower average cost, whereas if MC exceeds AC, increasing production raises average cost. The MC curve typically rises steeply as firms encounter diminishing returns to scale.

d. Relationship Between MC and AVC:

- When **MC < AVC**, the AVC is falling, as producing an additional unit cost less than the average variable cost. MC will be below AVC
- When **MC = AVC**, the AVC reaches its minimum point.
- When **MC > AVC**, the AVC starts rising, as each additional unit of production adds more to variable cost than the average. MC will be above AVC

The relationship between Average Cost (AC) and Marginal Cost (MC) is key for understanding how costs behave as production levels change. A firm aims to operate at the level where MC = AC, as this minimises cost per unit produced.

- When **MC < AC**, the AC curve is falling, indicating that production is becoming more efficient.
- When **MC = AC**, the firm is operating at optimal efficiency, and the average cost is at its minimum.
- When **MC > AC**, the AC curve begins to rise, reflecting inefficiencies in production.

Understanding the interaction between AC, MC, AVC, and ATC is key for businesses to determine efficient production levels and pricing strategies.

2.1.2.3 Relationship between Total Revenue, Average Revenue, and Marginal Revenue

The relationship between Total Revenue (TR), Average Revenue (AR), and Marginal Revenue (MR) is key in understanding how firms generate revenue and how these changes as output varies, especially in different market structures. Total Revenue refers to the total income earned by a firm from the sale of its goods or services, calculated as the product of price and quantity sold. Average Revenue is the revenue earned per unit of output, often equivalent to the price of the good under perfect competition. Marginal Revenue, on the other hand, measures the additional revenue generated from selling one more unit of a good.

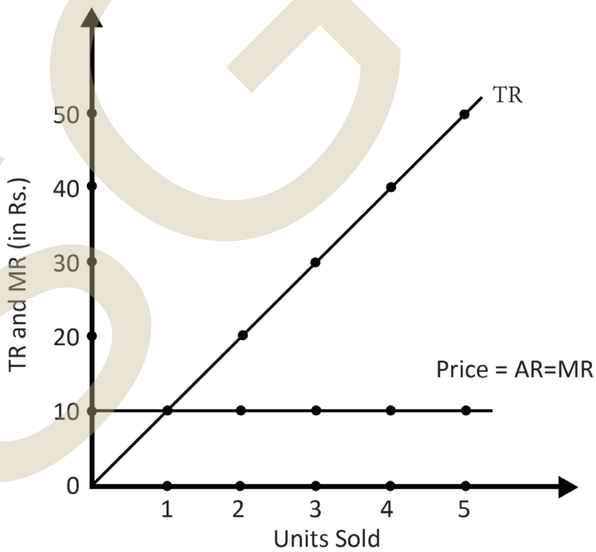


Fig 2.1.7 Revenue in Perfect Competition

For a firm in a competitive market, the price remains constant as output increases.In perfectly competitive markets, AR and MR are constant and equal to the market price.

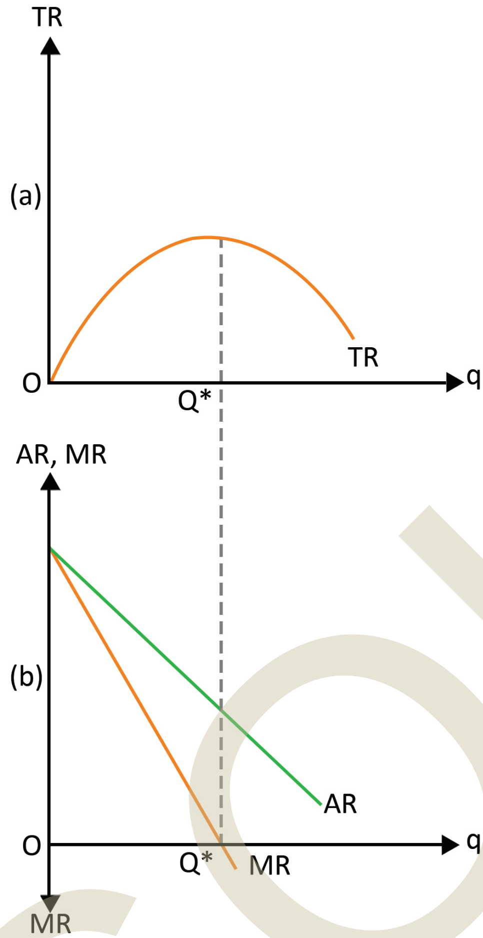


Fig 2.1.8 Revenue in Imperfect Competition

The TR curve increases as output increases, but after a certain point, it starts to flatten (indicating maximum total revenue). The MR curve initially rises, but then it begins to decline and can even become negative after the point where TR is maximised. The AR curve is downward sloping in imperfect competition, but it is always above the MR curve (except when MR = AR).

1. Relationship Between TR and AR:

Since AR is the price per unit, and TR is the total amount earned from selling all units, AR essentially determines TR. If AR increases, TR increases as long as quantity also increases. In a perfectly competitive market, AR is constant, so the TR curve is a straight line that rises linearly with quantity. Example: If AR is constant at ₹10 per unit, TR will increase by ₹10 for each additional unit sold. In imperfect markets, AR decreases as output increases. The firm lowers the price to sell more units, so the TR curve becomes steeper at first and then flattens as it reaches a maximum.

2. Relationship Between TR and MR:

- **MR is the rate of change of TR**: MR shows how TR changes as output increases. If MR is positive, TR increases as more units are sold. If MR is negative, TR starts to decrease as additional output is sold.

- **TR increases when MR is positive**: As long as MR is greater than zero, the total revenue is increasing. This is because each additional unit sold adds to the revenue.

- **TR reaches its maximum when MR is zero**: TR is at its peak when MR is zero. At this point, any additional output does not change the total revenue. This is the point of **revenue maximisation**.

- **TR decreases when MR is negative**: If MR becomes negative, selling additional units decreases the total revenue. This usually happens when the firm is lowering its price so much that it cannot compensate for the reduced price per unit with the extra units sold.

3. Relationship Between AR and MR:

AR and MR are related but have different behaviours as output increases, particularly in imperfect markets. In perfectly competitive markets, AR = MR because the price remains constant. The firm can sell any quantity at the same price, so both AR and MR are constant and equal to the price level. The TR curve is a straight line, and the MR curve is also a straight line (horizontal). In imperfect markets, MR falls faster than AR because the firm must lower the price to sell more units. As output increases:

- **MR declines faster**: Each additional unit of output contributes less to TR because the price must be lowered.

- **AR falls at a slower rate**: The price reduction required to sell an additional unit is reflected in the AR curve, but at a slower rate than the MR curve.

Firms use this relationship to decide the optimal level of output to maximise revenue or profit. For instance, they aim to produce at the level where MR = 0 if revenue maximisation is the goal. The downward-sloping AR and MR curves reflect imperfect competition's pricing and demand dynamics, helping firms' strategies regarding their pricing policies.

Recap

- Marginal utility is the additional satisfaction gained from consuming one more unit of a good, which decreases as more units are consumed.
- The Law of Diminishing Marginal Utility explains that the additional satisfaction from each unit decreases as consumption increases.
- Marginal product refers to the extra output from using one more unit of input, while other factors remain constant.
- Marginal cost represents the additional cost incurred when producing one more unit, calculated by dividing the change in total cost by the change in quantity produced.
- Marginal revenue is the additional income earned from selling one more unit, often diminishing as output increases.
- Marginal Rate of Substitution (MRS) measures the amount of one good a consumer is willing to give up to acquire one more unit of another good, maintaining the same level of utility.
- Indifference curves are convex for normal goods, straight for perfect substitutes, and right-angled for perfect complements.
- When TP increases at an increasing rate, MP rises, reflecting high efficiency. With diminishing returns, MP decreases, leading to inefficiency. When TP peaks, MP falls to zero. If TP declines, MP becomes negative.
- The relationship between AP and MP shows how the output per unit changes with more inputs. AP rises when MP > AP, peaks when MP = AP, and falls when MP < AP. Even when MP is negative, AP remains positive until TP falls.
- Average Cost (AC) and Marginal Cost (MC) relate to how the cost per unit changes as output increases. When MC < AC, AC decreases; when MC = AC, AC reaches its minimum; and when MC > AC, AC increases.
- AVC behaves similar to AC in relation to MC: AVC falls when MC < AVC, AVC reaches its minimum when MC = AVC, and AVC rises when MC > AVC.
- Total Revenue (TR), Average Revenue (AR), and Marginal Revenue (MR) help understand firm revenue. In perfect competition, AR = MR. TR increases with output until a maximum, after which MR declines and can become negative.

- AR and MR are connected: in perfect competition, AR = MR. In imperfect markets, MR declines faster than AR, and firms aim to produce where MR = 0 for revenue maximisation.

Objective Questions

1. What does the Law of diminishing marginal utility state?
2. What is the formula for calculating marginal utility?
3. What is the formula for calculating marginal product?
4. What happens to marginal product as more workers are employed?
5. What is the formula for calculating marginal cost?
6. What is the formula for calculating marginal revenue?
7. What shape do indifference curves take for normal goods?
8. What is the shape of MRS when goods are perfect substitutes?
9. What shape does the indifference curve take for perfect complements?
10. What happens to Marginal Product when Total Product increases at an increasing rate?
11. What is the relationship between Average Product and Marginal Product when Average Product is rising?
12. When is the Average Product maximised?
13. What happens when Marginal Product is negative?
14. What happens when Marginal Cost is less than Average Cost?
15. What happens when Marginal Cost is less than Average Variable Cost?
16. What happens when Marginal Revenue is positive?
17. When does Total Revenue reach its maximum?
18. What is the relationship between Average Revenue and Marginal Revenue in a perfectly competitive market?
19. What happens to Total Revenue when Marginal Revenue is negative?

Answers

1. MU decreases as consumption increases
2. $MU = \frac{\Delta TU}{\Delta Q}$
3. $MP = \frac{\Delta TP}{\Delta X}$
4. Decreases
5. $MC = \frac{\Delta TC}{\Delta Q}$
6. $MR = \frac{\Delta TR}{\Delta Q}$
7. Convex
8. Constant
9. Right angle
10. Increases
11. MP > AP
12. MP = AP
13. Decreases Total Output
14. Average Cost Decreases
15. AVC Falls
16. Total Revenue Increases
17. When MR = 0
18. AR = MR
19. Total Revenue Decreases

Assignments

1. Define marginal utility and explain the Law of Diminishing Marginal Utility with an example.

2. Using the formula for marginal utility, calculate the marginal utility for a consumer with the following total utility for chocolate bars.

Chocolate Bar	Total Utility (TU)
1	10
2	18
3	24
4	28
5	30

3. Calculate the marginal product given the following production data for a company producing widgets with labor input:

Number of Workers	Total Production (Units)
1	20
2	45
3	65
4	80
5	90

4. Explain marginal cost and calculate the marginal cost when total production cost increases from ₹1,000,000 for 24 units to ₹1,050,000 for 25 units.

5. Using the formula for marginal revenue, calculate the marginal revenue when total revenue changes from ₹1,000 for 100 units sold to ₹1,100 for 115 units sold.

6. Define the Marginal Rate of Substitution (MRS). How does the shape of the indifference curve change for normal goods, perfect substitutes, and perfect complements?

7. In a production scenario where 5 workers produce 90 units of output, calculate the marginal product if the output increases to 100 units with an additional worker.

8. Explain how marginal revenue helps companies in making pricing and production decisions, and calculate marginal revenue for a firm where total revenue increases from ₹500 to ₹550 when 10 units are sold and then ₹550 to ₹600 when 12 units are sold.

9. Explain the relationship between Total Product (TP), Average Product (AP), and Marginal Product (MP). Use diagrams to illustrate each stage of the relationship.

10. Discuss the behaviour of Average Cost (AC) and Marginal Cost (MC) when MC is less than AC, equal to AC, and greater than AC. Use graphical representations to explain the U-shape of the AC curve.

11. Explain the relationship between Total Revenue (TR), Average Revenue (AR), and Marginal Revenue (MR) in a perfectly competitive market. Use an example to show how TR increases and reaches its maximum as output increases.

12. Analyse the interaction between Average Cost, Marginal Cost, and Average Variable Cost (AVC) in a firm's production process.

References

1. Mankiw, N. G. (2021). *Principles of Economics* (9th ed.). Cengage Learning.

2. Dewett, K. K., & Verma, S. L. (2007). *Modern Economic Theory* (11th ed.). S. Chand & Co.

3. Varian, H. R. (2014). *Intermediate Microeconomics: A Modern Approach* (9th ed.). W.W. Norton & Company.

4. Pindyck, R. S., & Rubinfeld, D. L. (2018). *Microeconomics* (9th ed.). Pearson Education.

Suggested Readings

1. Rangarajan, C. (2012). *Macroeconomics: A Contemporary Introduction*. Tata McGraw-Hill Education.

2. Nicholson, W., & Snyder, C. (2018). *Microeconomic Theory: Basic Principles and Extensions* (12th ed.). Cengage Learning.

3. Salvatore, D. (2017). *Microeconomics: Theory and Applications* (8th ed.). Oxford University Press.

UNIT 2

Elasticity: Types and Economic Significance

Learning Outcomes

After completing this unit, learners will be able to:

- define price elasticity, income elasticity, and cross elasticity of demand
- differentiate between the types of price elasticity and their significance
- understand the factors influencing elasticity in economic behaviour

Prerequisites

Imagine you are at a local market, browsing through a variety of fruits. You notice that the price of apples has suddenly increased, but the demand for them does not seem to change much. On the other hand, when the price of oranges drops, you see more people rushing to buy them. Have you ever wondered why people still buy apples even when they cost more, but buy more oranges when they are cheaper? This is where the concept of elasticity comes into play.

Elasticity is a simple yet powerful idea that helps us understand how people react to changes in prices or income. It is like a rubber band: some things stretch easily when you pull them, while others stay firm no matter how much you pull. In the world of economics, elasticity helps explain how sensitive people are to changes in prices, income, and the prices of other products.

Think about the last time you received a small raise at work. Did you rush to spend all of it on something you did not need, or did you spend it more wisely? Your response to the increase in income reflects the concept of income elasticity. Just like your spending habits, different products and services react differently to price and income changes.

Keywords

Elasticity, Price Elasticity, Income Elasticity, Cross Elasticity, Demand Responsiveness

Discussion

2.2.1 Elasticity of Demand

Elasticity of demand measures the responsiveness of the quantity demanded of a good or service is to changes in other economic factors, such as price or consumer benefits. When a good demanded has high elasticity, it indicates that consumers are highly sensitive to changes in the associated factor. On the other hand, goods with low elasticity suggest that changes in these factors have minimal impact on quantity demanded. Elasticity can be calculated by dividing the percentage change in one variable (e.g., quantity demanded) by the percentage change in another variable (e.g., price). For instance, if the demand for a product is highly responsive to changes in its price, it is considered elastic. Conversely, if demand remains unaffected by price changes, it is considered inelastic.

Types of Elasticity

1. **Price Elasticity of Demand**: Measures the responsiveness of quantity demanded to changes in price.
2. **Income Elasticity of Demand**: Measures the responsiveness of quantity demanded to changes in price in consumer income.
3. **Cross-Price Elasticity of Demand**: Measures the responsiveness of quantity demanded to changes in price change of another related good (e.g., substitutes or complements).

2.2.2 Price Elasticity of Demand

Price Elasticity of Demand (PED) measures the responsiveness of the quantity demanded of a good or service to a change in its price, all other factors remaining constant.

Mathematically, it is defined as:

$$Price\ Elasticity\ of\ Demand\ (PED) = \frac{\%\ Change\ in\ Quantity\ Demanded}{\%\ Change\ in\ Price}$$

PED provides a way to assess how sensitive consumers are to price changes, which can vary depending on the type of good or service.

Types of Price Elasticities of Demand

The value of PED can vary, leading to five major classifications:

1. **Perfectly Elastic Demand (PED = ∞):** Even a small price change results in an infinite change in quantity demanded. Example: Highly competitive markets for homogeneous goods like agricultural produce, where consumers switch completely to cheaper alternatives.

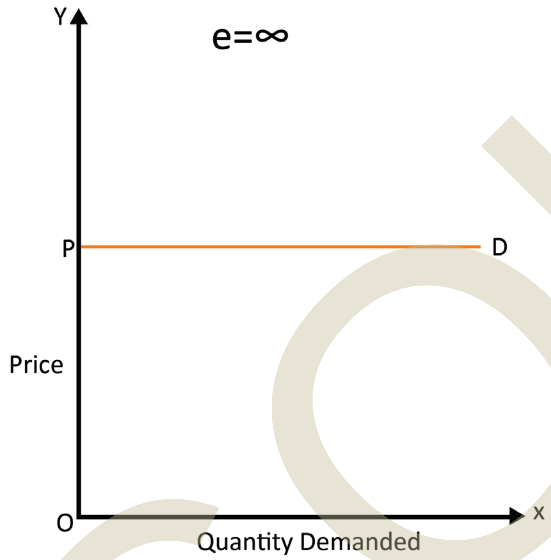


Fig 2.2.1 Perfectly Elastic Demand

2. **Relatively Elastic Demand (PED > 1):** The percentage change in quantity demanded exceeds the percentage change in price. Example: Luxury goods like designer clothing or high-end electronics, where consumers are sensitive to price fluctuations.

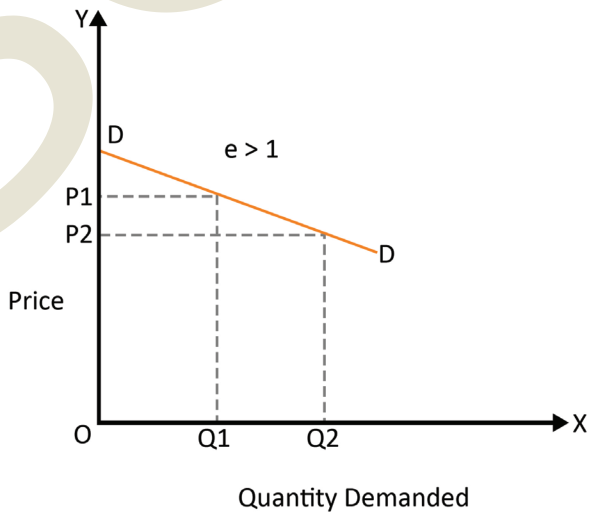


Fig 2.2.2 Relatively Elastic Demand

3. **Unitary Elastic Demand (PED = 1):** The percentage change in quantity demanded is equal to the percentage change in price. Example: Some mid-range goods or services may exhibit unitary elasticity.

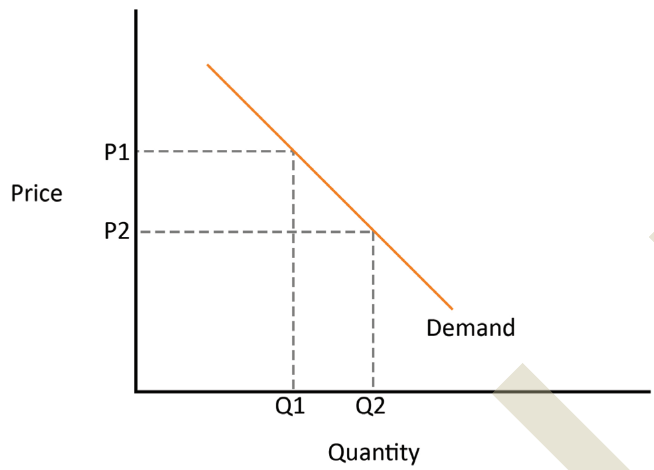


Fig 2.2.3 Unitary Elastic Demand

4. **Relatively Inelastic Demand (PED < 1):** The percentage change in quantity demanded is less than the percentage change in price. Example: Necessities like food, basic utilities, or petrol, where demand does not significantly decrease despite price increases.

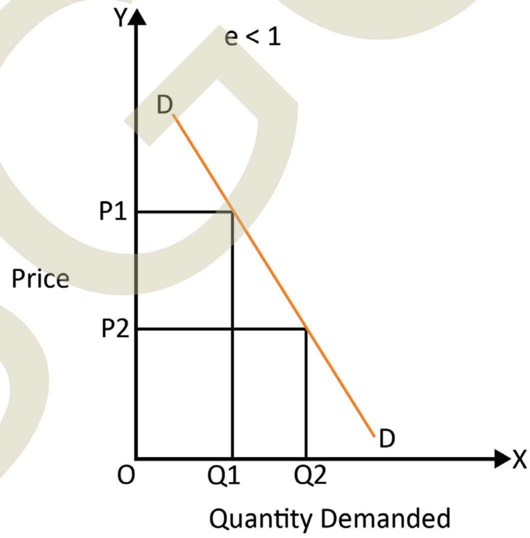


Fig 2.2.4 Relatively Inelastic Demand

5. **Perfectly Inelastic Demand (PED = 0):** The quantity demanded remains constant regardless of price changes. Example: Life-saving medicines, where consumers must purchase them irrespective of price.

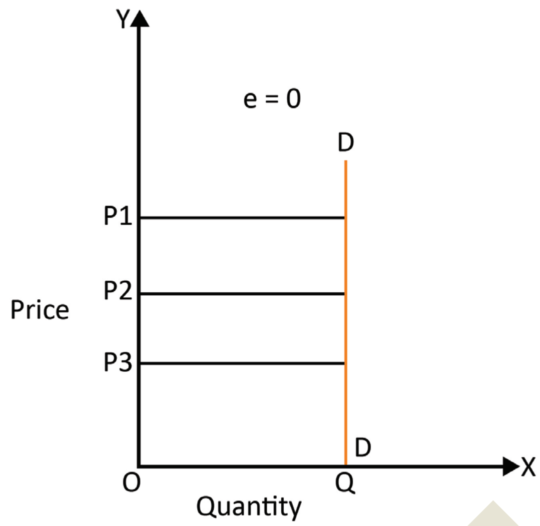


Fig 2.2.5 Perfectly Inelastic Demand

2.2.1.1 Determinants of Price Elasticity of Demand

The elasticity of demand for a product is influenced by various factors, some of which can be influenced by businesses, while others depend on prevailing market conditions. These determinants are as follows:

1. Availability of Substitutes: When suitable alternatives exist for a product, demand tends to be more elastic. Consumers can easily switch to substitute products if the price of one rises. For instance, if people enjoy coffee and tea equally, a price hike in coffee might tempt them to purchase tea instead, reducing the demand for coffee. This elasticity arises because coffee and tea act as substitutes.

2. Urgency or Necessity of Purchase: The degree of necessity affects elasticity significantly. Products considered discretionary, such as a new washing machine when the current one still works, often show more elastic demand. Consumers are likely to delay purchases if prices increase. Conversely, products that are essential or lack viable substitutes, such as life-saving medicines or addictive items like cigarettes, tend to have inelastic demand. For these products, price increases do not significantly reduce the quantity demanded. Similarly, brand-specific products, like a particular printer's ink cartridge, show low elasticity because alternatives are not compatible.

3. Proportion of Income Spent: The share of a consumer's income allocated to a product plays a key role in determining its elasticity. Products that consume a small portion of income generally have inelastic demand, as price changes are less noticeable. On the other hand, items that represent a significant part of the budget tend to have more elastic demand because price changes impact purchasing decisions more acutely. For instance, low-income households are more sensitive to price changes in essential goods than high-income households, where elasticity tends to be lower.

4. Time Frame: Elasticity varies depending on the duration consumers have to adjust to price changes. In the short term, demand is typically less elastic as consumers take time to find alternatives or adapt their habits. Over the long term, however, demand becomes more elastic as consumers identify substitutes or change their consumption behaviour. For example, a sudden rise in fuel prices may not immediately reduce car usage, but over time, people may opt for public transport or fuel-efficient vehicles.

5. Degree of Necessity: Products classified as necessities, such as food and housing, exhibit inelastic demand since consumers must continue buying them regardless of price changes. Substitutes are either unavailable or require significant effort and expense to find. Luxury items, often show higher elasticity. Consumers are more likely to reduce purchases of designer clothing or premium cars when prices rise, as these are not essential for daily living.

6. Brand Loyalty: Consumers loyal to a particular brand are less likely to switch even if prices increase, leading to inelastic demand. However, brand loyalty can change during economic downturns when customers prioritise affordability over brand preference. Income levels also influence brand loyalty which affects demand elasticity.

7. Competition Levels: The degree of competition in the market significantly impacts elasticity. In highly competitive markets, consumers have more options, and demand tends to be elastic. If a company raises its prices, customers can easily shift to alternatives. In less competitive markets, firms face reduced pressure to respond to consumer demand. With fewer substitutes available, demand for their products becomes inelastic, allowing them to charge higher prices without losing substantial market share.

8. Duration of Price Change: The elasticity of demand depends on whether a price change is temporary or prolonged. Short-term fluctuations, such as a one-day sale, may bringout an immediate response, while long-term price increases can lead to a more sustained change in demand as consumers adjust their buying habits. For instance, seasonal variations, such as higher swimsuit prices in summer, often do not discourage consumers from purchasing since demand is tied to seasonal needs.

9. Access to Information: With the availability of information online, consumers can easily compare prices, reviews, and alternatives. Greater access to such information makes demand more elastic, as well-informed buyers are more likely to adjust their purchasing decisions based on value. Businesses must capitalise on their digital presence to stay competitive and ensure positive consumer perception.

2.2.1.2 Methods to Calculate Price Elasticity of Demand (PED)

Price elasticity of demand (E_d) measures the responsiveness of the quantity demanded of a good to changes in its price. Several methods are used to calculate this elasticity, offering insights into consumer behaviour and market dynamics. The key methods are detailed below.

1. Ratio or Percentage Method

The ratio method, developed by Prof. Marshall, is a widely used approach that measures elasticity by comparing the percentage change in quantity demanded to the percentage change in price.

The formula is:

$$E_d = \frac{Percentage\ change\ in\ Quantity\ Demanded}{Percentage\ change\ in\ Price}$$

Alternatively, it can be expressed as:

$$E_d = \frac{\Delta Q}{Q} \div \frac{\Delta P}{P}$$

Where:

- Q: Original quantity demanded
- ΔQ: Change in quantity demanded
- P: Original price
- ΔP: Change in price

For instance, consider the following example:

If the original price of a product is ₹20, and it increases to ₹25, causing the quantity demanded to fall from 10 units to 9 units, calculate the price elasticity of demand. What does the result signify?

Using the formula for elasticity:

$$E_d = \frac{\Delta Q}{Q} \div \frac{\Delta P}{P}$$

Here,

Q= 10

ΔQ = 9-10 = - 1

P = 20

ΔP = 25 – 20 = 5

Substitute these values into the formula:

$$E_d = \frac{-1}{10} \div \frac{5}{20} = \frac{-1}{10} \times \frac{20}{5} = -0.4$$

Since $|E_d|$=0.4 (absolute value), demand is relatively inelastic, meaning the percentage change in demand is smaller than the percentage change in price. In this case, consumers are less sensitive to price changes, possibly due to the product being a necessity.

2. Total Expenditure Method

This method, also developed by Prof. Marshall, assesses elasticity by examining changes in total expenditure before and after a price change. Total expenditure is calculated as the product of price and quantity demanded.

The formula is:

Total Expenditure=Price × Quantity Demanded

Based on the relationship between price, demand, and total expenditure, three cases emerge:

- **Elastic demand (E_d>1)**: Total expenditure increases with a decrease in price.
- **Unitary elastic demand (E_d=1)**: Total expenditure remains constant regardless of price changes.
- **Inelastic demand (E_d<1)**: Total expenditure decreases with a decrease in price.

Consider the following example,

The price of a product changes across three cases as follows:

- Case A: Price decreases from ₹6 to ₹5, and the quantity demanded increases from 10 to 20 units.
- Case B: Price decreases from ₹4 to ₹3, and the quantity demanded increases from 30 to 40 units.
- Case C: Price decreases from ₹2 to ₹1, and the quantity demanded increases from 50 to 60 units.

Using the total expenditure method, determine the elasticity of demand for each case. Explain the outcomes.

Table 2.2.1 Price (₹) and Quantity Demanded

Case	Price (₹)	Quantity Demanded	Total Expenditure (₹)	Elasticity
A	6 → 5	10 → 20	60 → 100	$E_d > 1$ (Elastic)
B	4 → 3	30 → 40	120 → 120	$E_d = 1$ (Unitary)
C	2 → 1	50 → 60	100 → 60	$E_d < 1$ (Inelastic)

The total expenditure method clearly shows the relationship between price changes and consumer responsiveness. In Case A, consumers are highly sensitive, leading to increased total expenditure with a price drop. In Case B, demand changes proportionately to price changes, keeping expenditure constant. In Case C, demand changes only slightly, resulting in reduced total expenditure when prices fall.

3. Point or Geometric Method

This method measures elasticity at a specific point on the demand curve, overcoming the limitations of the ratio and total expenditure methods. The point method calculates elasticity as the ratio of the lower segment of the demand curve below a given point to the upper segment above that point.

The formula is:

$$E_d = \frac{\text{Lower Segment(distance to X} - \text{axis)}}{\text{Upper Segment (distance to Y} - \text{axis)}}$$

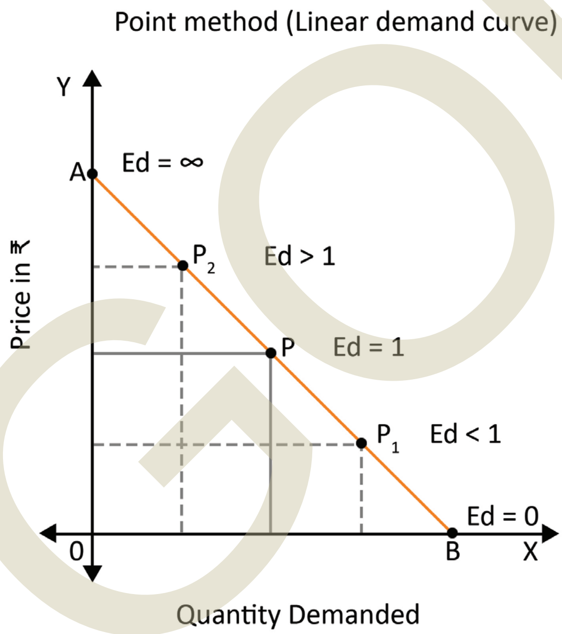


Fig 2.2.6 Point or Geometric Method

- For a **linear demand curve**, elasticity varies along its length:
  - At the midpoint: E_d=1 (Unitary elastic)
  - Above the midpoint: E_d >1 (Elastic)
  - Below the midpoint: E_d < 1 (Inelastic)

Consider a straight-line demand curve where the length from the point of interest to the X-axis is 6 units, and the length to the Y-axis is 4 units. Calculate the elasticity of demand at this point using the point method.

$$E_d = \frac{\text{Lower Segment(distance to X} - \text{axis)}}{\text{Upper Segment (distance to Y} - \text{axis)}}$$

Here,

- Lower segment = 6 units (distance to X-axis),
- Upper segment = 4 units (distance to Y-axis).

Substitute these values:

$$E_d = \frac{6}{4} = 1.5$$

Since E_d>1, demand is relatively elastic at this point. A small price change would lead to a larger proportional change in quantity demanded. This result aligns with consumer sensitivity to price when the curve's slope is relatively flat at the evaluated point. For non-linear demand curves, a tangent is drawn to estimate elasticity at a given point. The calculation follows the same principle as the linear curve.

2.2.3 Income Elasticity of Demand

Income elasticity of demand (Y_{ed}) measures the responsiveness of the quantity demanded for a good to changes in consumer income. It provides insights into how changes in income affect the consumption patterns of different goods, which is critical for businesses and policymakers.

Income elasticity of demand is defined as:

$$Y_{ed} = \frac{\textit{Percentage change in Quantity Demanded}}{\textit{Percentage change in Income}}$$

Alternatively, it can be expressed as:

$$Y_{ed} = \frac{\Delta Q}{Q} \div \frac{\Delta Y}{Y}$$

Where:

- Q: Original quantity demanded
- ΔQ: Change in quantity demanded
- Y: Original Income
- ΔY: Change in Income

Income elasticity of demand explains how changes in consumer income influence the demand for different goods. By examining the magnitude and sign of income elasticity, it becomes possible to classify goods as necessities, luxuries, or inferior items. This classification reflects consumer preferences and spending behaviour, offering insights into how demand shifts across varying income levels. The interpretation of income elasticity is essential for businesses aiming to align product strategies with target market segments and for policymakers planning income-driven economic policies.

- **$Y_{ed} > 1$ (Luxury Goods):** When the income elasticity is greater than 1, the good is classified as a luxury. A percentage increase in income leads to a more than proportionate increase in the quantity demanded.

- **$0 < Y_{ed} < 1$ (Normal Goods):** For normal goods, income elasticity is positive but less than 1. A percentage increase in income results in a less than proportionate increase in demand.

- **$Y_{ed} < 0$ (Inferior Goods):** For inferior goods, income elasticity is negative. As income rises, the quantity demanded decreases because consumers switch to better alternatives.

- **$Y_{ed} = 0$:** Demand is income-inelastic, and changes in income have no effect on the quantity demanded.

Consider the example given below

If a household's income rises from ₹40,000 to ₹45,000, his/her demand for milk increases from 20 litres to 23 litres. Calculate the income elasticity of demand.

Solution:

$$Y_{ed} = \frac{\Delta Q}{Q} \div \frac{\Delta Y}{Y}$$

$Q = 20$

$\Delta Q = 23\text{-}20 = 3$

$Y = 40000$

$\Delta Y = 45000 - 40000 = 5000$

Substituting the values:

$$Y_{ed} = \frac{3}{20} \div \frac{5000}{40000} = \frac{3}{20} \times \frac{40000}{5000} = 1.2$$

Since $0<Y_{ed}<1$, milk is a normal good, and demand moderately responds to income changes.

2.2.4 Cross Elasticity of Demand

Cross elasticity of demand (E_c) measures the responsiveness of the quantity demanded for one good to changes in the price of another related good. It is particularly useful in analysing the relationship between complementary and substitute goods.

The formula for cross elasticity of demand is:

$$E_c = \frac{\text{Percentage change in Quantity Demanded of Good X}}{\text{Percentage change in Price of Good Y}}$$

Alternatively, it can be expressed as:

$$E_c = \frac{\Delta Q_x}{Q_x} \div \frac{\Delta P_Y}{P_Y}$$

Where:

E_c: Cross elasticity of demand.

Q_x: Initial quantity demanded of Good X.

ΔQ_x: Change in quantity demanded of Good X.

P_Y: Initial price of Good Y.

ΔP_Y: Change in price of Good Y.

Cross elasticity of demand helps in understanding the relationship between two goods based on how the change in the price of one affects the demand for the other. The nature of the relationship determines the type of cross elasticity, which can be classified into three main types:

1. Positive Cross Elasticity of Demand (Ec > 0): Substitute Goods

When the cross elasticity of demand is positive, it indicates that the two goods are substitutes. A price increase in one good cause an increase in the demand for the other, as consumers switch to the alternative. For instance, if the price of coffee increases, some consumers may reduce their coffee consumption and purchase more tea, leading to a rise in the demand for tea. The degree of substitutability determines the magnitude of the cross elasticity. For closely substitutable goods like Pepsi and Coca-Cola, the value of E_c tends to be higher compared to goods that are weak substitutes like butter and margarine. Businesses producing substitute goods closely monitor cross elasticity to make competitive pricing decisions.

2. Negative Cross Elasticity of Demand (Ec<0): Complementary Goods

Negative cross elasticity is observed in complementary goods, which are consumed together. A price increase in one good results in a decrease in the demand for the other.

For example, an increase in petrol prices may lead to a drop in car sales, as petrol and cars are complementary goods. The stronger the complementarity between the goods, the more significant the negative elasticity. For instance, the relationship between gaming consoles and video games demonstrates stronger complementarity than the relationship between bread and butter. Firms often use this information to design bundle offers, such as discounts on printers when purchased with computers.

3. Zero Cross Elasticity of Demand (Ec=0): Unrelated Goods

When cross elasticity is zero, it means the goods are unrelated and changes in the price of one good have no effect on the demand for the other. For example, a change in the price of televisions is unlikely to impact the demand for bread, as these goods serve entirely different purposes. Zero cross elasticity highlights the lack of economic connection between goods. While this is less relevant for firms directly managing pricing strategies, it provides insight into consumer behaviour and unrelated market segments.

Consider the following example,

The price of petrol rises from ₹100 to ₹120 per litre, leading to a decrease in car sales from 1,000 to 800 units. Find the E_c. .

Using the formula:

$$E_c = \frac{\Delta Q_x}{Q_x} \div \frac{\Delta P_Y}{P_Y}$$

Here:

- Q_x=1000
- ΔQ_x = 800 – 1000 = -200
- P_Y = 100
- ΔP_Y = 120 – 100 = 20

Substituting the values:

$$E_c = \frac{-200}{1000} \div \frac{20}{100} = -0.2 \div 0.2 = -1$$

Since $E_c < 0$, petrol and cars are complements, and an increase in petrol prices reduces car sales.

Cross elasticity of demand provides valuable insights into the interdependence of goods. Positive elasticity indicates substitutes, negative elasticity shows complements, and zero elasticity reveals unrelated goods.

Recap

- Elasticity of demand measures how quantity demanded responds to changes in factors like price or income.
- Price Elasticity of Demand (PED) measures how quantity demanded changes with price changes.
- Types of PED include perfectly elastic (PED = ∞), relatively elastic (PED > 1), unitary elastic (PED = 1), relatively inelastic (PED < 1), and perfectly inelastic (PED = 0).
- Factors influencing PED include availability of substitutes, urgency of purchase, proportion of income spent, time frame, necessity of the product, brand loyalty, competition, price duration, and access to information.
- PED can be calculated using methods like the ratio or percentage method, total expenditure method, and point or geometric method.
- Income elasticity of demand (Y_{ed}) measures the responsiveness of demand to changes in income and helps classify goods as luxuries, normal, or inferior.
- If $Y_{ed} > 1$, goods are luxury, showing a greater demand response than income change.
- If $0 < Y_{ed} < 1$, goods are normal, with demand increasing less than income.
- If $Y_{ed} < 0$, goods are inferior, with demand decreasing as income rises.
- Cross elasticity of demand (E_c) shows how demand for one good changes with price changes in a related good.
- Positive E_c indicates substitute goods, where price increase in one leads to higher demand for the other.
- Negative E_c indicates complementary goods, where price increase in one reduces demand for the other.
- Zero E_c shows unrelated goods, where price changes have no impact on demand.

Objective Questions

1. What term is used to measure the responsiveness of quantity demanded to changes in other economic factors?
2. What type of elasticity indicates that consumers are highly sensitive to price changes?
3. What is the formula for calculating Price Elasticity of Demand (PED)?
4. If PED is greater than 1, what type of demand is it classified as?
5. If demand is perfectly inelastic, what is the value of PED?
6. What factor influences the elasticity of demand when there are alternatives available for a product?
7. Which type of product typically exhibits inelastic demand?
8. In the case of luxury goods, demand is typically what?
9. What happens to total expenditure when demand is elastic and price decreases?
10. In which method of calculating PED is the percentage change in quantity demanded compared to the percentage change in price?
11. What term is used for measuring elasticity by comparing price and quantity demanded at specific points on the demand curve?
12. What is the relationship between price and total expenditure when demand is inelastic?
13. What measures the responsiveness of the quantity demanded for a good to changes in consumer income?
14. The formula for income elasticity of demand can be expressed as:
15. If income elasticity is greater than 1, the good is classified as:
16. When the income elasticity of demand is negative, the good is classified as:
17. If income elasticity of demand is zero, demand is:
18. Cross elasticity of demand measures the responsiveness of the quantity demanded for one good to changes in the price of:

19. A positive cross elasticity of demand indicates the goods are:
20. A negative cross elasticity of demand indicates the goods are:
21. When cross elasticity is zero, the goods are:
22. The formula for cross elasticity of demand is:
23. A decrease in car sales due to an increase in petrol prices indicates the goods are:

Answers

1. Elasticity
2. Elastic
3. $PED = \frac{\%\ Change\ in\ Quantity\ Demanded}{\%\ Change\ in\ Price}$
4. Relatively Elastic
5. 0
6. Substitutes
7. Necessities
8. Elastic
9. Increases
10. Ratio Method
11. Point Method
12. Decreases
13. Income Elasticity of Demand
14. $Y_{ed} = \frac{\Delta Q}{Q} \div \frac{\Delta Y}{Y}$
15. Luxury Goods

16. Inferior Goods

17. Income-Inelastic

18. Another Related Good

19. Substitutes

20. Complements

21. Unrelated

22. $E_c = \frac{\Delta Q_x}{Q_x} \div \frac{\Delta P_Y}{P_Y}$

23. Complements

Assignments

1. Define elasticity of demand and explain its significance in economic analysis.

2. Calculate the Price Elasticity of Demand (PED) using the percentage method for the following data:

3. Original price: ₹50, New price: ₹60, Original quantity demanded: 100 units, new quantity demanded: 90 units.

4. Differentiate between elastic, unitary elastic, and inelastic demand with examples.

5. Discuss the factors that affect the price elasticity of demand.

6. Calculate the price elasticity of demand using the Point or Geometric method for a linear demand curve where the length from the point of interest to the X-axis is 10 units and the length to the Y-axis is 5 units.

7. Define income elasticity of demand and explain how it is calculated.

8. Explain the classification of goods based on income elasticity of demand with suitable examples.

9. Calculate the income elasticity of demand for a good with the following data: Initial income = ₹30,000, Final income = ₹33,000, Initial quantity demanded = 15 units, Final quantity demanded = 18 units.

10. What is cross elasticity of demand? How does it help businesses understand the relationship between two goods?

11. Calculate the cross elasticity of demand with the following data: The price of petrol increases from ₹90 to ₹100 per litre, and the demand for cars decreases from 1,500 to 1,200 units.

12. Suppose the price of tea increases from ₹50 to ₹55 per kg, and the demand for coffee increases from 200 to 220 units. Calculate the cross elasticity of demand for tea and coffee. What does the result imply about the relationship between the two goods?

References

1. Dewett, K. K., & Verma, S. L. (2007). *Modern Economic Theory* (11th ed.). S. Chand & Co.

2. Mankiw, N. G. (2021). *Principles of Economics* (9th ed.). Cengage Learning.

3. Varian, H. R. (2014). *Intermediate Microeconomics: A Modern Approach* (9th ed.). W.W. Norton & Company.

4. Pindyck, R. S., & Rubinfeld, D. L. (2018). *Microeconomics* (9th ed.). Pearson Education.

Suggested Readings

1. Jha, R. (2010). *Mathematical Economics*. McGraw-Hill Education.

2. Chiang, A. C., & Wainwright, K. (2005). *Fundamental Methods of Mathematical Economics* (4th ed.). McGraw-Hill Education.

3. Simon, C. P., & Blume, L. (1994). *Mathematics for Economists*. W.W. Norton & Company.

4. Nicholson, W., & Snyder, C. (2018). *Microeconomic Theory: Basic Principles and Extensions* (12th ed.). Cengage Learning.

BLOCK 3 Optimization

UNIT 1

Concavity and Convexity-Optimisation of Single/Multi Variable Functions

Learning Outcomes

After completing this unit, the learner will be able to:

- understand concave and convex functions
- apply optimisation techniques to single and multi-variable functions
- discuss critical points to determine maxima and minima

Prerequisites

Imagine you are hiking up a mountain, eager to reach the highest point. As you climb, you notice that some paths seem steep at first but then level out, while others keep rising steadily. You pause to look around have you reached the peak, or is there a higher point ahead? If you take the wrong path, you might end up going downhill instead of climbing higher. Making the right choice requires careful observation and understanding of the terrain.

Now, think about a bowl placed on a table. If you drop a small ball inside, it naturally rolls down and settles at the lowest point. No one has to tell the ball where to go - it simply follows the natural shape of the bowl. But what if the surface were uneven or had multiple dips? The ball might settle in one place, even if there is a deeper point somewhere else.

Just like in hiking or the movement of a ball, many things in life follow patterns of rise and fall. Whether it is predicting the stock market, designing a curved bridge, or adjusting the speed of a vehicle, understanding how things increase or decrease is important. It helps us make better decisions, avoid mistakes, and find the best possible outcomes.

In this unit, we will explore how to study these patterns mathematically. You will learn how to identify shapes, analyse their behaviour, and apply this knowledge to real-world problems. By the end, you will see how simple ideas from nature can be used in economics, business, and everyday decision-making.

Keywords

Concavity, Convexity, Local Maximum, Local Minimum, Critical Points, First-Order Condition, Second-Order Condition, Differentiability, Partial Derivatives, Hessian Matrix, Saddle Point

Discussion

Optimisation is the process or method of finding the maximum and minimum of a function. The optimisation techniques depends upon the types of function, ie whether the function is single variable function or multivariable function. The methods and conditions for optimisation are different for these two types of functions. This chapter deals with the different methods of optimisation and also the application of optimisation in economics.

3.1.1 Convexity and Concavity Function

The calculus is used for deciding the convexity and concavity of a function. A function is said to be concave upward or convex at X= a if the second derivative of the function is positive at x= a.If the second derivative is negative then the function is concave downward (Concave). The slope of the first derivative is irrelevant for concavity. Therefore the condition for convex function is f'' (x) >0 and for concave is f'' (x) < 0. Graph of the convex function lies above the tangent and for concave function lies below the tangent.

The conditions for a convex curve are that $f'(x) > 0$ and f'' (x) >0. $f'(x)$ >0 means that the value of the function tend to increase or the rate of change of y with respect to x is positive. f'' (x) >0 means the slope of the curve tend to increase or is positive. It is given in the figure. 1 and 2. A function is concave when the following conditions are satisfied, $f'(x) < 0$. and f'' (x) < 0. $f'(x) < 0$ means that the value of the function tend to decrease or the rate of change of y with respect to x is negative. f'' (x) < 0 means that the slope of the curve tend to decrease. It is given in figure 3 and 4. The sign of the second order condition determines the convexity or concavity of the function.

Convex functions

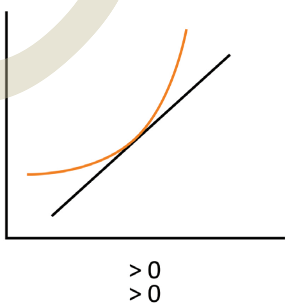


Fig 3.1.1 Convex Function - A

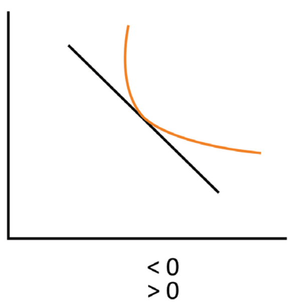
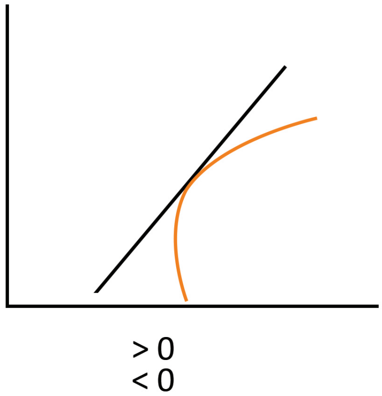


Fig 3.1.2 Convex Function - B

Concave functions



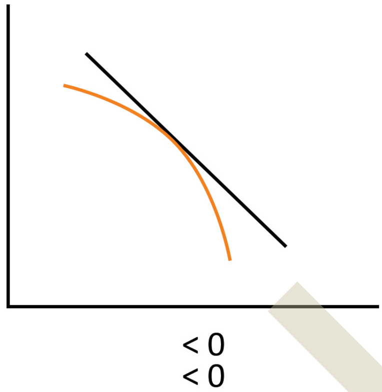


Fig 3.1.3 Concave Function - A

Fig 3.1.4 Concave Function - B

The conditions for concavity and convexity can be summarised as follows at x = a

1	$f'(a) > 0$ $f''(a) > 0$	F(x) Increasing and concave upward or convex
2	$f'(a) < 0$ $f''(a) > 0$	F(x) decreasing and concave upward or convex
3	$f'(a) > 0$ $f''(a) < 0$	F(x) increasing and concave downward
4	$f'(a) < 0$ $f''(a) < 0$	F(x) decreasing and concave downward

Determine the concavity and convexity of the following functions

Example 1

Y= - x²-120x-32

$\frac{dy}{dx} = \frac{d}{dx}$(- x²-120x-32) = -2x-120 =0; x = 60

$\frac{d^2y}{dx^2} = \frac{d}{dx}$(-2x-120) = -2 <0 the second derivative is negative, therefore the given function is concave

Example 2

y = x²+12x-32

$\frac{dy}{dx} = \frac{d}{dx}$ (**x^2+12x-32**) = 2x+12 =0 , x = -6

$\frac{d^2y}{dx^2} = \frac{d}{dx}$ (2x + 12) = 2> 0 the second derivative is positive, therefore the given function is convex

Example 3

Y = $2x^3 - 30x^2 + 126x + 59$

Examine whether the function is convex or concave

$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 30x^2 + 126x + 59) = 6x^2-60x+126 = 0 = 6(x-3)(x-7) = 0$

x=3 and x=7

$\frac{d^2y}{dx^2} = \frac{d}{dx}(6x^2-60x+126) = 12x-60$

at x = 3, $\frac{d^2y}{dx^2}$ = 12 (3) -60 = -24 < 0, => concave at x = 3

at x = 7 $\frac{d^2y}{dx^2}$ = 12 (7) -60 = 24 >0, => convex at x = 7

Example 4

Y = $3x + \frac{1}{x} - 5$

Examine whether the curve is convex or concave

$\frac{dy}{dx} = \frac{d}{dx}(3x+\frac{1}{x} - 5) = 3 - \frac{1}{x^2}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(3 - \frac{1}{x^2}) = \frac{2}{x^3} > 0$ for x>0

Therefore the given function is convex

$\frac{d^2y}{dx^2} = 2/x^3 > 0$ when x > 0: The curve is convex when x is greater than 0.

$\frac{d^2y}{dx^2} = 2/x^3 < 0$ when x < 0: The curve is concave when x is less than 0.

Conclusion:

The curve y = 3x + 1/x - 5 is:

Convex for x > 0

Concave for x < 0

The function is not defined at x = 0, so there is no concavity or convexity at that point.

Extreme Points

An extreme point of a function is a point where the function is at a relative maximum or minimum. The function x=a must be neither increasing nor decreasing at 'a' if the function is at relative maximum or minimum. A function is neither increasing nor decreasing at x=a if its first derivative is equal to zero. The second derivative is used for distinguishing the relative maximum or minimum.

Relative Maximum

The function is at relative maximum at x=a if the following two conditions are satisfied

1. First order condition –The first order derivative must be equal to zero ie $f'(a) = 0$
2. The second order condition – The second derivative must be less than zero . ie $f''(a) < 0$. This condition indicates that the function is concave downward. The graph of the function lies completely below its tangent line at x=a. This is shown in the following figure

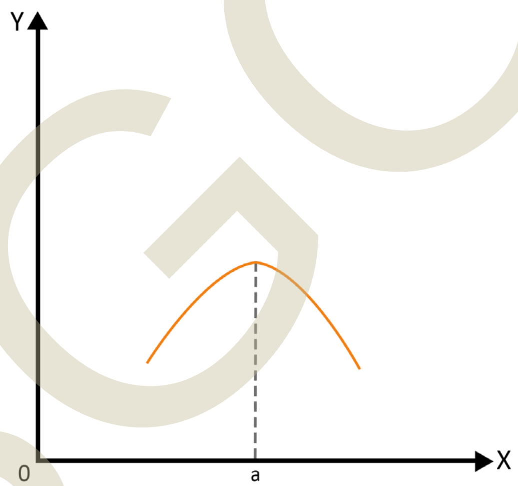


Fig 3.1.5 Relative Maximum

$f'a = 0$

$f''a < 0$

Relative Minimum

The function is at relative minimum at x=a if the following two conditions are satisfied

First order condition –The first order derivative must be equal to zero ie $f'(a) = 0$

1. The second order condition – The second derivative must be greater than

zero. ie $f''(a) > 0$. This condition indicates that the function is concave upward. The graph of the function lies completely above its tangent line at x= a. This is shown in the following figure.

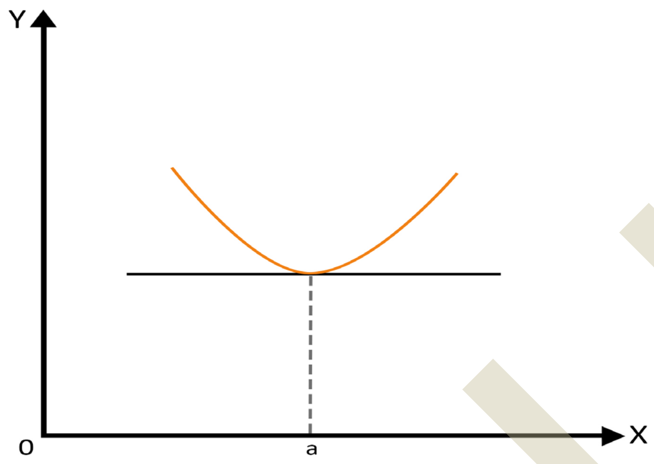


Fig 3.1.6 Relative Minimum

$$f'(a) = 0$$

$$f''(a) > 0$$

Determine the relative maximum and minimum

Example 5

$f(x) = 3x^2-12x+23$

Take the first derivative and set it equal to zero

$f'(x) = 6x-12 = 0$

$6x = 12$

$x = 2$

Take the second derivative and evaluate it at the critical value x=2 to determine the relative maximum or minimum

$f''(x) = 6 > 0$, Therefore it is concave upward

The given function has satisfied the two conditions for the relative minimum and therefore the function is a relative minimum function

Example 6

$f(x) = -4x^2+56x-32$

Take the first derivative and set it equal to zero

$f'(x) = -8x+56 = 0, -8x = -56$

$8x = 56$

$x = 7$

Take the second derivative and evaluate it at the critical value x=7 to determine the relative maximum or minimum

$f''(x) = -8 < 0$, Therefore it is concave downward

The given function has satisfied the two conditions for the relative maximum and therefore the function is a relative maximum function.

Example 7

$f(x) = -5x^3+22.5x^2+420x-85$

Take the first derivative and set it equal to zero

$f'(x) = -15x^2+45x+420 = 0$

$= -15(x^2-3x-28) = 0$

$= -15(x+4)(x-7) = 0$

Therefore, either x+4 =0 or x-7 =0

If x+4 = 0, x = -4

If x-7 = 0, x = 7

Therefore the critical values are -4 and 7

Take the second derivative and evaluate it at the critical value x =-4 and 7 to determine the relative maximum or minimum

$f''(x) = -30x+45$

$f''(x)$ at x = -4 = -30(-4)+45 = 165 >0, The second order condition at x=-4 is greater than zero and therefore it is concave upward and it is a point of relative minimum

$f''(x)$ at x = 7 = -30(7)+45 =-165 < 0, The second order condition at x=7 is less than zero and therefore it is concave downward and it is a point of relative maximum

Example 8

$f(x) = \frac{x^2+9}{x}$

Take the first derivative and set it equal to zero

$f(x)=\frac{x^2+9}{x} = x + \frac{9}{x}$

$f'(x) = 1 - \frac{9}{x^2}$

$x^2 = 9$

From the above equation $x^2 - 9 = 0$, $x^2 = 9$ or $x = \sqrt{9} = \pm 3$

Therefore the critical values are x = +3 and -3

Take the second derivative and evaluate it at the critical value x= +3 and -3 to determine the relative maximum or minimum

$f'(x) = 1 - \frac{9}{x^2}$

$f''(x) = \frac{18}{x^3}$

$f''(x)$ at x= +3 $= \frac{18}{(3)^3} = \frac{18}{27} = \frac{2}{3} > \mathbf{0}$

The second order condition at x=3 is greater than zero and therefore it is concave upward and it is a point of relative minimum

$f''(x)$ at x = -3 $= \frac{18}{(-3)^3} = \frac{18}{-27} = -\frac{2}{3} < \mathbf{0}$

The second order condition at x=-3 is less than zero and therefore it is concave downward and it is a point of relative maximum

Inflection Point

An inflection point is a point where the function changes from concave upward to concave downward or vice versa. The first order condition can be equal to zero, greater than zero or less than zero. It is the second order condition determines the point of inflection. The second order condition must be zero. The point of inflection can be explained clearly with the help of the following figure.

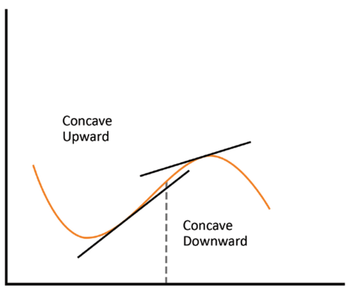


Fig 3.1.7 Point of Inflection

Inflection point at a where, f''a = 0, concavity changes at x=a

The changes in the nature of curve can also be like the following. In the following figures the curvature of the function changes and the various first and second order conditions can be explained as follows.

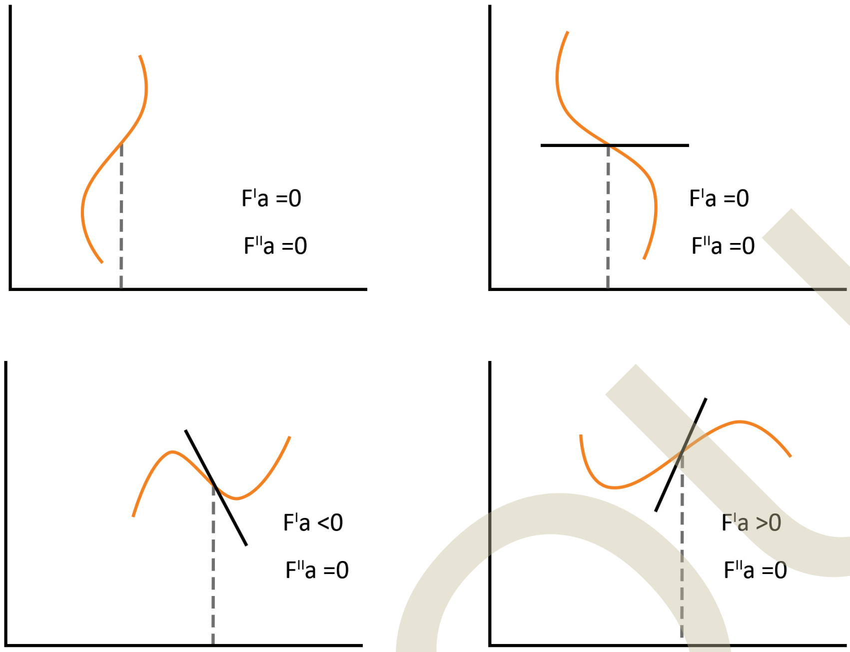


Fig 3.1.8 Point of Inflections

In all these situations the first order condition may differ in accordance with the nature of curvature. But the second order conditions are the same for all. The second order conditions are equal to zero at the point of inflection. The second order condition determines the point of inflection. It is the point where the function changes concavity

Example 9

$f(x) = 2x^3-54x^2+480x-1300$, find relative minimum or maximum

Take the first derivative and obtain the critical value

$f'(x) = 6x^2-108x+480 = 0$

$= x^2-18x + 80 = 0$

$= (x-8)(x-10) = 0$

Therefore x=8, x=10

The critical values are 8, 10

Derive the second derivative

$f''(x) = 12x-108$

The second derivative at critical values 8 and 10 are

At 8, 12 (8)-108 = -12 < 0, therefore concave downward and relative maximum point

At 10, 12(10) -108 = 12 >0 concave upward and relative minimum point

The point of inflection is obtained by setting the second derivative equal to zero ie

$f''(x)$ = 12x-108 = 0

= 12x= 108

x = 108/12 =9

i.e. at x= 9 the given function has an inflection point.

3.1.2 Optimisation

The optimisation is the method of finding the maximum or minimum values of a function. This is one of the important objective of a decision maker. In the case of utility, profit and revenue the objective is the maximisation. Objective of a decision maker with respect to cost is minimisation. The optimisation technique make use of calculus method for finding the maximisation or minimisation decisions of the maker. The optimisation of a function can be done with one independent variable and more than one independent variables. The optimisation methods with one independent variable and more than one independent variables are explained in the following sections.

3.1.2.1 Optimisation of Function-(Optimisation of Single Variable Function)

The optimisation technique which is used for finding the maximum or minimum values of a function can be applied for a function with one independent variable or with more than one independent variable.A single variable function is y = f(x) and multivariable function is y = f (x, t, u).

The optimisation is the process of finding the maximum or minimum value of a function based on two conditions. The two conditions for the maximisation or minimisation of a decision are denoted as first order and second order conditions. The first order condition is the necessary condition and the second order condition is known as the sufficient condition. The first order condition for maximisation and minimisation are same. The second order condition is different for maximisation and minimisation problems.

Conditions for Maximisation

The first order conditions or necessary condition is that the first derivative equals zero, ie, $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0 and second order condition or sufficient condition is that the second derivative must be negative ie $\frac{d^2y}{dx^2}$ < 0 or f''(x) <0 .

The first and second order conditions for maximisation are

1. $\frac{dy}{dx} = 0$ or $f'(x) = 0$
2. $\frac{d^2y}{dx^2} < 0$ or $f''(x) < 0$

Conditions for Minimisation

Similarly for minimisation the conditions are the first order conditions or necessary condition is that the first derivative equals zero, ie, $\frac{dy}{dx} = 0$ or $f'(x) = 0$ and second order condition or sufficient condition is that the second derivative must be positive ie $\frac{d^2y}{dx^2} > 0$ or $f''(x) > 0$.

The first and second order conditions for maximisation are

1. $\frac{dy}{dx} = 0$ or $f'(x) = 0$
2. $\frac{d^2y}{dx^2} > 0$ or $f''(x) > 0$

Maxima and Minima of Function

A function f(x) is said to have a maximum at x=a if f(x) ceases to increase and begins to decrease at this value of x. A function is said to be minimum at x=a if f(x) ceases to decrease and begin to increase at this value of x. This is explained in the following figure

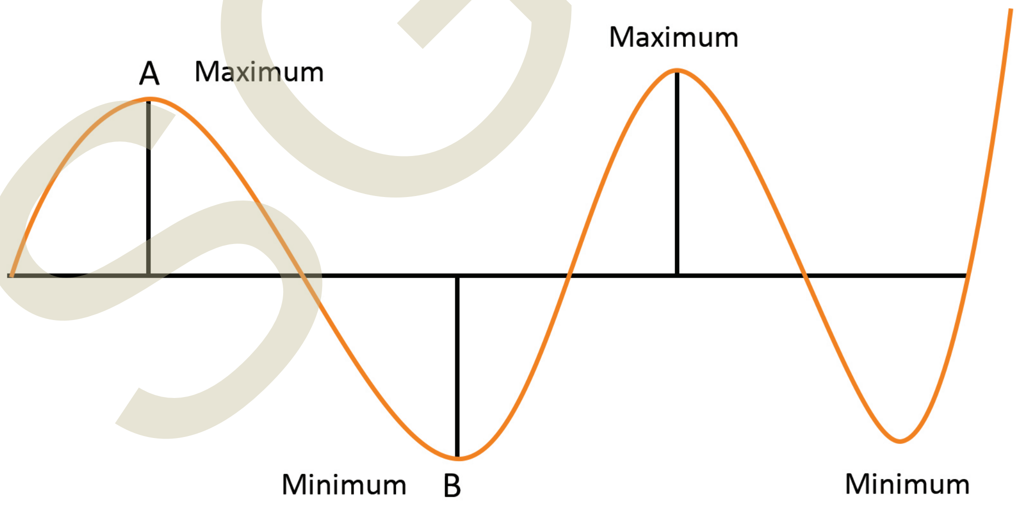


Fig 3.1.9 Maximum and Minimum

In the figure point A is the maximum point since it is the highest value than any value on either side of the A. This means that value increases upto A and it must fall after pont

A has been reached. At point A $\frac{dy}{dx} = 0$ or $f'(x) = 0$. After point A the slope of the curve is decreasing as x increases. That is $\frac{d^2y}{dx^2} < 0$ or $f''(x) < 0$ at point A.

Point B is the minimum point, since it is the lowest value than any value on either side of B. The value decreases upto point B and it must increase after point B. At point B also. At point B also, $\frac{dy}{dx} = 0$ or $f'(x) = 0$. After point B the slope of the curve is increasing. That is $\frac{d^2y}{dx^2} > 0$ or $f''(x) > 0$ at point B. Point A and B are the turning points. At these points the rate of change is constant and therefore $\frac{dy}{dx} = 0$.

Thus a function should satisfy two conditions inorder to decide about maximum or minimum value at a particular point. These conditions are known as order conditions. The order conditions for maximum and minimum are given below.

Conditions for Maximum Value

First order condition or Necessary condition - $\frac{dy}{dx} = 0$ or $f'(x) = 0$

Second order condition or Sufficient condition - $\frac{d^2y}{dx^2} < 0$ or $f''(x) < 0$

Conditions for Minimum Value

First order condition or Necessary condition - $\frac{dy}{dx} = 0$ or $f'(x) = 0$

Second order condition or Sufficient condition - $\frac{d^2y}{dx^2} > 0$ or $f''(x) > 0$

Steps to find the Maximum and Minimum Values of a Function

1. Find the first derivative and equate it to zero $\frac{dy}{dx} = 0$ or $f'(x) = 0$
2. Solve the equation $\frac{dy}{dx} = 0$ or $f'(x) = 0$. Find the values of x. We may get one or more values for x.
3. Find the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$
4. Substitute the values of x from step 2 in the second derivative
5. Values of x which makes the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$ negative, the function is maximum. Values of x which makes the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$ positive, the function is minimum
6. Substitute the respective values of x in the function to get the maximum and minimum values.

Example 10

Find the maximum and minimum values of the function

$y = x^3 - 3x + 1$

$\frac{dy}{dx}$ or $f'(x) = \frac{d}{dx}(x^3-3x+1) = 3x^2-3 = 0$ (1)

From equation (1) $3x^2-3 = 0$ $3(x^2-1) = 0$, $\therefore x = +1, -1$

$f''(x) = \frac{dy}{dx}(3x^2-3) = 6x$ (2)

Substitute x = +1, -1 in equation (2)

When x = +1, 6x= 6> 0. Therefore the given function is minimum at x= 1

When x= -1, 6x= -6<0. Therefor the given function is maximum at x= -1

Substitute X= 1 in the function $Y= x^3-3x+1$ to get the minimum value of function

x= 1, y= $1^3-3(1)+1$ = **-1**

Substitute X= -1 in the function $Y= x^3-3x+1$ to get the maximum value of function

x= -1, y= $-1^3-3(-1)+1$ = **3**

Example 11

Optimise the following function

$y = 3x^3-36x^2+135x-17$

Obtain the first order derivative and set it equal to zero and derive the critical values

$\frac{dy}{dx} = 9x^2-72x+135$

$= 9(x^2-8x+15) = 0$

$= 9(x-3)(x-5) = 0$, ie x=3, x=5

Therefore the critical values are 3 and 5

Take the second derivative

$\frac{d^2y}{dx^2} = 18x-72$

Evaluate the second derivative at its critical values X= 3, x=5

At x=3, 18 (3)-72 = - 18 <0,

The given function is a maximisation function at x=3, since the first order condition and second order conditions for maximisation was satisfied at this point

At x= 5, 18(5)-72 = 18>0

The given function is minimum at x=5, since the first order condition and second order conditions for minimisation was satisfied at this point

Example 12

Total revenue function of a firm is given by TR= $40x-x^2$, x is the output. Find the output at which total revenue is maximum.

TR = $40x-x^2$

$$\frac{dTR}{dx} = \frac{d(40x - x^2)}{dx} = 40 - 2x = 0 \text{ , } x = \frac{40}{2} = 20$$

$$\frac{d^2TR}{dx^2} = \frac{d^2}{dx^2}(40-2x) = -2 < 0$$

The first and second order conditions for maximum are satisfied. Therefore, the profit maximising level of output is 20

Example 13

The cost function of a firm producing x commodities is given as $5x^2-60x+100$. How many units of x to be produced to minimise the cost?

$$\frac{dTc}{dx} = \frac{d(5x^2 - 60x + 100)}{dx} = 10x- 60 = 0, x= \frac{60}{10} = 6$$

$$\frac{d^2Tc}{dx^2} = \frac{d^2}{dx^2}(10x- 60) = 10 > 0$$

The first and second order conditions for minimisation function are satisfied and therefore the cost minimising output is at x =6

Example 14

Given the total cost function TC= $2q^2+8q-50$, find out the level of output at which the average cost is minimum.

TC= $2q^2+8q-50$

$$AC = TC/q = \frac{2q^2 + 8q - 50}{q} = 2q+8- \frac{50}{q}$$

Minimisation of AC requires $\frac{d}{dq}(AC) = 0$, $\frac{d^2}{dq^2}(AC) > 0$

$$\frac{d}{dq}(AC) = \frac{d}{dq}(2q+8-\frac{50}{q}) = 2+0-\frac{50}{q^2} = 2-\frac{50}{q^2} = 2q^2 = 50$$

$$q^2 = \frac{50}{2} = 25, q = \sqrt{25} = \pm 5$$

The second order condition for minimisation is $\frac{d^2}{dq2}(AC) > 0$

$$\frac{d^2}{dq^2}(AC) = \frac{d^2}{dq^2}(2-\frac{50}{q^2}) = 0+\frac{100}{q^3} = \frac{100}{q^3}$$

At q = 5, $\frac{100}{q^3} = \frac{100}{5^3} = \frac{100}{125} = 0.8 > 0$

Therefore the cost minimising output is at q= 5 in which two conditions were satisfied.

Example 15

Given the total cost TC = $4q^3-16q^2+20q+8$, find the level of output at which average variable cost is minimum and also show that MC= AVC at this total output.

TC = $4q^3-16q^2+20q+8$

TVC = $4q^3-16q^2+20q$

AVC = $\frac{TVC}{q}$

AVC = $\frac{4q^3-16q^2+20q}{q}$ = $4q^2-16q+20$

For minimising AVC requires $\frac{d}{dq}$(AVC) =0, $\frac{d^2}{dq^2}$AVC) >0

$\frac{d}{dq}$(AVC) = $\frac{d}{dq}(4q^2-16q+20)$ = 8q-16 = 0, q = 2

$\frac{d^2}{dq^2}$(AVC) >0 = $\frac{d}{dq}(8q-16)$ = 8 > 0

Therefore, the output at which the AVC is minimum is at q= 2

MC = $\frac{d}{dq}$(TC)   = $\frac{d}{dq}(4q^3-16q^2+20q+8) = 12q^2-32q+20$

At q = 2, MC = $12(2^2)-32(2)-10$

= 48-64+20 = 4

At q =2 ,AVC = $4q^2-16q+20$

= $4(2^2)$-16(2)+20 16-32+20 = 4

At q = 2, MC= AVC

Example 16

The total cost function of a firm is given as TC = $200+20x^2$ and the total revenue function is given as TR =10x. Find the profit maximising level of output.

Profit= TR-TC

TR = 80x

TC= $200+20x^2$

Profit p = 80x – (200+20x²) = 80x-200-20x²

$$\frac{dp}{dx} = \frac{d(80x - 200 - 20x^2)}{dx} = 80-40x = 0$$

$$x = \frac{80}{40} = 2$$

$$\frac{d^2p}{dx^2} = \frac{d^2}{dx^2}(80-40x) = -40 < 0$$

The first and second order conditions for maximisation function are satisfied and therefore the profit maximising output is at x =2.

Example 17

Given total revenue $R= 6400Q-20Q^2$ and total cost $C= Q^3-5Q^2+400Q+52000$, maximise the profit for the firm.

Profit = Total revenue- Total cost, ie $\pi = R-C$

$\pi = (6400Q-20Q2) - (Q^3-5Q^2+400Q+52000)$

$= 6400Q-20Q2 - Q^3+5Q^2-400Q-52000$

$\pi = -Q^3-15Q^2+6000Q-52000$

Take the first derivative, set it equal to zero and obtain the value of Q

$$\frac{d\pi}{dx} = -3Q^2-30Q+6000$$

$= -3(Q^2 +10Q-2000) = 0$

$= -3 (Q+50) (Q-40) = 0$

Implies that $(Q+50)(Q-40) = 0$

ie Q = -50, Q = 40

Therefore the critical values are Q = -50 and Q = 40

Take the second derivative and evaluate it at the critical values

$$\frac{d^2\pi}{dx^2} = -6Q-30$$

The second derivative at crtical values Q = -50, Q = 40 are

At Q = -50 , -6(-50) -30 = 270 >0 minimises the profit

At Q = 40, -6(40) -30 = -270 < 0 maximises the profit

Therefore the profit is maximised at Q = 40 and

The maximum profit is

π = -(40) ^{3}-15 (40)^2+6000(40)-52000 = 100000

3.1.3 Optimisation of Multi Variable Function

The conditions for maximisation and minimisation of a function when two or more variables are used is explained in this section. For example the given function is U= 2x+3y^2-xy+y, it includes two variables x and y. In majority of the functions, the variables used will be more than one. This means that the majority of economic activities are using more than one variables in their decision making process. In such cases how the optimisation technique can be used for maximising or minimising a function are explained in this section.

Conditions for Maxima of a Function

The first and second order conditions for maximising a function U= f(x,y) are as follows. Since the function includes more than one variable, the partial differentiation techniques is used for the optimisation. The effect of one variable can be analysed by keeping other variable as constant if a function includes more than one variables. This can be analysed using partial differentiation of a function.

First Order Condition

Given U= f(x,y)

The first order partial differentials are $\frac{\delta U}{\delta x}$ and $\frac{\delta U}{\delta y}$, both these partial differential must be zero, ie

$\frac{\delta U}{\delta x} = 0$, or fx =0

$\frac{\delta U}{\delta y} = 0$, or fy = 0

The second order conditions

$\frac{\delta^2 U}{\delta x^2} < 0$ or fxx <0

$\frac{\delta^2 U}{\delta y^2} < 0$ or fyy< 0

$\frac{\delta^2 U}{\delta x^2} . \frac{\delta^2 U}{\delta y^2} - \left(\frac{\delta^2 U}{\delta x \delta y}\right)^2 > 0$, or fxx.fyy – (fxy)^2 > 0

Conditions for a Minima of a Function

Given U= f(x,y)

The first order partial differentials are $\frac{\delta U}{\delta x}$ and $\frac{\delta U}{\delta y}$, both these partial differential must be zero, ie

$\frac{\delta U}{\delta x} = 0$, or fx =0

$\frac{\delta U}{\delta y} = 0$, or fy = 0

The second order conditions

$\frac{\delta^2 U}{\delta x^2} > 0$ or fxx >0

$\frac{\delta^2 U}{\delta y^2} > 0$ or fyy > 0

$\frac{\delta^2 U}{\delta x^2} . \frac{\delta^2 U}{\delta y^2} - \left(\frac{\delta^2 U}{\overline{\delta x \delta y}}\right)^2 > 0$, or fxx.fyy – $(fxy)^2 > 0$

These are the conditions for the maxima and minima of a function with more than one variable. The first order conditions are the same for maxima and minima for the function with more than one variable also. It is the second order condition which determines the maximisation and minimisation of a function. If the second order condition is also $\frac{\delta^2 U}{\delta x^2}$ = 0 and $\frac{\delta^2 U}{\delta y^2}$ = 0 and $\frac{\delta^2 U}{\delta x^2} . \frac{\delta^2 U}{\delta y^2} - \left(\frac{\delta^2 U}{\overline{\delta x \delta y}}\right)^2 < 0$. It is the condition for saddle point. If the second order condition is $\frac{\delta^2 U}{\delta x^2}$ = 0 and $\frac{\delta^2 U}{\delta y^2}$ = 0 and $\frac{\delta^2 U}{\delta x^2} . \frac{\delta^2 U}{\delta y^2} - \left(\frac{\delta^2 U}{\overline{\delta x \delta y}}\right)^2 = 0$. It is the conditions for no information function.

Example 18

Evaluate the following function for maxima and minima

$z = 48-4x^2-2y^2+16x+12y$

First order conditions

$\frac{\delta Z}{\delta x} = -8x+16 = 0$

8x = 16

$x = \frac{16}{8} = 2$

$\frac{\delta Z}{\delta y} = -4y+12 = 0$

4y =12

$y = \frac{12}{4} = 3$

The function will be maximum or minimum at x= 2, y= 3. Take the second derivative and evaluate it at these points

$$\frac{\delta^2 z}{\delta x^2} = \frac{\delta(-8x+16)}{\delta x} = -8 < 0$$

$$\frac{\delta^2 z}{\delta y^2} = \frac{\delta(-4y+12)}{\delta y} = -4 < 0$$

$$\frac{\delta^2 z}{\delta x \delta y} = \frac{\delta}{\delta y}\left(\frac{\delta z}{\delta X}\right) = \frac{\delta}{\delta y}(-8x+16) = 0$$

$$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 = (-8 \text{ x } -4) - (0)^2 = 32 > 0$$

The second order derivative satisfies the conditions for maximum and therefore the given function is a maximisation function. The given function has a maximum value at x= 2 and y= 3. The maximum value of the function is

$z = 48-4x^2-2y^2+16x+12y$ at x=2, y=3

$48-4(2)^2-2(3)^2+16(2)+12(3)$

$= 48-16-18+32+36$

$= 116-34$

$= 82$

Example 19

Find out whether the following function has minimum or maximum values

$z = \frac{4}{3}x^3 + y^2 - 4x + 8y$

$\frac{\delta Z}{\delta x} = 4x^2 - 4 = 0$

$4x^2 = 4$

$x^2 = \frac{4}{4} = 1$, $x = \sqrt{1}$

x = +1 or -1

$\frac{\delta Z}{\delta y} = 2y + 8 = 0$

$= 2y = -8$

$y = \frac{-8}{2} = -4$

Therefore the critical values are x= 1 or -1 and y = -4

Thus two points will obtain (1,-4) and (-1, -4) evaluate these point in second derivative

$\frac{\delta^2 z}{\delta x^2} = 8x$

At x= (1,-4)

$\frac{\delta^2 z}{\delta x^2} = 8 > 0$

$\frac{\delta^2 z}{\delta y^2} = 2 > 0$

$\frac{\delta^2 z}{\delta x \delta y} = 0$

$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 = 8 \times 2 - (0)\ 2 = 16 > 0$

Therefore, at the points (1.-4) the given function is a minimisation function, since it satisfies the conditions for minimisation at first and second derivatives. The minimum value of the function is

$z = \frac{4}{3}x^3 + y^2 - 4x + 8y$

$= \frac{4}{3}(1)^3 + (-4)^2 - 4(1) + 8(-4)$

$= \frac{4}{3} + 16 - 4 - 32 = \frac{4}{3} + 16 - 36 = \frac{(4 + 48 - 108)}{3} = \frac{-56}{3}$

And at x= (-1, -4)

$\frac{\delta^2 z}{\delta x^2} = -8 < 0$

$\frac{\delta^2 z}{\delta y^2} = 2 > 0$

$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 = -8 \times 2 - (0)\ 2 = -16 < 0$

Since $\frac{\delta^2 z}{\delta x^2} < 0$, $\frac{\delta^2 z}{\delta y^2} > 0$ and $\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 < 0$ The given function will be at saddle point at the x= - 1 and y= -4.

Example 20

Find the maxima and minima for the function

$z = 10x + 20y - x^2 - y^2$

$$\frac{\delta Z}{\delta x} = 10-2x = 0$$

2x =10

$$x = \frac{10}{2} = 5$$

$$\frac{\delta Z}{\delta y} = 20-2y = 0$$

2y=20

$$y = \frac{20}{2} = 10$$

The critical values are x=5 and y=10. Take the second derivative to decide the maxima or minima of the function

$$\frac{\delta^2 z}{\delta x^2} = -2 < 0$$

$$\frac{\delta^2 z}{\delta y^2} = -2 < 0$$

$$\left(\frac{\delta^2 z}{\delta x \delta y}\right) = 0$$

$$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 = -2 \text{ x } -2 - (0)^2 = 4 > 0$$

The given function has satisfied the conditions for maxima and therefore the given function is a maximisation function.

The maximum value of the function at X= 5 and Y= 10 is

$Z= 10x+20y-x^2-y^2$

$Z= 10(5) +20 (10) -(5)^2 -(10)^2$

Z= 50+200-25-100

= 250-125 =125

Example 21

Find the maxima and minima for the function.

$Z= -x^2 +xy -y^2 +2x+y$

$$\frac{\delta Z}{\delta x} = -2x+y+2 = 0 \quad (1)$$

$$\frac{\delta Z}{\delta y} = x-2y+1 = 0 \quad (2)$$

Solve the equation (1) and (2) to obtain the critical values

(1) $= -2x+y+2 = 0$ (1)

(2) x -2 $= -2x+4y-2 = 0$ (3)

(1) – (3) = $0-3y+4 = 0$

$= 3y = 4$

$y = \frac{4}{3}$

Substitute $y = \frac{4}{3}$ in the equation (1)

$-2x+y+2 = 0$

$-2x+ 4/3+2 = 0$

$-2x + \frac{4+6}{3} = -2x + \frac{10}{3} = 0$

$2x = \frac{10}{3}$

$x = \frac{10}{6} = \frac{5}{3}$

Therefore the critical points are at $x = \frac{5}{3}$ and $y = \frac{4}{3}$. Evaluate it at the second derivative

$\frac{\delta^2 z}{\delta x^2} = -2 < 0$

$\frac{\delta^2 z}{\delta y^2} = -2 < 0$

$\left(\frac{\delta^2 z}{\delta x \delta y}\right) = 1$

$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 = -2 \times -2 - (1)^2 = 3 > 0$

The second derivative satisfies the conditions for maximisation and therefore the given function is maximum at $x = \frac{5}{3}$ and $y = \frac{4}{3}$

$z = -x^2 + xy - y^2 + 2x + y$

$= \left(-\frac{5}{3}\right)^2 + \frac{5}{3} \times \frac{4}{3} - \left(\frac{4}{3}\right)^2 + 2\left(\frac{5}{3}\right) + \frac{4}{3}$

$= -\frac{25}{9} + \frac{20}{9} - \frac{16}{9} + \frac{10}{3} + \frac{4}{3}$

$$= \frac{-25+20-16+30+12}{9} = \frac{62-41}{9} = \frac{21}{9} = 2.33$$

Example 22

A firm producing two goods x and y has the profit function $\pi = 64x-2x^2+4xy-4y^2+32y-14$, find the profit maximising level of output.

First order conditions are

$$\frac{\delta Z}{\delta x} = 64 - 4x+4y = 0 \quad (1)$$

$$\frac{\delta Z}{\delta y} = 4x-8y+32 = 0 \quad (2)$$

Solve the equations (1) and (2)

4x-4y = 64 (3)

-4x+8y =32 (4)

(3) + (4) = 4y = 96 , $y = \frac{96}{4} = 24$

Substitute y =24 in equation (3)

4x-4 (24) = 64

4x- 96 = 64

4x = 160

$$x = \frac{160}{4} = 40$$

$$\frac{\delta^2 z}{\delta x^2} = -4 < 0$$

$$\frac{\delta^2 z}{\delta y^2} = -8 < 0$$

$$\left(\frac{\delta^2 z}{\delta x \delta y}\right) = 4$$

$$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\delta x \delta y}\right)^2 = -4 \text{ x } -8 - (4)^2 = 3\ 2-16 = 16 > 0$$

The second derivative satisfies the conditions for maximisation and therefore the given function is maximum at x= 40 and y= 24

The maximum profit is

$\pi = 64x-2x^2+4xy-4y^2+32y-14$ at x= 40 and y= 24

$\pi = 64(40)-2(40)^2+4(40)(24)-4(24)^2+32(24)-14$

$=2560-3200+3840-2304+768-14 = 1650$

Example 23

A monopolists sells two products x and y for which the demand functions are $X = 25-0.5Px$ and $Y= 30-Py$ and the combined cost function is $TC= X^2+2xy+y^2+20$. Find the profit maximising level of output for each products, (b) the profit maximising price for each product and (c) the maximum profit

Profit (π)= TR1+TR2-TC

TR1 = PxX, TR2 =PyY

Therefore

Profit = PxX+PyY –TC

Given X= 25-0.5Px

Px= 50-2x (25- X = 0.5Px, $\frac{25}{0.5} - \frac{x}{0.5}$ = Px , 50-2x = Px)

PxX = (50-2x)X

Given Y= 30-Py

Py = 30-y

PyY = (30-Y)Y

$\Pi = (50-2x)X + (30-y)y - (X^2+2xy+y^2+20)$

$\Pi = 50X-3X^2+30Y-2Y^2-2XY-20$

Take the first order condition of Π with respect to X and Y

$\frac{\delta \Pi}{\delta X} = 50-6X-2Y$ (1)

$\frac{\delta \Pi}{\delta X} = 30-4y-2x$ (2)

Equation (1) and (2) can be written as

50= 6x+2y (3)

= 2x+4Y (4)

Equation (3) x 2 = 100 = 12x+4y (5)

Equation (5) – (4) = 70 = 10X, X= 7

Substitute X = & in Equation (4)

30 = 2(7) +4Y, 30-14 = 4Y, 16 = 4Y, Y= 4

Therefore X= 7 and Y = 4

Take the second order conditions

$$\frac{\delta^2\pi}{\delta x^2} = -6$$

$$\frac{\delta^2\pi}{\delta x^2} = -4$$

$$\frac{\delta^2\pi}{\overline{\delta x}\delta y} = -2$$

$$\frac{\delta^2 z}{\delta x^2} \cdot \frac{\delta^2 z}{\delta y^2} - \left(\frac{\delta^2 z}{\overline{\delta x}\delta y}\right)^2 = -6x\ -4 - (-2)^2$$

$= 24-4 = 20 > 0$

The second derivative satisfies the conditions for maximisation and therefore the given function is maximum at X= 7 and Y= 4

The maximum profit is

$\Pi = 50X-3X^2+30Y-2Y^2-2XY-20$

$= 50(7) -3(7^2) +30 (4) -2(4^2) -2(7)(4) -20 = 215$

Recap

- Optimisation - The process or method of finding the maximum and minimum of a function
- Convex function – If the function tend to increase or the rate of change is positive . The conditions are $f'(x) > 0$ and f'' (x) >0
- Concave function - If the function tend to decrease or the rate of change is Negative . The conditions are $f'(x) > 0$ and f'' (x) <0
- An extreme point of a function is a point where the function is at a relative maximum or minimum. The function x=a must be neither increasing nor decreasing at a if the function is at relative maximum or minimum. A function is neither increasing nor decreasing at x=a if its first derivative is equal to zero
- Relative maximum - The first order derivative must be equal to zero ie $f'(a) = 0$. The second derivative must be less than zero. ie $f''(a) < 0$. This condition indicates that the function is concave downward
- Relative minimum - The first order derivative must be equal to zero ie $f'(a) = 0$. The second derivative must be greater than zero. ie $f''(a) < 0$. This condition indicates that the function is concave upward
- An inflection point is a point where the function changes from concave upward to concave downward or vice versa. The first order condition can be equal to zero, greater than zero or less than zero. It is the second order condition determines the point of inflection. The second order condition must be zero
- A function f(x) is said to have a maximum at x=a if f(x) ceases to increase and begins to decrease at this value of x. A function is said to be minimum at x=a if f(x) ceases to decrease and begin to increase at this value of x. The conditions are $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$ for maximum and $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} > 0$ for minimum for single variable function

Objective Questions

1. The convexity of the function requires the second order condition must be -------
2. The concavity of the function requires the second order condition must be -----

3. --------technique is used for finding the maximum and minimum of a function

4. The conditions for maximising a function are -----

5. Point of inflexion is the point at which ----------------

Answers

1. Greater than zero

2. Less than zero

3. Optimisation

4. $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$

5. $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

Assignments

1. Define optimisation.

2. Explain the conditions for convexity.

3. Explain the conditions for concavity.

4. What the conditions for maximisation of a function?

5. What are the conditions for minimisation of a function?

6. Explain the application of optimisation without constraint in Economics.

References

1. Chiang, A. C., & Wainwright, K. (2013). *Fundamental Methods of Mathematical Economics* (4th ed.). McGraw-Hill.

2. Varian, H. R. (1992). *Microeconomic Analysis* (3rd ed.). W. W. Norton & Company.

3. Simon, C. P., & Blume, L. (1994). *Mathematics for Economists*. W. W. Norton & Company.

Suggested Readings

1. Mehta, B. C., & Madnani, G. M. K. (2017). *Mathematics for Economists*. Sultan Chand & Sons.

2. Sundaram, R. K. (1996). *A First Course in Optimisation Theory*. Cambridge University Press.

3. Allen, R. G. D. (1974). *Mathematical Analysis for Economists*. Macmillan.

UNIT 2

Constrained Optimisation

Learning Outcomes

After learning this unit, the learner will be able to:

- understand the Lagrange multiplier method for constrained optimisation
- know the economic significance of the Lagrange multiplier
- apply constrained optimisation to utility maximisation, cost minimisation, and profit maximisation problems

Prerequisites

Imagine you are planning a road trip with your friends. You have a destination in mind, but there is a catch - your car's fuel tank can only hold a limited amount of gas. You want to make the most of your journey, visit as many interesting spots as possible, and still ensure you do not run out of fuel before reaching your destination. How do you decide the best route to take? How do you balance your desire to explore with the constraints you have? This is a classic example of making the best decisions within limits, a problem we often face in life, whether it is managing time, resources, or even money.

Now, think about a farmer who wants to grow the best possible crop. He has a fixed amount of land, a limited budget for seeds and fertilizers, and only so much water available for irrigation. His goal is to maximize his harvest, but he has to work within these constraints. How does he decide how much to invest in each resource to get the best outcome? This is another real-life situation where we need to find the optimal solution while respecting the boundaries we are given.

In both these scenarios, the challenge is to achieve the best possible result - whether it's enjoying your road trip or maximizing a harvest - while working within certain limitations. This is where the concept of constrained optimisation comes into play. It is a powerful tool that helps us make the best decisions when we're faced with restrictions, whether in everyday life, business, or even science and engineering.

In this unit, we will explore how to solve such problems using a method called the Lagrange Multiplier. By the end of this unit, you will see how this method can be applied to various real-world situations, from managing resources to making financial decisions.

Keywords

Constrained Optimisation, Lagrange Function, Lagrange Multiplier, Constraint Equation, Objective Function, First-Order Conditions, Second-Order Conditions, Utility Function, Budget Constraint, Cost Function, Profit Function, Marginal Utility

Discussion

3.2.1 Constraint Optimisation

In optimisation, constrained optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables. Constraint optimisation mainly deals with the restrictions imposed the availability of resources and other requirements. Two methods are used in constraint optimisation technique. They are substitution method and Lagrange Method. If the constraint optimisation problem has only one constraint and two expalnatory variables then substitution method can be used. If the constrained problem has only equality constraints and two or more than two variables, the method of Lagrange multipliers can be used to convert it into an unconstrained problem. The method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints.

3.2.1.1 The Substitution Method

The substitution method can be used when the function has one constraint and two explanatory variables only. It is a two step procedure. In the first step the constraint is reduced in terms of one decision variable. This new equation is substituted in to the objective function. Thus the original constraint form is transformed into unconstraint function. Then the maximisation or minimisation condition for single variable function is applied. But it has the limitations like, it can be applied when there is only one constraint and two decision variables. When the number of decision variables are two or more and deals with more complicated constraints the Lagrange method is used. The substitution method is explained through the following example,

Example 1

Maximise

Z= XY +2X (1)

Subjected to the constraint

X+3Y = 18 (2)

From equation (2)

X= 18-3Y

Substitute this into equation (1)

Z= (18-3y)Y +2(18-3Y)

Z= $18Y-3Y^2 + 36-6Y$

Z= $12Y+36 -3Y^2$

This can be written as

Z= $-3Y^2+12Y +36$

The objective function (1) is reduced to one variable function. Apply the optimisation conditions for single variable function. ie

$\frac{dZ}{dy}$ = 0 = -6Y+12= 0

Y= $\frac{12}{6}$ = 2

The second order condition

$\frac{d^2z}{dx^2}$ = -6 <0 Therefore the second order condition for maximisation is also satisfied at Y= 2

Therefore at y=2, X= 18-3Y = 18-3(2) = 18-6 = 12

The critical values of the function are X= 12 and Y= 2

The maximum value of Z = XY+2X = 12(2)+2(12) = 24+24 =48

Therefore in the substitution method the constraint is reduced in terms of one explanatory variable and this function is substituted into the objective function. Then the objective function is optimised. The optimisation condition for single variable function is applied to get the critical values. But this method has its own limitation. It is applicable when the number of variables are two only. The lagrange method is used for optimisation when the number of variables are two or more and the constraints includes more complicated information. This method is explained below.

3.2.1.2 Lagrange Method

Another method for the constraint optimisation is the Lagrange method. This method is named after the mathematician Joseph Louis Lagrange. This method is used for the optimisation of constraint functions with two or more explanatory variables. The method of solution is given below.

The Method of Solution

This method is used when we want to find the optimal value of the objective function subject to the constraint. This method is used for finding the optimal values of utility function, cost function, profit function etc. In all these function the objective function is optimised based on certain constraints. For example the utility maxisation is possible on the basis of budget constraints. Cost minimisation and profit maximisations are based on the constraints of production function. The optimisation based on the Lagrange method can be explained as follows. Firstly for optimisation there should be an objective function . The objective function may be maximisation or minimisation problem and secondly there must be a constraints related with the objective function

Suppose

The objective function is given as

$Z = f(x, Y)$

The constraint is given by

$C = P_1 X + P_2 Y$

The Lagrange method is explained as follows- Steps

1. Transform all the constraints into a form that equals Zero. The given constraint $C = P_1 X + P_2 Y$ is transformed as $C - P_1 X - P_2 Y = 0$

   $C - P_1 X - P_2 Y = 0$ (1)

   This can be written as $P_1 X + P_2 Y - C = 0$ also.

2. $P_1 X + P_2 Y$ Multiply the transformed constraint using λ (Lamda). multiplying the transformed constraint by a constant which is termed as λ ie find λ C-P1 X-P2Y

   The λ is known as Lagrange multiplier.

   $\lambda\ C - P_1 X - P_2 Y$ (2)

3. New Objective function-The third step is creating a new objective function called Lagrangean function The new objective function is created by adding the objective function with the transformed constraint which is multiplied by λ. That is add the two function – Objective function $Z = f(x, Y)$ and the transformed constraint λ (C-P1 X-P2Y). Then this function can be written $L = f(x,y) + \lambda (C - P_1 X - P_2 Y)$. This function is known as Lagrange function.

The λ in this function measures the marginal change in the value of objective function from a one unit change in the constraint. Step three is

$$L=f(x,y) + \lambda (C-P_1 X-P_2 Y) \quad (3)$$

This function need to be optimised (maximised or minimised) based on the following conditions

Conditions for Optimisation using Lagrange Methods

The Lagrange function which is given as $L=f(x,y) + \lambda (C-P_1 X-P_2 Y)$ includes three variables ie X,Y and λ. Inorder to optimise the function, the first and second order condition must be satisfied.

First Order Condition

The first order condition is that differentiate the Lagrange function with respect to the variables included. In the above given Lagrange function the variables are X,Y and λ. That is differentiate L with respect to X,Y and λ and equate it to zero The first order differentials are

$$\frac{\delta L}{\delta X} = 0$$

$$\frac{\delta L}{\delta Y} = 0$$

$$\frac{\delta L}{\delta \lambda} = 0$$

The first order condition for the lagrange function $L=f(x,y) + \lambda (C-P_1 X-P_2 Y)$ is as follows

$$\frac{\delta L}{\delta X} = fx- \lambda P_1 = 0$$

$$\frac{\delta L}{\delta Y} = fy- \lambda P_2 = 0$$

$$\frac{\delta L}{\delta \lambda} = C-P_1 X-P_2 Y = 0$$

The first order condition for maximisation and minimisations are the same.

Second Order Condition

The second order condition is based on the bordered Hessian determinant $\overline{H}$. Hessian matrix is the square matrix of second order partial derivative of a function. Hessian determinant

Obtain the bordered Hessian determinant $[\overline{H}]$

For maximum the condition is that

$$[\overline{H}] = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} > 0$$

For minimization

$$[\overline{H}] = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} < 0$$

Significance of the Lagrange Multiplier- λ

The Lagrange multiplier λ approximates the marginal impact on the objective function caused by a small change in the constant of the constraint. for instance, a 1-unit increase (decrease) in the constant of the constraint would cause Z (the objective function) to increase (decrease) by approximately by λ units. Lagrange multipliers are often referred to as shadow prices. In utility maximization subject to a budget constraint the marginal utility of an extra dollar of income is measured by Lagrange multiplier.

(11) consider the following functions

Maximise Z= f(x,y)

Subject to

g(x,y) =0

First Order Condition

Step 1- Create a new function- Introduce a new variable λ (lambda). Multiply the constraint by the new variable λ. Add objective function and the constraint

L= f(x,y)+ λ g(x,y)

λ measures the marginal change in the value of the objective function resulting from a one unit change in the value of constraint

Second Order Condition

Obtain the bordered Hessian determinant $[\overline{H}]$

For maximum the condition is that

$$[\overline{H}] = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} > 0$$

Example 1

Use Lagrange Multiplier to optimise the following function subjected to the given constraint. Estimate the effect on the value of the objective function from a 1 unit change in the constant of the constraint.

$Z = 4x^2 - 2xy + 6y^2$ Subjected to $x + y = 72$

The Lagrange function $L = 4x^2 - 2xy + 6y^2 + \lambda(72 - x - y)$

$$\frac{\delta L}{\delta X} = 8x - 2y - \lambda = 0 \qquad (1)$$

$$\frac{\delta L}{\delta Y} = -2x + 12y - \lambda \qquad (2)$$

$$\frac{\delta L}{\delta \lambda} = 72 - x - y = 0 \qquad (3)$$

Substract equation (2) from (1) = $10x - 14y = 0$

Therefore $x = 1.4y$

Substitute $x = 1.4y$ in equation (3)

$= 72 - 1.4y - y = 0$

$72 = 1.4y + y$

$2.4y = 72$

$$y = \frac{72}{2.4} = 30$$

$x = 1.4y = 1.4 \times 30 = 42$

Substitute in equation (1) which will give the value of λ

$\lambda = 8(42) - 2(30)$

$\lambda = 336 - 60 = 276$

The critical values are $x = 42$, $y = 30$ and $\lambda = 276$

The value of $L = 4(42)^2 - 2(42)(30) + 6(30)^2 + 276(72 - 42 - 30) = 9936$

$\lambda = 276$ means a one unit increase in the constant of the constraint will lead to an increase of approximately 276 increase in the value of objective function L = 10212 (approximately)

Example 2

Maximise

$Z = f(x,y)$

Subjected to

$C = P_1x - P_2y$

The constraint can be expressed as $C - P_1x - P_2y = 0$

Step 1 Create a new function L by adding the objective function and constraint multiplied by λ

$L = f(x,y) + \lambda (C-P_1x-P_2y)$

Step 2- First order condition- Obtain the partial derivative with respect to x y and λ

$$\frac{\delta L}{\delta X} = fx- \lambda P1 =0$$

$$\frac{\delta L}{\delta Y} = fy- \lambda P2 =0$$

$$\frac{\delta L}{\delta \lambda} = C-P_1x-P_2y \ =0$$

Second order condition is that

$$[\overline{H}] = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{vmatrix} > 0$$

Example 3

Maximise $Y= X_1X_2+2X_1$ (1)

Subjected to

$X_1+2X_2 =20$ (2)

$L= X_1X_2+2X_1 + \lambda(20- X_1-2X_2)$

The first order condition requires that

$$\frac{\partial L}{\partial x_1} =0 = X_2+2 - \lambda= 0 \quad (3)$$

$$\frac{\partial L}{\partial x_2} =0 = X_1 - 2\lambda = 0 \quad (4)$$

$$\frac{\delta L}{\delta \lambda} = 0 = 20 - X_1-2X_2 =0 \quad (5)$$

From eqn (3) $X_2+2 = \lambda$

From eqn (4) $X_1 /2 = \lambda$

ie $X_2+2 = X_1 /2 = \lambda = X_1 -2X_2 = 4$ (6)

$[X_1= 2(X_2+2) = X_1= 2X_2+4 = X_1-2X_2= 4]$

Add eqn (2) and eqn (6)

$$
\begin{array}{l}
X_1 + 2X_2 = 20 \quad + \\
X_1 - 2X_2 = 4 \\
\hline
2X_1 = 24
\end{array}
$$

$X_1 = \frac{24}{2} = 12$ substitute in eqn (6) $12 - 2X_2 = 4$

$2X_2 = 8$

$X_2 = 4$

$X_1 = 12$, $X_2 = 4$

Second order condition requires

$$[\bar{H}] = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0$$

$L_1 = X_2 + 2 - \lambda = 0$ (3)

$L_2 = X_1 - 2\lambda = 0$ (4)

$L_\lambda = 20 - X_1 - 2X_2 = 0$ (5)

$L_{11} = 0$ $L_{12} = 1$ $g_1 = 1$

$L_{21} = 1$ $L_{22} = 0$ $g_2 = 2$

$$[\bar{H}] = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 4 > 0$$

$= 0(0-1) - 1(0-2) + 2(1-0)$

$= 0 + 2 + 2 = 4$

Since the second order condition was also satisfied the given function will maximize at $X_1 = 12$ and $X_2 = 4$

Maximum value will be at $y = Y = X_1X_2 + 2X_1 = 48 + 24 = 72$

Example 4

Minimise

$Z = x^2 - y^2 + xy + 5x$

Subject to

$x-2y = 0$

$L = x^2-y^2+xy+5x + \lambda (x-2y)$

$\frac{\delta L}{\delta X} = 2x + y + 5+ \lambda=0$ (1)

$\frac{\delta L}{\delta Y} = -2y + x-2 \lambda = 0$ (2)

$\frac{\delta L}{\delta \lambda} = x-2y = 0$ (3)

From equation (1) $-2x-y-5 = \lambda$

From equation (2) $-y + \frac{x}{2} = \lambda$

i.e $-2x-y-5 = -y + \frac{x}{2}$

$= -2x- \frac{x}{2} -5 = 0$

$= -2.5x =5$

$x = -2$

Substitute $x = -2$ in equation (3)

$-2-2y = 0$

$-2 = 2y$

$y = -1$

Therefore the critical values are $x = -2$ and $y = -1$

The second order condition is that

$$[\bar{H}] = \begin{vmatrix} 0 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

Therefore the second order condition is also satisfied. The given function is a minimisation function and it will be minimised at $x =-2$ and $y = -1$

Example 5

Given the utility function $U= (x+2) (y+1)$ and the budget constraint $2x+5y=51$, find the optimal level of X and Y purchased by the consumer.

The function is Maximise

$U= (x+2) (y+1)$

Subject to

2x+5y = 51

The Lagrange function is

Z= (x+2)(y+1) + λ(51-2x-5y)

$\frac{\delta L}{\delta X}$ = Y+1 -2 λ = 0 (1)

$\frac{\delta L}{\delta Y}$ = X+2-5 λ = 0 (2)

$\frac{\delta L}{\delta \lambda}$ =51-2x-5y = 0 (3)

From equation (1)

Y+1 = 2 λ or $\frac{Y+1}{2}$ = λ (4)

From equation (2)

X+2 = 5 λ or $\frac{X+2}{5}$ = λ (5)

From equation (4) and (5)

$\frac{Y+1}{2}$ = λ and $\frac{X+2}{5}$ = λ

we get $\frac{Y+1}{2} = \frac{X+2}{5}$ (6)

from equation (6) Y= $\frac{2X+4}{5} - 1$

$= \frac{2X+4-5}{5}$

$= \frac{2X-1}{5}$

Substituting in equation (3)

51-2x-5($\frac{2X-1}{5}$)

= 51-2x-2x-1 = 0

52 - 4x = 0

x = $\frac{52}{4}$ = 13

Therefore y = 51-2(13)-5y =0

$51-26-5y=0$

$25-5y=0$

$y = \frac{25}{5} = 5$

$\frac{5+1}{2} = \lambda, \quad \lambda = 3$

$x = 13, y = 5, \lambda = 3$

Second order condition is

$L_1 = Y+1-2\lambda \quad =0 \qquad (1)$

$L_2 = X+2-5\lambda \quad =0 \qquad (2)$

$L\lambda = 51-2x-5y \quad =0 \qquad (3)$

$L_{11} = 0 \qquad L_{12} = 1 \qquad g_1 = 2$

$L_{21} = 1 \qquad L_{22} = 0 \qquad g_2 = 5$

$$[\bar{H}] = \begin{vmatrix} 0 & 2 & 5 \\ 2 & 0 & 1 \\ 5 & 1 & 0 \end{vmatrix} = 20 > 0$$

Therefore the second order condition for maximization is also satisfied and the utility for the given function will be maximized at x = 13, y = 5.

Example 6

Maximise Utility $U = Q_1Q_2$, subject to $P_1 = 10$, $P_2 = 2$ and M= 240. What is the marginal utility of money?

The Lagrange function L is

$L = Q_1Q_2 + \lambda(240-10Q_1-2Q_2)$

$\frac{\partial L}{\partial Q_1} = Q_2 - 10\lambda = 0 \qquad (1)$

$\frac{\partial L}{\partial Q_2} = Q_1 - 2\lambda = 0 \qquad (2)$

$\frac{\partial L}{\partial \lambda} = 240-10Q_1-2Q_2 = 0 \qquad (3)$

From Equation (1) $\frac{Q_2}{10} = \lambda$ and from equation (2) $\frac{Q_1}{2} = \lambda$

$$\frac{Q_2}{10} = \frac{Q_1}{2}$$

$= 2Q_2 = 10Q_1$ or $Q_2 = 5Q_1$

Substitute in equation (3)

$240 - 10Q_1 - 2Q_2 = 0$

$= 240 - 10Q_1 - 10Q_1 = 0$

$240 - 20Q_1 = 0$

$20Q_1 = 240$

$$Q_1 = \frac{240}{20} = 12$$

Therefore $Q_2 = 5Q_1 = 5(12) = 60$

$$\lambda = \frac{Q_2}{10} = \frac{60}{10} = 6$$

The critical values are Q1 = 12, Q2 = 60 and $\lambda = 6$

Therefore the marginal utility of money at $Q_1 = 12$, $Q_2 = 60$ is 6

3.2.2 Economic Application of Constraint Optimisation

The constraint optimisation is used for the optimisation of many important decisions in economics. In this section application of constraint optimisation in utility maximisation, cost minimisation and profit maximisation are given.

1. Utility Maximisation

The optimisation technique is used for finding the utility maximising levels of various commodities consumed by the consumer. The utility is maximised subject to the budget constraint. The consumer can maximise his utility subject to the income or budget of the consumer. Therefore, maximisation of utility is subject to the constraint and constraint optimisation techniques are used for finding the utility maximising levels of goods. The Marshallian concept of utility maximisation condition is that consumer has to allocate his total budget for consumption in such a way that the ratio of marginal utilities to prices are equals to each commodities.' Hicks states the consumer utilisation conditions for two commodities as point where the slope of budget line equals slope of indifference curve. It states that the budget line should be tangent to the indifference curve at the point of equilibrium. The indifference curve must be convex to origin also at the point of equilibrium to maximise the utility. The slope of budget line shows the price ratios of two commodities (Px/Py) and the slope of indifference curve shows the ratios of marginal rate of substitution (MRSxy) between two commodities. MRS is the ratios of marginal utilities of two commodities, MUx/MUy. That is at the equilibrium point the price ratio must be equal to ratios of MRS between two commodities.

The utility maximisation condition can be stated as follows

The consumer is in equilibrium when he maximises his utility given his income and market prices (ie budget line)

The utility is maximised at the point where the slope of indifference curve must be equal to the slope of budget line or

Marginal rate of substitution must be equal to Price ratio ie, MRS $= \frac{MUx}{MU_y} = \frac{Px}{P_y}$

The utility maximisation condition can be shown as in the figure given below

The consumers utility maximisation point is given at point E. At point E the utility is maximised since the slope of indifference curve and budget lines are equal.

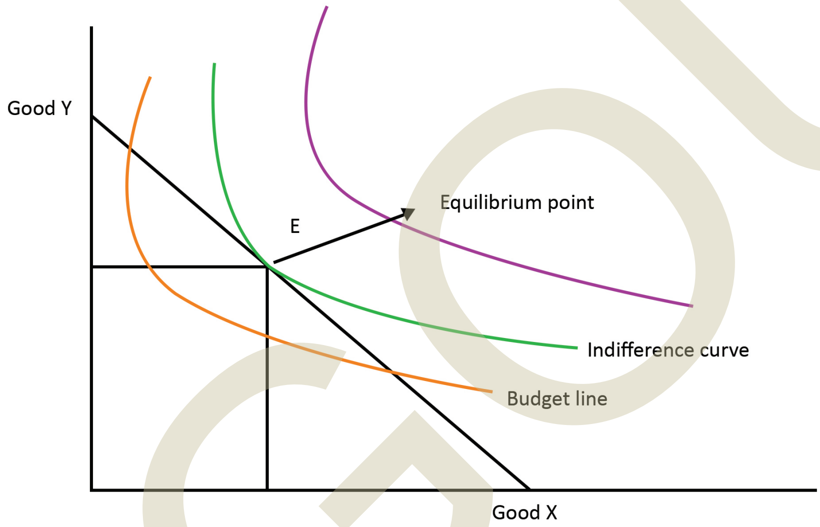


Fig 3.2.1 Consumers Equilibrium- Utility Maximisation Conditions

At the point of tangency the slope of budget Line and slope of indifference curves are equal

i.e at point E, MRS $= \frac{MUx}{MU_y} = \frac{Px}{P_y}$

Derivation of Utility Maximization Conditions

The utility maximisation conditions can be derived mathematically using Lagrange method as follows

Proof

Given an indifference curve

U= U(X,Y)

And Budget constraint $M = P_x X + P_y Y$

Step 1 - Rewrite the constraints to equate to zero

$M - P_x X - P_y Y = 0$

Step 2 - Multiply the constraint by $\lambda = \lambda (M - P_x X - P_y Y)$

Step 3 - Formulate the lagrange function –L. Add the objective function and new constraint

$$\mathbf{L = U(X,Y) + \lambda\ (M - P_x X - P_y Y)}$$

Find the partial derivative, with respect to x, y and λ

$$\frac{\partial L}{\partial X} = \frac{\partial U}{\partial X} - \lambda Px = 0 \qquad (1)$$

$$\frac{\partial L}{\partial Y} = \frac{\partial U}{\partial Y} - \lambda Py = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = M - P_x X - P_y Y = 0 \qquad (3)$$

Divide eqn (1) by eqn (2) $= \dfrac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = \dfrac{Px}{P_y}$

$$= \frac{MUx}{MU_y} = \frac{Px}{P_y}$$

Or the consumer is in equilibrium when $\dfrac{MUx}{Px} = \dfrac{MUy}{Py} = \ldots\ldots\ldots\ldots\ldots \dfrac{MUn}{Pn}$

The second order condition requires that at point of tangency the indifference curve must be convex to the origin. Ie the determinant of bordered Hessian must be greater than Zero

$$[\overline{H}] = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0$$

2. Cost Minimisation Conditions

Cost minimisation involves the question of how to produce a given output at minimum total cost. With a given production function the producer minimises the total cost of production by choosing the optimal combination of factors. The cost should be minimised by selecting the optimum combination of various factors. The cost minimisation conditions can be explained through the isoquants and isocost curves. The cost is minimised at the point where the slope of isoquants and cost curves are equal. Cost is minimised at the tangency point of isoquants and isocost where the slopes of both are equal. The slope of isoquants are known as marginal rate of technical substitution (MRTS). MRTS is the ratios of marginal products of inputs, ie MRTS= MPL/MPK. The slope of cost curves represents the input price ratios. w/r. Thus the cost minimsation point is at the point where MRTS= W/R or slope of isoquants equals slope of cost curves.

Cost minimisation condition can be stated as follows

1. Cost is minimised at the point where the slope of isoquant is equal to the slope of cost curves or

2. Cost is minimised at the point where where $MRTS_{LK}= MP_L/MP_K = w/r$ and the isoquants must be convex to the origin

The cost minimisation conditions are shown in the following diagram

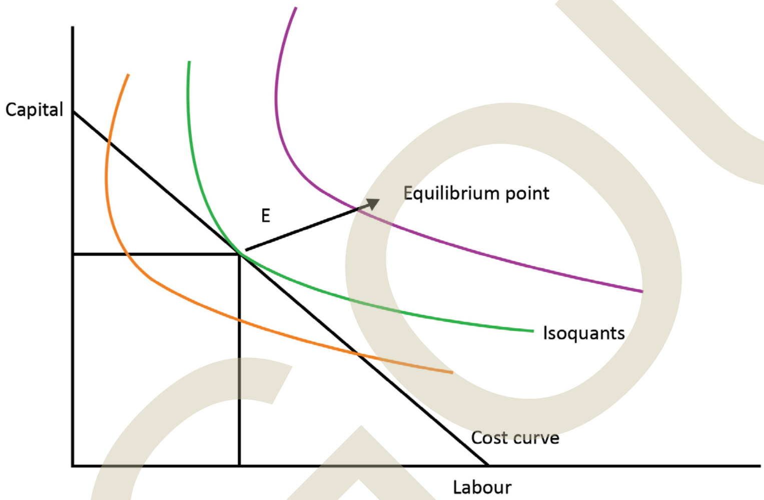


Fig 3.2.2 Cost Minimisation Conditions

The cost is minimum at point E where the slope of isoquants and isocost curves are equal. At this point the condition $MRTS_{LK}= MP_L/MP_K = w/r$ is satisfied. At this point the isoquants are convex to the origin also. Therefore it is the least cost combination of inputs.

Derivation of Cost Minimization Conditions

The objective of the producer is the minimisation of cost of production subjected to the constraint production function

The cost function is given as $C = wL+rK$

the production function is given as $Q=f(K,L)$

i.e.

Minimise $C = wL+rK$

Subject to Q = f(k,L), ie f(k, L - Q) = 0

Multiply the constraint by λ(f(k,L)- Q) = 0

The lagrange function Z = wL+rK + λ(f(k,L) - Q)

Differentiate the Z function with respect to L,K, and λ

Z = wL+ rK+ λ(f(k,L) - Q)

$$\frac{\partial Z}{\partial L} = 0, \frac{\partial Z}{\partial k} = 0, \frac{\partial Z}{\partial \lambda} = 0$$

$$\frac{\partial Z}{\partial L} = w+\lambda fL = 0 \qquad (1) \text{ where } fL = \frac{\partial f(L,K)}{\partial L} = \frac{\partial Q}{\partial L} = MP_L$$

$$\frac{\partial Z}{\partial k} = r + \lambda fK = 0 \qquad (2) \text{ where } fK = \frac{\partial f(L,K)}{\partial K} = \frac{\partial Q}{\partial K} = MP_K$$

$$\frac{\partial Z}{\partial \lambda} = f(k,L) - Q = 0 \qquad (3)$$

From equation (1) w + λfL = 0 = w = λfL = w/fL = - λ (4)

From equation (2) r + λfK = 0 = r = λfK r/ fk = - λ (5)

Ffrom equation (4) and (5) we will get w/fL = r/ fk or w/r = fL/fK

w/fL = r/ fk or w/r = fL/fK

fL/fK= MPL/MPk = MRTSLK or it is the slope of isoquants

w/r = ratios of input prices = slope of isocost curve

i.e. At equilibrium the slope of isoquants is equal to slope of isocost line

Second order condition requires that the isoquant must be convex to the origin at the point equilibrium. This sufficient condition is shown through the bordered Hessian determinant. The bordered Hessian determinants must be negative .

$$[\overline{H}] = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{vmatrix} < 0$$

3. Derivation of Profit Maximisation Conditions

Producer aims at maximisation of his profit. Constraint optimisation problem can also be used for the maximisation of profit. The profit is maximised subject to the constraints of production function. Profit maximisation conditions are given below. The profits are maximised at the point where the price ratios are equal to marginal products.

Given TR= PQ

TC = wL+ rK

Π = TR-TC = PQ - (wL+rK)

Thus the objective function is maximize the profit function Π = PQ - (wL+rK)

The firm has to face the constraint Q = f(k,L)

The lagrange function Z= PQ - (wL+rK) +λ f (K,L)- Q

Proof

Z= PQ - (wL+rK) + λf (K,L)- Q

$$\frac{\partial Z}{\partial Q} = P - \lambda \qquad (1)$$

$$\frac{\partial Z}{\partial L} = -w + \lambda fL = 0 \qquad (2)$$

$$\frac{\partial Z}{\partial K} = -r + \lambda fL = 0 \qquad (3)$$

$$\frac{\partial Z}{\partial \lambda} = f(K,L) - Q = 0 \qquad (4)$$

Proof

from eqn (1) p = λ

substitute p = λ in eqn (2) and eqn (3)

-w + λfL = w = pfL = w/fL= P

-r + λf = r = pfK = r/fK = p

w/fL = r/fK

w/r = fL/fK

The input price ratio equal to ratio of marginal products. This is the condition for profit maximisation

Second order condition requires

$$[\bar{H}] = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & z_{11} & z_{12} \\ g_2 & z_{21} & z_{22} \end{vmatrix} > 0$$

Examples 1

Given the utility function U= $f(q_1q_2)$ and the budget constraint Y= $q_1p_1+q_2p_2$ derive the conditions for consumers equilibrium

U= $f(q_1q_2)$

Budget constraint is given as

$Y = q_1p_1 + q_2p_2$

The new Lagrange function can be written as

$L = U - \lambda (q_1p_1 + q_2p_2 - Y)$

The first order condition is that

$$\frac{dL}{dq_1} = \frac{dU}{dq_1} - \lambda P_1 = 0 \quad \text{-------------(1)}$$

$$\frac{dL}{dq_2} = \frac{dU}{dq_2} - \lambda P_2 = 0 \quad \text{-----------------(2)}$$

$$\frac{dL}{d\lambda} = (q1p1 + q2p2 - Y) = 0 \quad \text{-------------(3)}$$

From equation (1)

$\frac{dL}{dq_1} = \lambda P_1$ (4), $\frac{dL}{dq_1}$ is the marginal Utility of $q_1 = MU_1$

$MU_1 = \lambda P_1$ or $\lambda = \frac{MU_1}{P_1}$ (5)

From equation (2)

$$\frac{dU}{dq_2} = \lambda P_2 \quad (6)$$

$$\frac{dU}{dq_2} = MU_2$$

$MU_2 = \lambda P_2$ or $\lambda = \frac{MU_2}{P_2}$ (7)

From equation (5) and (7) $\lambda = \frac{MU_1}{P_1} = \frac{MU_2}{P_2}$

i.e,

$\frac{MU_1}{P_1} = \frac{MU_2}{P_2}$ or

$$\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$$

This is the equilibrium condition for consumer. At the equilibrium the ratios of MUs (ie Marginal Rate of Substitution MRS) must be equal to ratio of prices

The second order condition required is that

$$[\bar{H}] = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0$$

Example 2

Given the utility function $U= 2q_1+q_2+2q_1q_2$ and budget constrain is $100=2q_1+q_2$. Find the equilibrium quantities of q_1 and q_2 that will maximize consumer utility function.

$L= 2q_1+ q_2+ 2q_1q_2 +\lambda (2q_1+ q_2-100)$

$\frac{dL}{dq_1} = 2 + 2q_2 + 2\lambda = 0$ (1)

$\frac{dL}{dq_2} = 1+ 2q_1 + \lambda = 0$ (2)

$\frac{dL}{d\lambda} = 2q_1+ q_2-100 = 0$ (3)

Proof

From eqn (2) $\lambda = -2q_1-1$, substitute this into eqn (1)

$2+2q_2 +2 (-2q_1-1) = 2+2q_2- 4q_1-2$

$- 4q_1 + 2q_2 = 0$ (4)

From eqn (3) $(2q_1+q_2-100)$, $2q_1+ q_2 = 100$ (5)

Eqn (5) × 2= $4q_1+ 2q_2 = 200$

Add eqn (4) and (5)

$- 4q1 + 2q_2 = 0$

$4q_1+2q_2 = 200$

$0 + 4q_2 = 200$

$q_2 = 50$

Substitute $q_2 = 50$ in eqn (4), $- 4q_1+ 2(50)= 0$, $100 = 4q_1$, $q_1= \frac{100}{4} = 25$

$\therefore q_1= 25, q_2= 50$

Second order condition

From eqn (1) and (2)

$L_{11}= 0, L_{12}=2, L_{21}=2, L_{22}= 0, g_1= 2\ g_2= 1$

The bordered Hessian matrix can be written as

$$[\bar{H}] = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} > 0$$

$$[\bar{H}] = \begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} > 0$$

$[\bar{H}]$ = 0(0-4) -2(0-2)+1(4-0) = 4+4 = 8 >0

The second order condition is also satisfied and therefore q_1= 25 and q_2= 50 will maximize the function and the maximum value of the function

U= 2(25)+50+2(25)(50) = 2600

Example 3

The cost functions of the firm is given as C = $8x_1^2+9x_2^2$ and the production constraint is X_1+X_2= 54. Find the value of x1 and x2 that minimizes the cost

Minimize

C = $8x_1^2+9x_2^2$

Subject to

$x_1+ x_2$= 54

The Lagrangian function is $\mathcal{L} = 8x_1^2 + 9x_2^2 + \lambda(54 - x_1 - x_2)$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 16x_1 - \lambda = 0$$

$$16x_1 = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 18x_2 - \lambda = 0$$

$$18x_2 = \lambda$$

From the first two equations

$$16x_1 = 18x_2$$

$$\frac{x_1}{x_2} = \frac{18}{16} = \frac{9}{8}$$

$$x_1 = \frac{9}{8}x_2$$

Substituting into constraint equation

$$\frac{9}{8}x_2 + x_2 = 54$$

$$\frac{17}{8}x_2 = 54$$

$$x_2 = \frac{54 \times 8}{17} = \frac{432}{17} \approx 25.41$$

$$x_1 = \frac{9}{8} \times 25.41 = \frac{9 \times 432}{8 \times 17} = \frac{3888}{136} \approx 28.59$$

Thus, the cost minimizing values are

$$x_1^* \approx 28.59, \quad x_2^* \approx 25.41$$

The bordered Hessian Matrix is

$$H_B = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 16 & 0 \\ -1 & 0 & 18 \end{vmatrix}$$

$$H_B = 0 \times \begin{vmatrix} 16 & 0 \\ 0 & 18 \end{vmatrix} - (-1) \times \begin{vmatrix} -1 & 0 \\ -1 & 18 \end{vmatrix} + (-1) \times \begin{vmatrix} -1 & 16 \\ -1 & 0 \end{vmatrix}$$

$$H_B = 0 - (-1) \times (-18) + (-1) \times (16)$$

$$H_B = -18 - 16 = -34$$

The optimal values that minimize cost, subject to the production constraint, are:

$x_1 \approx 28.59$,$x_2 \approx 25.41$

The bordered Hessian method confirms that this is a cost-minimizing solution.

Example 4

Minimise the costs for a firm with the cost function $C = 5x^2 + 2xy + 3y^2 + 800$ subject to the production quota $x + y = 39$

$Z = 5x^2 + 2xy + 3y^2 + 800 + \lambda (39-x-y)$

$\frac{\partial Z}{\partial x} = 10x + 2y - \lambda = 0$ (1)

$\frac{\partial Z}{\partial y} = 2x + 6y - \lambda = 0$ (2)

$\frac{\partial Z}{\partial \lambda} = 39 - x - y = 0$ (3)

Substract equation (2) from (1)

$= 8x-4y = 0$

$x = 0.5 y$

Sustitute in $x = 0.5y$ in equation (3)

$39-0.5y-y = 0$

$y = 39/1.5 = 26$

$x = 0.5y = 0.5(26) = 13$

Inorder to get the value of λ, substitute $x = 13$ and $y = 26$ in equation (1)

$\lambda = 10 (13)+2(26) = 130+52 = 182$

The critical values are $x = 13$, $y = 26$, $\lambda = 182$

$Z = 5x^2 + 2xy + 3y^2 + 800 + \lambda (39-x-y)$

$= 5(13)^2+2(13)(26)+3(26)^2+800 +182(39-13-26)$

$= 845+676+2028+800+182(0) = 4349$

Since $\lambda = 182$, an increase in additional production quota will increase the cost additionally by 182.

The second order condition is that

$$[\bar{H}] = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 10 & 2 \\ 1 & 2 & 6 \end{vmatrix} < 0$$

$0(10x6-2x2) -1(6-2) +1(2-10) = 0 -4-8 = -12 < 0$

Therefore the second order conditions for minimisation is also satisfied.

Recap

- In optimisation, constrained optimisation is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables
- If the constrained problem has only equality constraints, the method of Lagrange multipliers can be used to convert it into an unconstrained problem
- The method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints
- The consumer is in equilibrium when he maximizes his utility given his income and market prices (ie budget line)
- The conditions for utility maximization is that The utility is maximised at the point where the slope of indifference curve must be equal to the slope of budget line or Marginal rate of substitution must be equal to Price ratio ie, MRS $= \frac{MUx}{MU_y} = \frac{Px}{P_y}$
- Cost is minimized at the point where the slope of isoquant is equal to the slope of cost curves
- Cost is minimized at the point where where $MRTS_{LK} = MP_L/MP_K = w/r$ and the isoquants must be convex to the origin

Objective Questions

1. ------------method is used for finding the optimisation function with constraint
2. The Lagrange method was developed by
3. The utility maximisation condition requires that -------------
4. conditions for Increasing function and concave upward are -------------
5. Cost minimisation requires -

Answers

1. Lagrange method of optimisation
2. Joseph Louis Lagrange
3. MRS = $\frac{MUx}{MU_y} = \frac{Px}{P_y}$
4. $f'(a) > 0, f''(a) > 0$
5. Slope of isoquant must be equal to the slope of cost curves

Assignments

1. Explain the term utility.
2. What is a constraint optimisation?
3. What is a Lagrange method?
4. What are the steps for the lagrange method for optimisation?
5. What are the conditions for maximisation using Lagrange method?
6. What are the conditions for minimisation using Lagrange method?
7. Derive the equilibrium conditions for utility maximisation using Lagrange method.
8. Derive the conditions for cost minimisation using Lagrange method.
9. Derive the conditions for profit maximisation using Lagrange method.
10. Explain the application of optimisation with constraint in Economics.
11. Find the optimum value when the utility function is U= q1.5q2 and budget constraint is $3q_1+4q_2 = 90$. Answer $q_1= 9$ and $q_2= 18$
12. Find the optimum value given $Z= 20x+20y-x_2-y_2$ given the constraint $2x+5y = 10$ Answer $x = 5.86$ and $y = -0.345$
13. Given the utility function $U = x_1x_2$ and the budget constraint is $200-4x_1-10x_2$. Find x_1 and x_2 at which utility is maximum. Answer $x_1= 25$

and $x_2 = 10$

14. Given the utility function $U = 4x_1^{1/2}x_2^{1/2}$ and the budget constraint $60 = 2x_1 + x_2$ find x_1 and x_2 at which utility is maximum. Answer $x_1 = 15$, $x_2 = 30$

References

1. Chiang, A. C., & Wainwright, K. (2013). *Fundamental Methods of Mathematical Economics* (4th ed.). McGraw-Hill.
2. Varian, H. R. (1992). *Microeconomic Analysis* (3rd ed.). W. W. Norton & Company.
3. Simon, C. P., & Blume, L. (1994). *Mathematics for Economists*. W. W. Norton & Company.

Suggested Readings

1. Mehta, B. C., & Madnani, G. M. K. (2017). *Mathematics for Economists*. Sultan Chand & Sons.
2. Sundaram, R. K. (1996). *A First Course in Optimisation Theory*. Cambridge University Press.
3. Allen, R. G. D. (1974). *Mathematical Analysis for Economists*. Macmillan.

4

BLOCK

Production Function

UNIT 1

Production function

Learning Outcomes

After completing this unit, the learner will be able to:

- mathematically express the functional relationship between input and output
- familiarise with the diagrammatic and mathematical representation of different types of production function

Prerequisites

Production is a term constantly used in association with a firm/organisation or an industry. They are involved in producing goods that we see around us. These may be books, clothes, dams, cars, agricultural goods or any essential commodities. For producing goods we need factors of production such as land, labour, capital, raw materials and other resources known as inputs which is to be combined in a certain proportion. The final good that is produced with the help of these inputs is termed as output. Let us consider the case of a notebook manufacturing firm. For the production of notebook which represent the output, we need inputs such as land, labour, machinery, raw materials constituting paper, thread, and bind. Thus, without factor inputs and without some way of combining these factor inputs we cannot produce output. Therefore there exists a functional relationship between input and output which is studied under the topic production function. Here in this unit, we provide a mathematical expression of production function and its different forms. The mathematical modelling can be introduced in any organisation or unit to identify the most efficient way to utilise resources, to assess the productivity of inputs used in the production process as well as to project the production of various goods.

Keywords

Input, Output, Marginal product, Isoquant, Short Run, Long Run, Production Function, Marginal Rate of Technical Substitution

Discussion

4.1.1 Production function

The production function for any commodity is a mathematical expression, table, or graph that illustrates the maximum quantity of a commodity that can be produced within a given period, based on various input levels, assuming the most efficient production methods are utilised. Thus, it represents the functional relationship between physical inputs (or factors of production) and output. It assumes inputs as the explanatory or independent variable and output as the dependent variable. Mathematically we may write this as follows.

$$Q = f(L,K,M)$$

Where Q represents output, L,K, and M represents inputs such as labour, capital and raw material respectively. Output is the final good that is produced with the help of inputs. Input is any good or service used for the creation of output. Without factor inputs and without some way of combining these factor inputs we cannot produce output i.e. there exist a relation between input and output. This relationship is studied under the topic production function. It shows how factor inputs are combined and the maximum amount of output we can produce with given quantities of inputs. Now let us understand the concept with the help of an example of a simple production function for a company that manufactures chairs using labour (workers) and capital (machines).

- **Labour (L)**: Number of workers
- **Capital (K)**: Number of machines

The production function shows the maximum number of chairs (Q) that can be produced based on different combinations of labour and capital.

Table 4.1.1 Production Function

Labour (L)	Capital (K)	Number of Chairs Produced (Q)
1	1	10
2	1	18
3	1	24
1	2	20
2	2	35
3	2	45
2	3	50

In this example:

- With **1 worker** and **1 machine**, the company can produce **10 chairs** in a given time period.

- With **2 workers** and **1 machine**, they can produce **18 chairs**.
- If the company increases both labour and capital (e.g., **2 workers** and **2 machines**), they can produce **35 chairs**.

This shows how the output (number of chairs) changes with variations in labour and capital. The production function captures the relationship between inputs and output, which is key to understanding production efficiency.

Production function is of two types- Short run and Long run production function. When the quantities of some inputs such as capital and land are kept constant and the quantity of one input such as labour is varied, this kind of production function [$Q = f(L,\bar{K})$] is called short run production function. The long run production function analyses the relationship between input and output when all inputs are varied in the same proportion, it can be expressed as Q=f(L,K,M). The derivative of production function with respect to L is known as Marginal Product of Labour, which measures the rate at which output changes as the number of workers increases. Thus, we have $MP_L=dQ/dL$. The marginal product of capital(MP_K) is the rate of change of output relative to capital and is defined as dQ/dK. Thus, MP_L is approximately the change in output resulting from a one unit increase in labour.

Example

For the production function

$Q=4L^{1/2}K$

Find the Marginal product of labour. Determine the output and the marginal product of labour when

1. L=1
2. L=4
3. L=100

The marginal product of labour is found by differentiating $Q=4L^{1/2}$. The second order derivatives help to determine whether marginal product is increasing or decreasing.

$MP_L=dQ/dL = 4*(1/2)\ L^{(1/2)-1} = 2\ L^{-1/2} = 2/L^{1/2}$

$d^2Q/dL = -1/2\ 2\ L^{-1/2-1} = -L^{-3/2}$

As the result of second order differentiation obtained is negative MP is decreasing when more labourers are employed

1. When L=1, we get output as Q=4, since $Q=4L^{1/2}$ by substituting 1 for L we get Q=4. Similarly, MP_L obtained by differentiating $Q=4L^{1/2}$ is $2/L^{1/2}$. Thus, the marginal product of labour when L=1 is $MP_L=2$

2. When L=4, Q=8 and MP_L=1

3. When L=100, Q=40 and MP_L=0.2

As L increases from 0, output also increases. However, MP_L decreases and output increases at a decreasing rate.

4.1.2 Homogenous Production Function

The concept of Homogenous production function is related with long run production function where the law of returns to scale operates and all factors of production are varied in the same proportion. Homogeneity of production function implies that multiplying all inputs of a production by a constant, 'λ' increases output by λ^n. In other words, if each of the inputs is multiplied by a real constant 'λ' then 'λ' can be completely factored out of the function, then the new level of output Q^* can be expressed as a function of 'λ' (to any power n) and the initial level of output. In general, they are multiplicative rather than additive although a few exceptions exist.

$$Q=f(K,L)$$

$$Q^*=f(\lambda K, \lambda L)$$

$$Q^*=\lambda^n f(K,L)$$

$$Q^*= \lambda^n Q$$

Where Q is the initial level of output, L and K are the initial quantities of labour and capital respectively. Suppose that the scale of production of L and K is increased by 'λ' proportion then changed production function becomes $Q^*=f(\lambda K, \lambda L)$. Here 'n' representing the power of 'λ' is called the degree of homogeneity and n measures the returns to scale.

Properties

1. If production function is homogenous of degree n, the marginal productivities of the factors will be of degree n-1.

2. If the production function is homogenous of degree one then the marginal and average productivities will depend on capital-labour ratio.

3. Homogenous production function satisfies Euler's theorem. The theorem states that if the factors are rewarded according to their Marginal product, then the Total Product will be exhausted.

4. If production function is linear homogenous and if isoquant is convex then marginal productivities will be diminishing.

5. Returns to scale under homogenous production function constitute three types-Constant returns to scale, increasing returns to scale and decreasing

returns to scale.

4.1.3 Non Homogenous Production Function

Non homogeneous production function is a function such that if each of the inputs is multiplied by a real constant 'λ' then 'λ' cannot be completely factored out.

Example

$Q=LK+20$

Let us change L and K by 'λ' proportion

So $Q^*=(\lambda L)(\lambda K)+20$

$= \lambda^2 LK+20$

Here $Q^* = \lambda^2 Q$

Where Q is the initial level of output, L and K are the initial quantities of labour and capital respectively. Suppose that the scale of production of L and K is increased by 'λ' proportion then changed production function becomes $Q^* = \lambda^2 LK+20$. Here the returns to scale (i.e. λ^2) cannot be factored out, therefore it represents a non-homogenous production function.

4.1.4 Degree of Homogeneity and Returns to scale

Returns to scale represents long run production function where all inputs are increased proportionately. For example, if a farmer has 1 acre of land, 20 workers and one tractor, then to expand the scale of operation by a factor of 2, would require that the farmer purchase an additional 1acre land, an additional tractor and hire 20 more workers with exactly the same skill. A long run production function can be represented by a two factor model where the magnitude of both factors can be varied. It can be represented as

$Q=f(L,K)$

Where Q=output, K=Capital, L=Labour

Mathematically, returns to scale depend on degree of homogeneity of the production function. When inputs are multiplied by λ then output will be multiplied by λ^n. Here 'n' is any constant that measures the degree of homogeneity of the production system. The value of 'n' may be zero, equal to, less than or greater than unity. If n=0 then $Q^* = \lambda^0 Q = Q$. Here if we increase input by a certain amount output will not change or production function is homogenous of degree zero.

4.1.4.1 Isoquants

The returns to scale is diagrammatically represented with the help of isoquants.

Isoquant is also called Equal Product curve or Production Indifference curve. Isoquant is the locus of all combinations of two factors of production which yield the same level of output. Since it is a two factor model, the factors are assumed to be labour and capital. The concept can be understood from the table 4.1.2. Suppose two factors capital and labour are employed to produce a commodity. Each of the factor combinations Q, R, S, T and U produces the same level of output, say 100 units. To start with factor combination Q constituting 1 unit of labour and 15 units of capital produces the given 100 units of output. In the same way factor combination R consisting of 2 units of labour and 10 units of capital, combination S consisting of 3 units of labour and 6 units of capital, combination T consisting of 4 units of labour and 3 units of capital, combination U consisting of 5 units of labour and 2 units of capital can produce the same level of output i.e., 100 units. By plotting these combinations in a graph, we obtain an isoquant showing that each combination produces 100 units of output as given in figure 4.1.1. An isoquant map representing a set of isoquants is shown in figure 4.1.2. Here each set of isoquant bears different levels of output and higher isoquants represent higher levels of output. In the figure we have drawn an isoquant map with a set of three isoquants representing 100 units, 200 units and 300 units of output respectively. At the same time the input combination at point B on higher isoquant IQ_2 is greater than input combination at point A on isoquant IQ_1. So the quantity of output at point B is greater than at point A.

Table 4.1.2 Factor Combinations to Produce a Given Level of Output

Factor combinations	Labour	Capital
Q	1	15
R	2	10
S	3	6
S	4	3
T	5	2

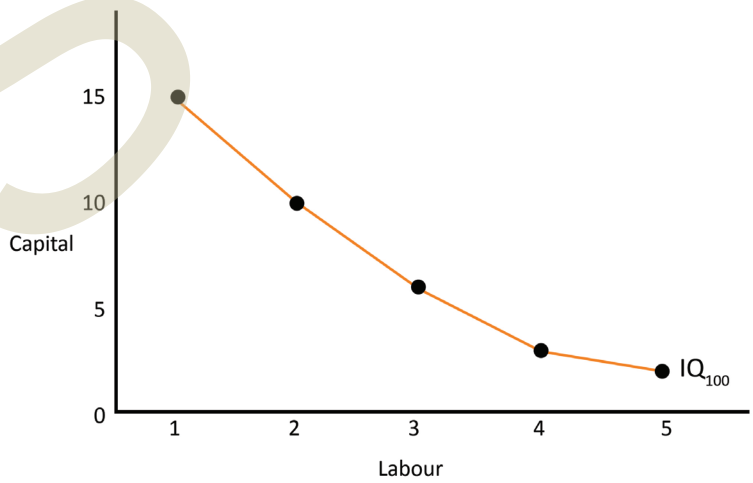


Fig 4.1.1 Isoquant

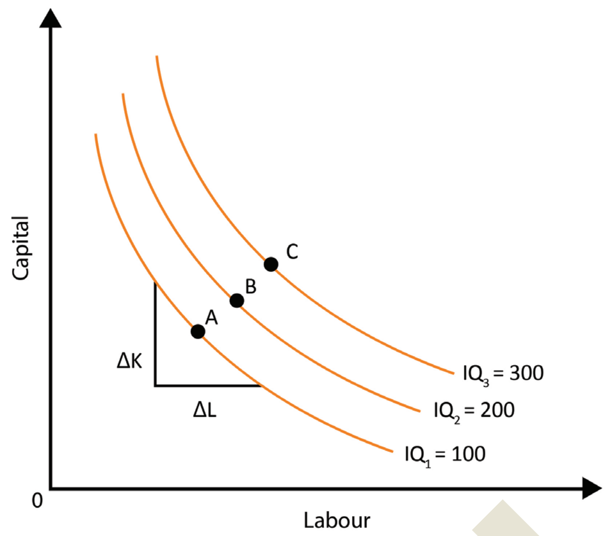


Fig 4.1.2 Isoquant map

4.1.4.2 Marginal Rate of Technical Substitution

Slope of an isoquant is called Marginal rate of technical substitution. Generally, a decrease in labour ($\Delta L<0$) requires an increase in capital ($\Delta K >0$) to maintain the value of output at a constant level. Therefore $\Delta K/ \Delta L$ is normally negative.

$$MRTS_{LK} = - \Delta K/ \Delta L = MP_L/MP_K$$

i.e., $MRTS_{LK}$ is equal to the ratio of marginal physical products of two factors. It shows the amount of capital the producer is willing to sacrifice to employ one more labour and still remain on the isoquant. $MRTS_{LK}$ diminishes as the amount of labour used is increased and the quantity of capital employed is reduced. As seen in figure 4.1.1 the slope of isoquant diminishes as the employment of labour increases.

We will now explain the concept of returns to scale by assuming that only two factors capital and labour are used in the production process, in terms of isoquants. It refers to the change in output to proportionate change in all inputs used in production process. Returns to scale with various degrees under a homogenous production function may be constant, increasing or decreasing.

Constant Returns to scale

If n=1 then $Q^* = \lambda^1 Q = \lambda f(K,L)$ where n represents the degree of homogeneity. This implies that if inputs are multiplied by 'λ' proportion then output is also multiplied by the same proportion. It exhibits the properties of linear homogenous production function with degree of homogeneity equal to 1. If increase in all inputs in the same proportion results in same and proportionate increase in output then production function exhibits Constant Returns to Scale (CRS), the distance between successive isoquants, representing equal increment in output, remains constant i.e., OA=AB=BC as shown in figure 4.1.2.

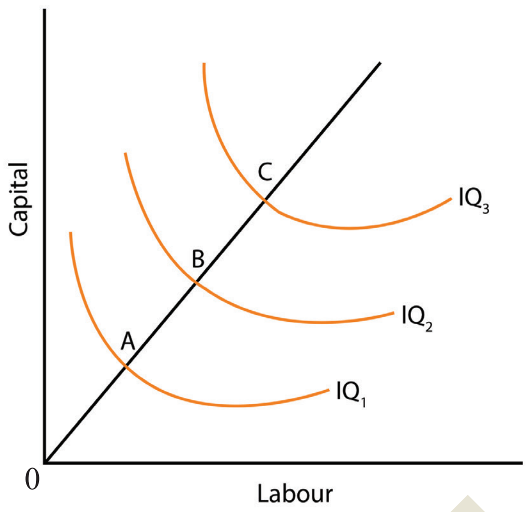


Fig 4.1.3 Constant returns to scale

For example, if $\lambda = 2$ and $Q=f(K,L)$ and $n=1$; then $Q^* = \lambda f(K,L)$; since $\lambda = 2$, $Q^*= 2^1 Q$

Example 1

Identifying degree of homogeneity

$Q=20K^{0.5}.L^{0.5}$

Where Q=Output

$20K^{0.5}.L^{0.5}$=Input with K=Capital and L=Labour

Thus if input is increased λ times output increases by λ^n.

$Q=20K^{0.5}L^{0.5}$

$Q^*= \lambda 20K^{0.5}\lambda L^{0.5}$

$=20(\lambda K)^{0.5} (\lambda L)^{0.5}$

$= \lambda^{0.5+0.5} 20K^{.5}L^{0.5}$

$= \lambda^1 Q$

Thus, the production function is homogenous of degree one, exhibiting constant returns to scale.

Increasing returns to scale

If $n > 1$ then $dQ/d\lambda > 1$. This implies that if inputs are multiplied by 'λ' proportion then output rises more than proportionately. When increase in inputs in the same proportion results in more than proportionate increase in output, production function exhibits increasing returns to scale, where degree of homogeneity is greater than 1. When IRS operates, the distance between successive isoquants, representing equal increments in output goes on decreasing i.e., OA>AB>BC>CD For example, if $\lambda = 2$ and $Q=f(K,L)$ and $n=1.3$ then $Q^* = \lambda^{1.3} f(K,L)$ and $Q^* = 2^{1.3} Q$. Thus under increasing returns to scale, n representing the degree of homogeneity is greater than 1.

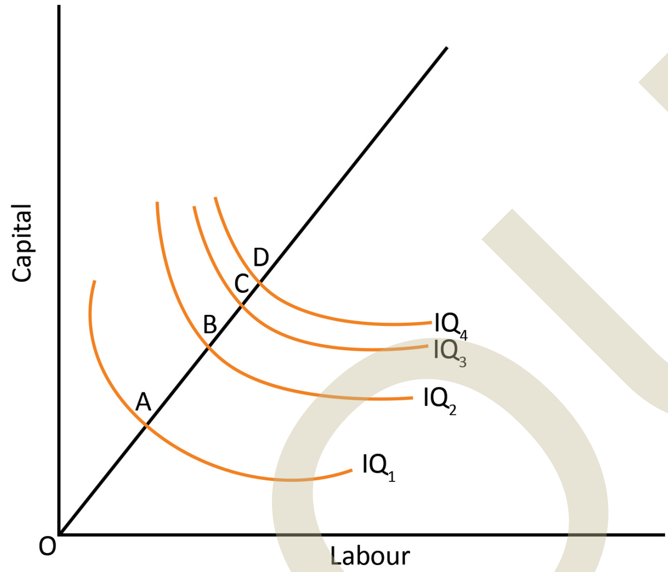


Fig 4.1.4 Increasing returns to scale

Example 1

Identifying degree of homogeneity

$Q=20K^{1.5}.L^{0.5}$

Where Q=Output

$20K^{1.5}.L^{0.5}$=Input with K=Capital and L=Labour

Thus if input is increased λ times output increases by λ^n.

$Q=20K^{1.5}L^{0.5}$

$Q^*= \lambda 20K^{1.5}\lambda L^{0.5}$

$=20(\lambda K)^{1.5} (\lambda L)^{0.5}$

$=\lambda^{1.5+0.5} 20K^{1.5}L^{0.5}$

$= \lambda^2 Q$

Thus, the production function is homogenous of degree two, exhibiting increasing returns to scale.

Diminishing returns to scale

If n<1 then dQ/dλ<1 increase in all inputs in the same proportion results in less than proportionate increase in output, then DRS operate with degree of homogeneity less than 1. When DRS operates, distance between successive isoquants representing equal increments in output goes on increasing. OA<AB<BC<CD

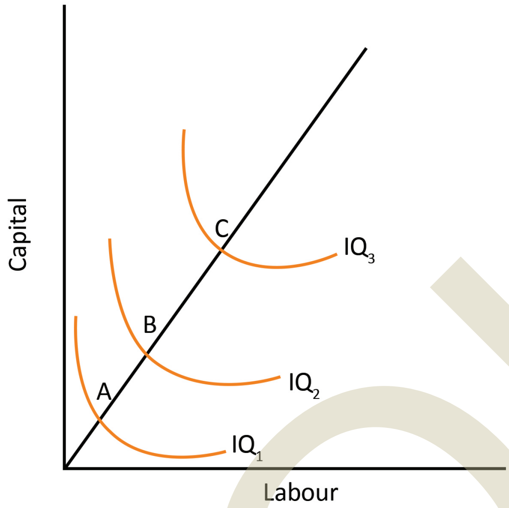


Fig 4.1.5 Decreasing returns to scale

For example, if $\lambda = 2$. and Q=f(K,L) and n = 0.8, then $Q^* = \lambda^{0.8}$ f(K,L) and $Q^* = 2^{0.8}$ Q.

Example 1

Identifying degree of homogeneity

$Q=2K^{0.5}L^{0.3}$

$Q^*= \lambda 2K^{0.5}\ \lambda L^{0.3}$

$=2(\lambda K)^{0.5}\ (\lambda L)^{0.3}$

$= \lambda^{0.5+0.3}\ 2K^{0,5}.L^{0.3}$

$= \lambda^{0.8}\ Q$

Thus the production function is homogenous of degree 0.8, exhibiting decreasing returns to scale.

Thus under decreasing returns to scale degree of homogeneity is less than 1.

4.1.4.3 Linear Homogenous Production Function

A homogenous production function of degree 1 represents a linear homogenous production function with constant returns to scale. In this, with a proportionate change in all factors of production, the output also increases in the same proportion. If there are two factors X and Y, then homogenous production function of first degree can be mathematically expressed as:

$$mQ=f(mX,mY)$$

where Q stands for output and m is any real number.

The above function means that if factors X and Y are increased by m-times, output also increases by m-times.

The isoquants will lie equidistant from each other.

Example

$Q=4K^{0.5}L^{0.5}$

$Q^{*}= \lambda 4K^{0.5}\, \lambda L^{0.5}$

$= 4(\lambda K)^{0.5}\, (\lambda L)^{0.5}$

$= \lambda^{0.5+0.5}\, 4K^{0.5}.L^{0.5}$

$= \lambda^{1}\, Q$

Here the production function is homogenous of degree 1, exhibiting constant returns to scale with neither economies nor diseconomies of scale.

Example

Given the production function $Q(K,L)=3K^2+2K+2L^2$, evaluate MP_L and MP_K for K=2, L=3. Hence, Write down the value of MRTS.

Solution

$MP_L= dQ/dL= 4L$

Differentiating output with respect to labour we get $MP_L= 4L$

$MP_K= dQ/dK=6K+2$

$MRTS= MP_L/MP_K =4L/(6K+2) =4L/2(3K+1)=2L/3K+1$

When K=2and L=3,

then $Q=3K^2+2K+2L^2$

Substituting 2 in place of K and 3 in place of L we get the result as follows:

Q=3*2*2+2*2+2*3*3

Q=34

When K=2and L=3,

$MP_L= dQ/dL= 4L$

Substituting 2 in place of K and 3 in place of L we get the result as follows:

MP_L=4*3=12

MP_K= dQ/dK=6K+2

MP_K=6*2+2

MP_K=14

MRTS= MP_L/MP_K =2L/3K+1

Substituting 2 in place of K and 3 in place of L we get the result as follows

MRTS=2*3/3*2+1

MRTS= $\frac{6}{6+1} = \frac{6}{7}$

Recap

- The functional relationship between physical inputs (or factors of production) and output is called production function
- Production function can be mathematically expressed as Q=f(K,L,M)
- When the quantities of some inputs such as capital and land are kept constant and the quantity of one input such as labour is varied, this kind of production function [$Q=f(L,\bar{K})$] is called short run production function
- The long run production function analyses the relationship between input and output when all inputs are varied in the same proportion
- The derivative of production function with respect to L is known as Marginal Product of Labour, where $MP_L = dQ/dL$
- The returns to scale is diagrammatically represented with the help of isoquants
- If increase in all inputs in the same proportion results in same and proportionate increase in output then production function exhibits CRS
- A homogenous production function of degree 1 represents a linear homogenous production function with constant returns to scale
- If increase in all inputs in the same proportion results in less than proportionate increase in output, then DRS operate with degree of homogeneity less than 1

Objective Questions

1. What is a production function?
2. How can production function be mathematically expressed?
3. What is an isoquant?
4. What is the slope of an isoquant?
5. What is MRTS?
6. Identify the degree of homogeneity for the production function $Q=25K^{0.5}L^{0.5}$

7. Given the production function $Q(K,L)=2K^{1/3}L^{1/2}$, evaluate $MP_{L,}$ $MP_{K,}$ and $MRTS_{LK}$?
8. What is a linear homogenous production function?
9. State the returns to scale that operate if increase in all inputs in the same proportion results in less than proportionate increase in output?
10. State the degree of homogeneity under constant returns to scale?
11. If a production function is homogenous of degree 2, then identify its returns to scale?

Answers

1. The functional relationship between physical inputs (or factors of production) and output is called production function.
2. Production unction can be mathematically expressed as Q=f(K,L,M)
3. Isoquant is the locus of all combinations of two factors of production which yield the same level of output.
4. MRTS
5. Marginal rate of technical substitution representing the slope of isoquant is equal to the ratio of marginal physical products of two factors.
6. Degree of homogeneity is equal to one.
7. $MP_L=L^{1/2}\ K^{1/3}$, $MP_K=2/3K^{2/3}\ L^{1/2}$,$MRTS=3/2\ K^{-1/3}$
8. A homogenous production function of degree 1 represents a linear homogenous production function.
9. Decreasing returns to scale.
10. Degree of homogeneity is equal to one.
11. Increasing returns to scale.

Assignments

1. The production function is given as $Q=2L^{1/4}K^{3/4}$. Find the MP_L, MP_K, and $MRTS_{KL}$. Also identify the degree of homogeneity and its associated returns to scale.

2. Identify whether the following production function exhibits increasing , decreasing or constant returns to scale?

   a)$Q=5KL$ b)$Q=2K+4L$

References

1. Salvatore, D (2011), *Microeconomic theory*, Schaum's Outline Series, McGrawHill.(3rd Edition

2. Bradley, Teresa and Paul Patton, (2002) *Essential Mathematics for Economics and Business*, John Wiley & Sons Ltd

Suggested Readings

1. Ramesh Chandra Das(2017), *Microeconomics Theory and Practice*(Second edition), Kunal Books, New Delhi.

2. Henderson, J. M. and R.E. Quandt (1980) *Microeconomic Theory: A Mathematical Approach*, McGraw Hill, New Delhi.

UNIT 2

Types of Production Function

Learning Outcomes

After completing this unit, the learner will be able to:

- familiarise with Cobb Douglas production function and its mathematical properties.
- understand the various properties of Cobb Douglas Production function

Prerequisites

Production is the act of creating wealth and production function shows the maximum output that can be produced with given quantity of inputs. Economic units or firms use production function to decide best possible combination of labour and capital to produce a given amount of output. When the producers are choosing how much to produce they find that the production will be subject to increasing, decreasing and constant returns to scale, which we have discussed in the previous unit. Having explained the homogenous production function in the previous unit we now turn to explain the various forms of production function. We will examine the cobb- douglas production function along with CES and VES production functions. We shall also explain the estimation of these three types of production function. Inorder to explain the resource allocation problem, the Production possibility curve is also introduced in this unit.

Discussion

4.2.1 Types of Production function

There are many forms of production function, that describes the specific relationship among the factors used in the production process. In the previous unit we had made a distinction between short run and long run production function and also homogenous and non homogenous production function. The mathematical relationship between inputs and output in the long run can take diverse forms. Three such important forms of production function are Cobb Douglas, CES and VES production functions. Cobb Douglas production function, a neo classical production function, assumes that output is a function of only two inputs namely capital and labour which is linear homogenous of degree one. Another popular neo classical production function is CES production function where any change in the input factor, results in constant change in the output. By relaxing the assumption of constant elasticity we arrive at variable elasticity of substitution.

4.2.1.1 Cobb-Douglas Production Function

Economists in the past have formulated several production functions on the basis of statistical analysis of the relation between changes in physical input and output. A most popular empirical production function is the Cobb Douglas production function associated with the name of Charles Cobb and Paul Douglas. It is a widely used production function in Econometrics. It assumes that output is a function of only two inputs namely capital and labour.

Origin

The Cobb-Douglas Production Function owes its origin to Douglas observations:

He observed that

$W/Y=a$	$w.l/Y=a$

$W=a(Y/L)$

W=wage bill

a= a constant

Y=value of national output

w= money wage rate

l=total labour employed

Douglas observed that the wage bill was a constant proportion of the value of national output. According to the Marginal Productivity theory in competitive equilibrium, the money wage rate W is equal to the Marginal Product of labour.

$$W = dY/dL$$

$$dY/dL = a(Y/L)$$

and that MP_L is a constant proportion of AP_L.

General Form

The general form of the Cobb-Douglas Production Function is

$$Q = AL^{\alpha}K^{\beta}$$

Where Q = Output

A = Efficiency Parameter

α = Elasticity of output with respect to labour

β = Elasticity of output with respect to capital

L = labour input

K = capital input

$A>0, \alpha>0, \beta>0, \alpha+\beta=1$

Properties

1. Homogeneity

If all factors are increased in the same proportion output also increases in the same proportion.

The production function has inputs labour and capital

$Q=f(K,L)$

Multiplying both sides by a constant λ

$\lambda Q=A(\lambda L)^{\alpha}.(\lambda K)^{\beta}$

$A.\lambda^{\alpha}.L^{\alpha}.\ \lambda^{\beta}.K^{\beta}$

$A.\ L^{\alpha}.K^{\beta}.\ \lambda^{\alpha+\beta}$

$Q\lambda=Q\lambda$

Thus it is proved that if inputs are increased by λ times output is also increased by λ times. From this it can be concluded that,

If $\alpha+\beta=1$, constant returns to scale operate and is homogenous of degree 1

If $\alpha + \beta < 1$, decreasing returns to scale operate

If $\alpha + \beta > 1$, increasing returns to scale operate

2. The exponents of labour and capital of the production function represent output elasticity of labour and capital respectively

The elasticity of output with respect to L is the ratio of the proportionate change in total quantity of output with respect to the proportionate change in the quantity of labour use.

Elasticity of output with respect to labour is:

$$e_L = (dQ/dL).(L/Q) = (A\,\alpha L^{\alpha} L^{-1} K^{\beta}).(L/\ Q = AL^{\alpha}K^{\beta}) = \alpha$$

Elasticity of output with respect to Capital is

$$e_K = (dQ/dK).(L/Q) = (A\,\beta L^{\alpha} K^{-1} K^{\beta}).(K/\ Q = AL^{\alpha}K^{\beta}) = \beta$$

3. Marginal Product of factors

(a) MP_L

$Q = A.L^{\alpha}.K^{\beta}$

$MP_L = dQ/dL$

$= A.\ \alpha.L^{\alpha-1}.K^{\beta}$

$= A.\ \alpha.L^{\alpha}.L^{-1}.\ K^{\beta}$

$= Q/L.\ \alpha$

$= \alpha AP_L$

$MP_L = .\ \alpha\ AP_L$

(b) MP_K

$Q = A.L^{\alpha}.K^{\beta}$

$MP_k = dQ/dK$

$= A.\ \beta\ .L^{\alpha}.K^{\beta-1}$

$= A.\ \beta.L^{\alpha}.\ K^{\beta}.\ K^{-1}$

$= Q/K.\ \beta$

$= \beta\ AP_K$

$MP_K = \beta\ AP_K$

4. Marginal rate of substitution

It represents the slope of the isoquant and it is negatively sloped.

$MRS_{LK} = MP_L/MP_K$

$MP_L = Q/L.^{\alpha}$ $MP_K = Q/K.\ \beta$

$= \alpha/\beta.K/L$

5. Elasticity of factor substitution is unity

It is defined as percentage change in K/L ratio divided by percentage change in MRTS

$\varepsilon = [d(K/L)/K/L)] / [dMRTS/MRTS]$

$\varepsilon = [d(K/L)/K/L)] / [d(MP_L/MP_K) / (ML_L/MP_K)]$

$\varepsilon = 1$

The elasticity of substitution as per Cobb Douglas production function is equal to unity.

6. Average product

$AP_L = A[K/L]^{\beta}$ or Q/L

$AP_K = A[L/K]^{\alpha}$ or Q/K

7. Function coefficient

$\alpha + \beta = 1$

Denotes constant returns to Scale

8. Exhaustion of Product

The Cobb-Douglas Production Function can prove the celebrated Eulers theorem. The theorem states that if the factors are rewarded according to their Marginal Product, then the Total Product will be exhausted.

i.e. $MP_L.L = MP_K.K = Q$

Such a proposition can be proved with the help of linear homogenous variety of Cobb-Douglas production function.

$dQ/dL.L = dQ/dK.K = Q$

$\alpha.Q/L.L + \beta.Q/K.K = Q$

$\alpha.Q + \beta.Q = Q$

$\alpha + \beta = 1$

This ensures that the total product is exhausted.

9. Factor intensity

In a Cobb-Douglas production function the factor intensity is measured by α/β ratio. The higher ratio more labour intensive the technique and lower the ratio more capital intensive the technique is.

10. Efficiency of production

In a Cobb-Douglas production function the efficiency of production is measured by the coefficient A. A more efficient firm will have a larger A.

11. Expansion Path

It represents the locus of points of tangency between isoquants and isocost lines. The expansion path of Cobb-Douglas production function is linear and passes through origin.

$Q = A.L^{\alpha}.K^{\beta}$

$dQ/dL = \alpha.Q/L \qquad dQ/dK = \beta.Q/K$

1st order condition of optimisation

$MP_L/MP_K = P_L/P_K$

$[\alpha.Q/L]/[\beta.Q/K] = P_L/P_K$

$\alpha.K.\,P_K = \beta.L.P_L.$

$= \alpha.K.\,P_K - \beta.L.P_L = 0$

The equation represent that the expansion path generated by Cobb-Douglas production function is linear homogenous and passes through the origin.

12. The two factors complement to each other

It assumes constant returns to scale under which all inputs are changed in equal proportion. If one of the input is zero, naturally all other inputs will also be held zero.

For constant levels of K, the output labour relation is shown as a series of curved lines.

If either input is zero(K=0 or L=0), output is zero. Thus both inputs are necessary in the production process. The curve is such that MP falls as input grows. There is no ceiling beyond which production cannot grow, but the rate of increase in output decreases at higher levels of inputs.

13. For Cobb-Douglas production function the isoquants will be downward sloping and convex to the origin

The isoquants of a Cobb-Douglas production function are convex to the origin. Convexity implies that the MP of an input decreases as the quantity of output increase. The first order derivatives are the marginal functions, referred to as the marginal products. The second derivatives(the rate of change of the first derivatives) indicate whether marginal products (MP_L, MP_K) is increasing or decreasing.

Consider input L

$dQ/dL = A .\alpha.L^{\alpha}.L^{-1}. K^{\beta}$

$d^2Q/dL^2 = A. L^{\alpha-2}. K^{\beta}. \alpha .\alpha-1$

$= \alpha. (\alpha -1). A. L^{\alpha-2}. K^{\beta}$

As 0<α -1, the expression (α-1) is negative and the whole expression which stands for the change in MP_L is negative. Since MP_L is declining we can conclude that the isoquant is convex to the origin. A Cobb Douglas production function exhibits diminishing returns to each factor. This is confirmed by the negative second derivatives.

Worked example

1. Given the Cobb Douglas production function

$Q=50L^{0.3}K^{0.5}$

$Q=50L^{0.4}K^{0.6}$

$Q=50L^{0.5}K^{0.6}$

(i) Calculate the level of output when L = 10, K = 15.

(ii) Calculate the level of output when both inputs double,

Comment on the returns to scale.

Production function	L = 10, K = 15	L = 20, K = 30	
$Q=50L^{0.3}K^{0.5}$	$Q_1=50L^{0.3}K^{0.5}$ $Q_1=50(10)^{0.3}(15)^{0.5}$ = 50(2.0)(3.9) = 390	$Q_2=50L^{0.3}K^{0.5}$ $Q_2=50(20)^{0.3}(30)^{0.5}$ =50(2.46)(5.48) =674 <390*2 $<Q_1$*2	Production function exhibits decreasing returns to scale as (α+ β)<1 Degree of homogeneity=0.8; <1
$Q=50L^{0.4}K^{0.6}$	$Q_1=50L^{0.4}K^{0.6}$ $Q = 50(10)^{0.4}(15)^{0.6}$ = 50(2.51)(5.08) = 637	$Q_2=50L^{0.4}K^{0.6}$ $Q_2=50(20)^{0.4}(30)^{0.6}$ =50(3.31)(7.70) =1274 =637*2 i.e. $= Q_1$*2	Production function exhibits constant returns to scale as (α+ β)=1 Degree of homogeneity=1

$Q=50L^{0.5}K^{0.6}$	$Q_1=50L^{0.5}K^{0.6}$ $Q=50(10)^{0.5}(15)^{0.6}$ $= 50(3.3)(5.1)$ $=816$	$Q_2=50L^{0.5}K^{0.6}$ $Q_2=50(20)^{0.5}(30)^{0.6}$ $=50(4.47)(7.69)$ $=1719$ $>816*2$ $> Q_1*2$	Production function exhibits increasing returns to scale as $(\alpha+ \beta)>1$ Degree of homogeneity = 1.1; >1

Recap

- The general form of the Cobb-Douglas production function is $Q=AL^{\alpha}K^{\beta}$
- The isoquants of a Cobb-Douglas production function are convex to the origin
- The Eulers theorem states that if the factors are rewarded according to their Marginal Product, then the Total Product will be exhausted.
- In a Cobb-Douglas production function the factor intensity is measured by α/ β ratio

Objective Questions

1. What is the general form of a Cobb Douglas production function?
2. What is the shape of isoquant under Cobb Douglas production function?
3. What is Eulers theorem?

Answers

1. The general form of the CDPF is $Q=AL^{\alpha}K^{\beta}$
2. The isoquants of a CDPF are convex to the origin.
3. The Eulers theorem states that if the factors are rewarded according to their Marginal Product, then the Total Product will be exhausted.

Assignments

1. Derive the important properties of Cobb-Douglas production function.
2. Given the Cobb-Douglas production function $Q=50L^{0.3}K^{0.5}$. Calculate and comment on the values of the AP_L and the MP_L at l=2,4,6 and when K remains constant at K=10.
3. Determine whether each of the following production functions have increasing, decreasing or constant returns to scale. Also examine the degree of homogeneity.

   (a)$Q=10K^{0.5}L^{0.5}$ (b) $Q=10K^{0.7}L^{0.6}$ (c) $Q=10K^{0.3}L^{0.5}$
4. Define homogenous production function with example.
5. Find the first and second partial derivatives for the production function.

   $Q = 80K^{0.2}L^{0.5}$

References

1. Salvatore, D (2011), *Microeconomic theory*, Schaum's Outline Series, McGrawHill.(3rd Edition
2. Bradley, Teresa and Paul Patton, (2002) *Essential Mathematics for Economics and Business*, John Wiley & Sons Ltd

Suggested Readings

1. Ramesh Chandra Das(2017), *Microeconomics Theory and Practice*(Second edition), Kunal Books, New Delhi.

2. Henderson, J. M. and R.E. Quandt (1980) *Microeconomic Theory: A Mathematical Approach*, McGraw Hill, New Delhi.

5

BLOCK

Linear Programming and Input Output Analysis

UNIT 1

Linear Programming

Learning Outcomes

After completing this unit, learners will be able to:

- narrate the basic concepts of linear programming
- identify the method to be followed in solving linear programming problems
- get a basic idea in solving LP problems

Prerequisites

Optimisation is the way of life. The economy has limited resources and time which is to be utilised competently to attain our goals. By optimisation we mean the process of making the best or most effective use of the available resources. Linear Programming is one of the simplest ways to perform optimisation. The origin of linear programming was during the Second World War, when England faced the problem of resource scarcity. At the same time their objective was to win the war with these limited resources. Therefore in order to utilise the resources effectively and to solve the problem linear programming was developed. Whenever we want to allocate existing limited resources for various competing activities for achieving our desired objective the technique that helps us is linear programming. It is an important tool for analysing a wide range of real world decision problems. It is used in business and industry in various types of scheduling, production planning, transportation and routing. In industrial sector the model is used for minimising cost of production, maximising profit, increasing productivity and effective utilisation of scarce resources.

Keywords

Decision Variables, Objective Function, Constraints, Feasible Solution, Optimal Solution

5.1.1 Linear Programming

Linear Programming employs a mathematical model to characterise optimisation problem. A model represents the features of an object, system or problem, and mathematical models are represented in the form of parameters, variables and functions. These models give an idea of how the real system works under various conditions. An optimisation problem seeks to maximise or minimise an objective function. In the purview of business economics, LP is an efficient tool that may help a firm to make proficient decisions. A firm faces two important problems while undertaking production. Firstly for producing a particular commodity, the producer has to choose the production process from various alternative processes which minimise its cost of production. Secondly the firm needs to take decision on what amount of output to be produced so as to maximise its profit. Thus the managers face decision making problem in allocating resources constituting inputs like land, labour and capital, efficiently in order to minimise cost and in choosing the optimum product mix which maximises profit. Moreover an important assumption applied to the theory of the firm under LPP is that the prices of product and inputs will remain constant. Let us go through an example:

If a textile manufacturer has 1000m^2 of cotton and 1250 m^2 of silk. The production of 1 saree needs 3m^2 of silk and 2m^2 of cotton. Each kurta needs 1.5 m^2 silk and 1 m^2 cotton. The price of saree is fixed as Rs60 and kurta as Rs 30. Now suppose the objective is to find the number of sarees and kurtas to be produced and sold by the firm which will maximise its revenue. Here saree and kurta constitute the decision variables. The available resources comprises of 1000m^2 of cotton and 1250 m^2 of silk, which act as constraints in the production process. The constraints can be mathematically depicted as:

$3x+1.5y \leq 1250$….1

$2x+1y \leq 1000$….2

The equation 1 represents the constraint in the availability of silk and the second constraint represents the limited accessibility of cotton.

Now the objective of LPP is to maximise the numerical value i.e the sales revenue which is obtained by finding out the optimum product mix constituting number of sarees and kurtas to be produced and sold by the firm. Let x and y represent saree and kurta. Thus the objective function is maximise Z= (price per unit of x)* (number of units of x)+ (number of units of y))*(price per unit of y) Max Z=60x+30y.

Thus such problem has an objective function and one or more constraints. The firms while choosing their optimum product mix which maximises their profit faces the constraint of input availability, at the same time while choosing the input mix which minimises their cost of production faces the minimum quality constraints. Thus LPP is an efficient tool for calculating optimum product mix and input mix which maximises profit and minimises cost subject to constraints. Dr George B. Dantzig who developed this technique in 1947, is considered as the father of Linear Programming. In 1975 Tjalling Koopmans and Leonid Kantorovich won the Nobel prize in economics for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role.

LPP considers only linear relationship between two or more variables. The relationship is linear because it yields a straight line when plotted graphically. A linear program may take different forms. We have maximisation and minimisation problems depending on whether the objective function is to be minimised or maximised. The optimal value of linear function (output, cost, profit) is subject to several constraints. Constrained optimisation models helps in finding the best solution subject to a set of constraints. When the constraints are restricted to two variables graphical method is used for obtaining optimal solution. On the other hand when the variables in the constraints constitute more than two variables, the optimal value is found by simplex method. It includes problems dealing with maximising profit, minimising costs and minimal usage of resources. Thus LPP is a technique used to solve maximisation and minimisation problems, especially relating to production through systematic planning called programming. Thus in linear programming we formulate our real life problem to a mathematical model. It is used in business and industry in various types of scheduling, production planning, transportation and routing.

Let us look at a Linear programming problem

A firm has given quantity of 3 factors, Labour, Capital and land, with which it produces two commodities X and Y. The problem of the firm is choosing the optimum product mix which will maximise its profit. Suppose X makes a profit of Rs 2 per unit and Y yields a profit of Rs 1 per unit. The resource position can be put as follows.

L= 400 units(hours)

K=300 units(machine hours)

S=1000 units(eg feet)

Where L= labour, K = capital and S = land

The firm produces commodities x and y. For the production of x and y the firm uses two processes. For producing 1 unit of commodity x the firm is required to employ 4 labourers, 1 unit of capital and 2 units of land and 1 unit of commodity y requires 1 labourer, 1 capital and 5 units of land.

Process A	Process B
$L_x = 4$	$L_y = 1$

$K_x=1$ $K_y=1$

$S_x=2$ $S_y=5$

The optimum product mix which will maximise profit Z=2x+1y; where x and y are the number of units produced of x and y respectively and the numbers 2 and 1 represent the profit per unit of x and y respectively as stated earlier.

With respect to the above example we can discuss the basic concepts of LPP.

5.1.1.1 Basic Concepts

The three major components of LPP are decision variables, objective function and constraints.

Decision Variables

These are variables over which we have control. A solution to LPP is explained in terms of decision variables. In the above example the decision variables are the quantities of X and Y that will be produced. The objective of solving a linear programming problem is to find a set of decision variables that will produce the optimal output. The decision variables must always have non-negative value, which is given by non negative constraints.

Objective Function

The decision variables when arranged in a mathematical equation form constitute an objective function. The objective function also known as criterion function defines the quantity that we wish to optimise and specifies the direction of optimisation i.e. either to maximise or minimise. It is expressed as a linear equation.

In the above problem we wish to maximise profit. Total profit designated as z is written as follows:

Z = (profit per unit of x) × (number of units of x produced) + (profit per unit of y) × (number of units of y produced)

Therefore our objective function is:

$$Z = 2x+1y$$

Constraints

As discussed earlier the maximisation as well as minimisation of objective function is subject to certain constraints or restrictions. As per the above example in order to make Z as large as possible, infinite quantity of x and y should be produced. But there are certain constraints which act as a barrier in attaining infinite profit. In the above example the availability of land, labour and capital is limited which act as a constraint in the production capacity of the firm. These are expressed in the form of mathematical equalities and inequalities that represent physical, technical, economical or other restrictions. In order to achieve the objective function, the firm faces many constraints. The constraints are the restrictions or limitations on the total amount of a particular resource, required to carry out the activities, which would decide the level of achievement in the decision variables. If the basic problem is maximisation then all constraints associated with the objective function must have 'less than equal to' restrictions with the resource availability. On the other hand if the basic problem whose solution is to be attempted is minimisation then all constraints associated with it must have 'greater than equal to' sign. The constraints can be grouped into two; technical and non negativity constraints or restrictions.

Technical constraints are set by the state of technology and availability of factors. The constraint being the "the quantity of factors used in production of commodities cannot exceed the availability of factors". Technical constraints form the inequality constraint. The technical constraints in the above example can be expressed as:

$4x+1y\leq400$

$1x+1y\leq300$

$2x+5y\leq1000$

Since in the above problem we wish to maximise profit, the constraints associated with it has 'less than equal to sign'. If in the maximisation problem the constraints have '$\geq$'sign then multiply both sides by '-1' and convert the inequality sign to '$\leq$'

Non-negativity constraints means that the level of production of X and Y should not be negative.

$x\geq0$

$y\geq0$

Maximise $Z=2X+1Y$

Subject to $4x+1y\leq400$

$1x+1y\leq300$

$2x+5y\leq1000$

$X,Y\geq0$

This type of model is called linear programming model. Thus a linear program model has a linear objective function and linear constraints.

5.1.3 Nature of Feasible, Basic and Optimal Solution

Feasible Region

The feasible region is determined by the constraints. It contains those solutions which meet or satisfy the constraints of the problem. The region of feasible solutions depends upon the nature of constraints.

Feasible Solutions

A feasible solution is one in which the values of decision variables satisfies all constraints. In the above example any value of x and y which satisfies all four constraints constitutes a feasible solution. A deeper understanding of feasible solution is possible from looking at the linear programming problems solved in the subsequent unit.

Basic Solution

A basic solution for which all the basic variables are non-negative is called the basic feasible solution.

Optimum Basic Feasible Solution

The best of all feasible solutions is the optimal solution. In other words any point in the feasible region of a linear programming problem that maximises or minimises the objective function is called an optimal feasible solution. It may have the largest objective function value (for a maximisation problem and smallest objective function value (for minimisation problem). For example if the objective function is to maximise profits from the production of two goods, then the optimal feasible solution constitute the combination of two goods which maximise the profit of the firm. Similarly if the objective is to minimise the costs of firm the optimal solution constitutes the process which minimises the cost for a given level of output. Moreover the optimum solution must lie within the region of feasible solutions. The optimal solution in linear programming is obtained from two methods-graphical method and simplex method. The graphical method for solving LPP is detailed in the coming unit.

Assumptions

Linearity

Linearity constitutes the principal attribute of a linear programming problem. By linearity we mean that linear programming problems must be formulated in terms of linear functions which are also subject to a set of linear constraints of which some are expressed as inequalities. Therefore the linear programming problem which is to be analysed is expressed in the form of linear equations and inequalities. Moreover the objective function and the constraints must be linear

Proportionality

Proportionality means that the change in objective function and left hand side of the constraints is proportional to the value of the decision variables.

Additivity

Additivity means both the objective function and left hand side of the constraints are expressed as the sum of individual contributions of the decision variables.

Divisibility

Divisibility means the decision variables may take any value, not just integral or discrete values.

Certainty-means that the coefficients of objective function and constraints should be known with certainty.

The decision variables must be continuous.

5.1.4 Steps in Problem Formulation

1. The first step in formulating a LPP is to categorise the decision variables completely along with the units in which they are measured. For example the time in hours, quantity in tons, units per day etc.

2. The second step is the formulation of objective function. The decision variables should be arranged in a mathematical form which constitute the objective function. Also identify whether the function needs to be minimised or maximised.

3. Elucidate the constraints that act as a limitation to the production capacity of the firm. Express them in mathematical equalities and inequalities as well as group them to technical and non-negativity constraints.

4. Ensure that the decision variables are greater than or equal to zero.

5. Solve the LPP using graphical or simplex methods.

Formulation of LPP with **Different Type of Constraints**.

Worked examples

Example 5.1.1

A toy manufacturing company manufactures 2 types of dolls a basic version doll A(x_1) and a deluxe version doll B (x_2). Each doll of type B takes twice as long as to produce as 1 doll of type A and the company would have time to make a maximum of 2000 if it produced only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B). The deluxe version requires a fancy dress of which are available 600 per day. The company makes profit of Rs 3 and Rs 5 per doll respectively on doll A and B. Write the problem in linear Programming form.

Solution

Let us describe x_1 and x_2 as the number of units of A and B manufactured daily. Since the purpose is to maximise the total profit of the toy manufacturing company the linear programming problem is given as:

Objective function Max $Z=3x_1+5x_2$

Since it is a maximisation problem then all technical constraints associated with the objective function must have 'less than equal to' sign

Constraints

Time constraint

$x_1 + 2x_2 \leq 2000$

Supply constraint

$x_1 + x_2 \leq 1500$

Fancy dress constraint

$x_2 \leq 600$

Non negativity constraints

$x_1 \geq 0 \quad x_2 \geq 0$

Maz $\quad x_1 + 2x_2 \leq 2000$

Sub to $\quad x_1 + 2x_2 \leq 2000$

$x_1 + x_2 \leq 1500$

$x_2 \leq 600$

$x_1 \geq 0 \quad x_2 \geq 0$

Example 5.1.2

Three nutrient components namely calories, carbohydrates and Protein are found in a diet of two food items A and B. The amount of each nutrient in grams is given below.

Nutrient	A	B
Calories	0.15	0.10
Carbohydrate	0.75	1.70
Protein	1.30	1.10

The cost of food A and B are Rs 2 per gram and Rs 1.70 per gram respectively. the minimum daily requirements of these nutrients are at least 1.00 gram of calories, 7.50 gram of carbohydrate and 10.00 g of protein. Write the problem in linear programming form.

Solution

Let us define by x_1 and x_2 the number of units of A and B respectively purchased every day. Since the purpose is to minimise the total cost of the food items and to satisfy the minimum daily requirement of nutrient the linear programming problem is given by:

Objective function

Minimise $Z=2\ x_1+1.7x_2$

Subject to

$0.15\ x_1+0.10x_2 \geq 1.0$

$0.75\ x_1+1.70x_2 \geq 7.5$

$1.30\ x_1+1.10x_2 \geq 10.0$

$x_1 \geq 0,\ x_2 \geq 0$

Recap

- LP is an efficient tool that may help a firm to make proficient decisions
- Linear programming is a technique used to solve maximisation and minimisation problems
- The linear function which has to be maximised and minimised under LPP is known as linear objective function
- A solution to LPP is explained in terms of decision variables
- The decision variables when arranged in a mathematical equation form constitute an objective function
- The optimal value of linear function (output, cost, profit) is subject to several constraints
- The objective of solving a linear programming problem is to find a set of decision variables that will produce the optimal output
- The decision variables when arranged in a mathematical equation form constitute an objective function
- In Linear programming the objective function and the constraints must be linear
- A feasible solution is one which satisfies all constraints

Objective Questions

1. State the unique technique used to obtain optimum use of productive resources.
2. Which function in a linear program has to be maximised or minimised?
3. State the common region determined by all the linear constraints in a LPP.
4. What is an objective function?
5. Identify the sign of constraints for a maximisation problem.
6. What is a feasible solution?
7. What is a decision variable?

Answers

1. Linear Programming.
2. Objective function.
3. Feasible region.
4. The decision variables when arranged in a mathematical equation form constitute an objective function.
5. Less than or equal to sign.
6. A feasible solution is one which satisfies all constraints.
7. A solution to LPP is explained in terms of decision variables.

Assignments

1. What are the steps followed in the formulation of a linear programming problem?
2. A company manufactures two products X and Y, which require the following resources. The resources are the capacities of machine M_1,M_2 and M_3. The available capacities are 50, 25 and 15 hours respectively in the planning period. Product X requires 1 hour of machine M_2 and 1 hour of machine M_3. Product Y requires 2 hours of machine $M_{1,}$ 2 hours of machine M_2 and 1 hour of machine M_3. The profit contribution of products X and Y are Rs 5 and Rs 4 respectively. Formulate LPP.
3. Reddy Mikks produces both interior and exterior paints from two raw materials, M and M. the following table provides the basic data of the problem:

	Tons of raw material Exterior paint	 Interior paint	Maximum daily availability(tons)
M	6	4	24
M	1	2	6
Profit per ton	5	4	

The daily demand for interior paint cannot exceed that of exterior paint by more than one ton. Also the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum(best) product mix of exterior and interior paints that maximises the total daily profit. Formulate Linear Programming Problem.

References

1. Murthy P Rama(2007), *Operations Research,* New Age International Publishers, New Delhi

2. Hamdy A Taha(2017), *Operations Research-An introduction,* Pearson Education Limited, England

3. Mike Rosser(2003), *Basic Mathematics for Economists,* Routledge London

4. Teresa Bradley and Paul Patton (2002), *Essential Mathematics for Economics and Business,* John Wiley & Sons Ltd, England.

5. Dowling, Edward T. *Introduction to Mathematical Economics, Schaum's Outline Series, McGrawHill. 2nd edition*

6. Ahuja H. L. (2018), *Advanced Economic Theory – Microeconomic Analysis* (Twenty-First Edition), S Chand and Company Limited, New Delhi.

UNIT 2

Graphical Solution to Linear Programming

Learning Outcomes

After completing this unit, learner will be able to:

- solve linear programming problems by applying graphical methods
- familiarise with duality in LPP
- apply linear programming models for decision making

Prerequisites

In the previous unit we have discussed the basic concepts of linear programming and the method for formulating the LPP. In this unit we move on to the graphical method of solving linear programming problems, focusing on maximisation and minimisation problems along with the duality analysis. The objective function may have two or more decision variables. Graphical method is adopted for solving linear programming problems with two decision variables. For example, if a company manufactures two types of dolls say X and Y. The profit per unit of X and Y is given and the aim of the company is to maximise profit. Here the decision variables constitute the number of units of X and Y that the company should produce to maximise profit. Since there are only two decision variables X and Y the problem is solved through graphical method. Similarly, there is minimisation problem associated with minimising the cost of production. After learning the procedure of maximisation and minimisation problems, we discuss the concept of duality with worked examples. Every linear programming problem has an associated problem called the dual problem, which provides the same optimal solution as the primal.

Keywords

Maxima, Minima, Feasible Region, Duality, Primal, Dual

5.2.1 Graphic Solution in Linear Programming

Linear programming with two decision variables can be solved graphically. The objective function and the constraints must be linear. Under graphical method, the inequalities are measured as equations, as one cannot draw a graph for inequality. We have only two decision variables in the problem, as we can draw straight line graphs in two-dimensional plane having X axis and Y axis. Besides the non-negativity constraints in the problem should be greater than zero (i.e. positive), since the solution to the problem should be on the first quadrant of the graph.

Steps

1. Formulate the linear programming problem as stated in section 5.1.4.
2. Read all the constraints as equalities.
3. Draw straight lines corresponding to the equations obtained in step 2 by joining the two axes. So there will be as many straight lines, as there are equations.
4. Plot the feasible region defined by the constraints. If the constraint is 'less than equal to' then the feasible area lies towards the origin. On the other hand if constraints have 'greater than equal to' sign then the feasible zone lies towards the origin.
5. The feasible region has many points. The corner point of the figure is to be located and the coordinates at these points is to be measured.
6. Calculate the value of the objective function corresponding to each corner point.
7. Find the coordinates of objective function and plot it on the graph.
8. Find the solution point.

5.2.2 Maxima

A resource allocation problem in which the firm decides to produce the optimum output mix which maximises its profit subject to certain constraints is a maximisation problem in linear programming. The objective function is maximised subject to limited input availability. Here we have to find the maximum value of objective function. Under maximisation problem all constraints associated with the objective function must have 'less than equal to' sign. Here feasible solution region lies towards the origin and the point that lies at the furthermost point of feasible zone will give the maximum profit.

Example 1

A firm manufactures two goods X and Y using three inputs K, L and S. The firm has at its disposal 40 units of S, 150 units of K and 120 units of L. The net profit contributed by each unit sold is \$ 4 for X and \$ 1 for Y. Each unit of X produced requires 3 units of K, 2 units of S and 4 units of L. Each unit of Y produced requires 5 units of K, 3 units of L and none of S. What combination of X and Y should the firm manufacture to maximise profits given these constraints on input availability? Solve graphically.

Solution

	Product X	Product Y	Total Available/ Factors
Profit	\$4	\$1	
Land(S)	2	0	40
Labour(L)	4	3	120
Capital(K)	3	5	150

Solution

We introduce the decision variables x and y, where x and y represents the quantities of x and y that are produced. The objective function can be expressed as:

$$Z = 4x+1y$$

which is to be maximised.

The constraints corresponding to Land, labour and capital are:

$$2x \leq 40$$

$$4x+3y \leq 120$$

$$3x+5y \leq 150$$

Non negativity constraints

$$X \geq 0$$

$$Y \geq 0$$

Now turning on the graph in figure 5.2.1 which measures commodity X on x-axis and Y on y-axis. The first step is to mark out the feasible region. This will contain all values of X and Y which satisfy the above constraints.

Reading all the constraints as equalities.

$2x=40$

$4x+3y=120$

$3x+5y=150$

$x=0, y=0$

The linear constraints can be marked out by joining the intercepts in two axes. Obtain the values for x and y for each constraint, by setting x=0 and y=0 to obtain the values of y and x for each constraint.

1. $2x = 40$

   $x = 40/2,$

   $x = 20$ (x,y)=(20,0)

2. $4x+3y =120$ | $4x+3y =120$

   $x = 0,$ | $y = 0$

   $3y =120$ | $4x =120$

   $y =120/3$ | $x =120/4$

   $y = 40$ | $x =30$ (x,y)=(30,40)

3. $3x+5y =150$ | $3x+5y =150$

   $x = 0$ | $y = 0$

   $5y =150$ | $3x =150$

   $y = 150/5$ | $x =150/3$

   $y =30$ | $x = 50$ (x,y)=(50,30)

Plot the equations in a graph by locating two distinct points. In the figure 5.2.1the horizontal axis x and the vertical axis y represent the good x and y respectively. The first step in graphical solution is to mark out the feasible region. This will contain all values of x and y that satisfy all the above constraints. The linear constraints are marked out by joining the two axes. The non negativity constraints mean that the solution must lie on or above the x axis or to the right of y axis. For equation 4x+3y=120, when x=0 then y = 40 and when y = 0 then x = 30. In the same manner we deduce the values of x and y for the remaining two equations and plot on the graph.

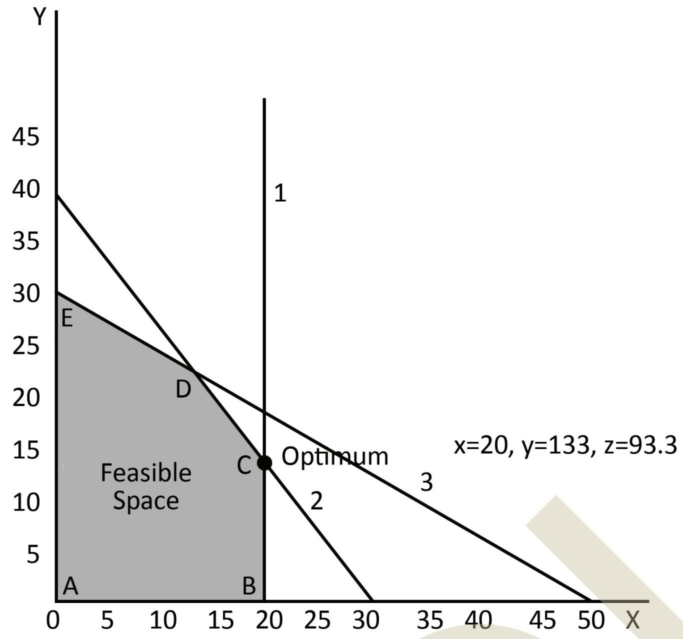


Fig 5.2.1 Feasible Solution

The constraints produce the feasible solution space ABCDE in which all constraints are satisfied. All points outside the boundary of the area ABCDE are infeasible. Here the firm is seeking to maximise profit subject to the constraints of land, labour and capital. Therefore the direction in which the profit function increase is determined by assigning arbitrary increasing values to Z. The value of Z at each point in the feasible space is computed and the solution with the maximum profit is the feasible point in the solution space.

Point	X	Y	Z = 4x+1y
A	0	0	0
B	20	0	Z = 4*20+0 = 80
C	20	13.33	Z = 4*20+13.33 = 93.3
D	13.63	21.82	Z= 4*13.63+21.82 =76.35
E	0	30	Z= 0+30 = 30

The above table depicts the values of x and y on each feasible points and the corresponding optimum values of objective function. The optimum values of x and y associated with the feasible point C and D are determined by solving equations associated with lines 1 and 2 for point C and lines 2 and 3 for point D. The solution is obtained at the point where the two constraints intersect. The maximum profit of Z=4x+y is 93.3 which is at point C the solution is 20 units of x and 13.3 units of y.

Example 2

A toy company manufactures two types of dolls a basic version doll A(x) and a deluxe version doll B(y). Each doll of type B takes twice as long as to produce 1 type A doll and the company would have time to make a maximum of 2000, if it reduced only the basic version. The supply of plastic is insufficient to produce 1500 dolls per day(both A and B). The deluxe version requires a fancy dress of which there are 600 per day available. If the company makes profit of Rs 3 and Rs 5 per doll respectively on doll A and B, how many of each should be produced per day in order to maximise profit.

Solution

Let A and B be the two dolls

$A = x$

$B = y$

Maximise $Z=3x+5y$

Time constraint

$x+2y \leq 2000$

Supply constraint

$x+y \leq 1500$

Fancy dress

$y \leq 600$

Non negativity constraints

$x \geq 0$

$y \geq 0$

Maximise $Z=3x+5y$

Subject to $x+2y \leq 2000$

$x+y \leq 1500$

$y \leq 600$

$x \geq 0$

$y \geq 0$

Reading constraints as equations

$x+2y = 2000$

$x+y = 1500$

$y = 600$

x = 0

y = 0

Obtain the values of x and y for each constraint by setting x=0 and y=0 to obtain the values of y and x for each constraint.

1. x+2y = 2000	x+2y = 2000	
x = 0	y = 0	
2y = 2000	x = 2000	
y = 2000/2		(x,y)=(2000,1000)
y = 1000		
2. x+y = 1500	x+y =1500	
x = 0,	y = 0	
y =1500	x =1500	(x,y)=(1500,1500)
3. y = 600	x = 0	(x,y)=(0,600)

By plotting the values of x and y of each equation in a graph we can mark out the feasible region.

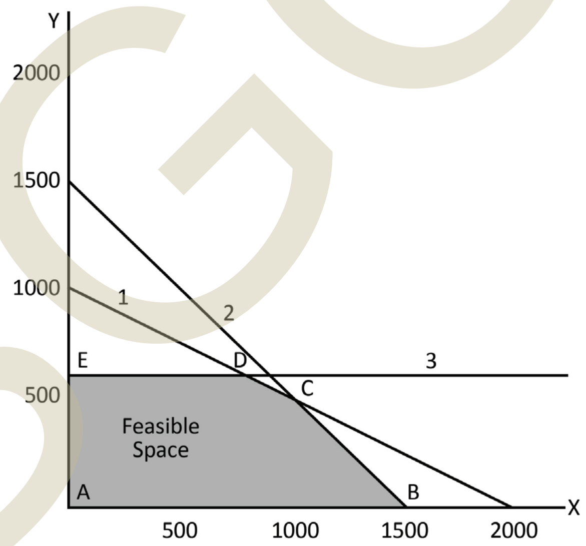


Fig 5.2.2 Feasible Solution

The constraints produce the feasible solution space ABCDE in which all constraints are satisfied. All points outside the boundary of the area ABCDE are infeasible. Here the firm is seeking to maximise profit subject to time supply and fancy dress constraints. Therefore the direction in which the profit function increase is determined by assigning arbitrary increasing values to Z. The value of Z at each point in the feasible space is computed and the solution with the maximum profit is the feasible point in the solution space.

Point	X	Y	Z=3x+5y
A	0	0	0
B	1500	0	Z=3*1500+0 = 4500
C	1000	500	Z=3*1000+5*500 = 5500
D	800	600	Z=3*800+5*600 = 5400
E	0	600	Z=0+5*600 = 3000

The above table depicts the values of x and y on each feasible points and the corresponding optimum values of objective function. The optimum values of x and y associated with the feasible point C and D are determined by solving equations associated with lines 1 and 2 for point C and lines 1 and 3 for point D. The solution is obtained at the point where the two constraints intersect. The maximum profit of Z=3x+5y is 5500 which is at point C the solution is 1000 units of x and 500 units of y.

Example 3

Maximise $Z=5x_1+4x_2$

Subject to

$6x_1+4x_2 \leq 24$

$x_1+2x_2 \leq 6$

$-x_1+x_2 \leq 1$

$x_2 \leq 2$

$x_1 \geq 0,\ x_2 \geq 0$

First replace each inequality with an equation.

$6x_1+4x_2=24$

$x_1+2x_2=6$

$-x_1+x_2=1$

$x_2=2$

Graph the resulting straight line of each equation by locating two distinct points. For example after replacing $6x_1+4x_2=24$, two distinct points are determined by setting $x_1=0$ to obtain $x_2=24/4=6$. Thus the line $6x_1+4x_2=24$ passes through (0,6) and (4,0) as shown in the figure 5.2.3.

1) $6x_1+4x_2=24$

when $x_1=0$

$4x_2=24$, $x_2=\frac{24}{4}=6$

when $x_2=0$, $6x_1=24$

$x_1=\frac{24}{6}=4$

(x_1, x_2) (4,6)

2) $x_1+2x_2= 6$

when $x_1= 0$ $2x_2= 6$

$x_2=\frac{6}{2}=3$

when $x_2= 0$, $x_1= 6$

(x_1, x_2) (6,3)

3) $-x_1+x_2=1$

when $x_1= 0$

$x_2=1$

when $x_2= 0$,

$x_1= -1$

(x_1, x_2) (-1,1)

4) $x_2=2$

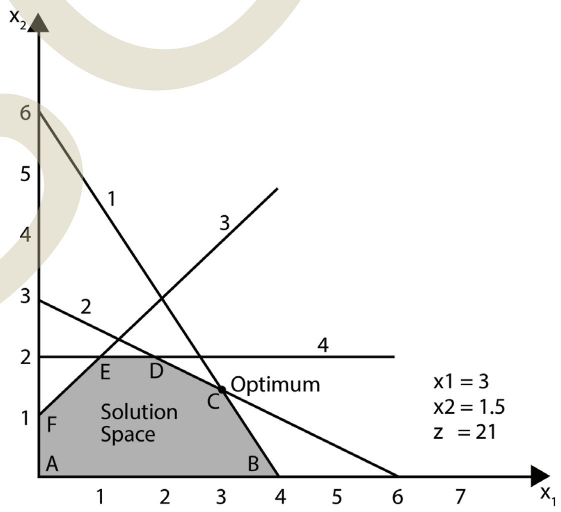


Fig 5.2.3 Feasible Solution

In the figure the horizontal axis represents x_1 and the vertical axis represents x_2. To account for the four constraints first replace each inequality with an equation and graph the resulting straight line by locating two distinct points. The constraints produce the feasible space ABCDEF in which all constraints are satisfied. All constraints outside the boundary of the area ABCDEF are infeasible.

Determination of the optimum solution

The number of solution points in the feasible space ABCDEF is infinite. First the direction in which the profit function increases is determined by assigning arbitrary increasing values to Z. The value of Z at each point on the feasible space is computed, and the solution with the maximum profit is the feasible point in the solution space.

Point	x_1	x_2	$Z=5x_1+4x_2$
A	0	0	0
B	4	0	Z=5*4+0=20
C	3	1.5	Z=5*3+4*15=21
D	2	2	Z=5*2+4*2=18
E	1	2	Z=5*1+4*2=12
F	0	1	Z=0+4=4

The values of x_1 and x_2 associated with point C are determined by solving equations associated with lines (1) and (2).

$$6x_1+4x_2=24\text{….(1)}$$

$$x_1+2x_2=6\text{…..(2)}$$

Multiplying equation (2) with 2.. $2x_1+4x_2=12$

Subtracting equation (1) $\quad 6x_1+4x_2= 24$

$$4x_1 = 12$$

$$x_1=\frac{12}{4}=3$$

Substituting $x_1=3$ **in equation (2)** $x_1+2x_2= 6$ we get $3+2*x_2= 6$

$$x_2=\frac{3}{2}=1.5$$

Therefore the maximum value of $Z=5x_1+ 4x_2$ is 21 which is at point C. The solution is 3 of x_1 and 1.5 of x_2.

5.2.3 Minima

A resource allocation problem in which the firm decides to produce the optimum input mix which minimises the cost of production subject to certain constraints is a minimisation problem in linear programming. Here we have to find the minimum value of objective function. Here the feasible area is usually above the constraint line and one needs to find the objective function line that is nearest to the origin within the feasible region.

Example 1

A firm makes a product that has minimum input requirements for the four ingredients W, X,Y and Z. these cannot be manufactured individually and can only be supplied as part of the composite inputs A and B.

1 litre of A includes 20 g of C, 5 g of D, 5 g of E and 20 g of F

1 litre of B includes 90 g of C, 7 g of D and 4 g of E but no F

One drum of the final product must contain at least 7200 g of C, 1400 g of D, 1000 g of E and 1200 g of F. (The volume of the drum is fixed and not related to the volume of inputs A and B as evaporation occurs during the production process.) If a litre of A costs \$9 and a litre of B costs \$16 how many litres of A and B should the firm use to minimise the cost of a drum of the final product? Assume that all other costs can be ignored.

Solution

Let us define x and y as the number of units of A and B respectively. Since the purpose is to minimise the cost of a drum of the final product, the liner programming problem is given by:

Objective function

Minimise $Z=9x+16y$

Subject to $20x+90y\geq 7200$

$5x+7y\geq 1400$

$5x+4y\geq 1000$

$20x\geq 1200$

$x,y\geq 0$

First replace each inequality with an equation.

$20x+90y=7200$

$5x+7y=1400$

$5x+4y = 1000$

$20x = 1200$

Graph the resulting straight line of each equation by locating two distinct points.

1. $20x+90y = 7200$

   When $x = 0$, $90y = 7200$

   $y = \frac{7200}{900} = 80$

   Similarly when $y = 0$ $20x = 7200$

   $x = \frac{7200}{20} = 360$

   (x,y)(360,80)

2. $5x + 7y = 1400$

   When $x = 0$, $7y = 1400$

   $y = \frac{1400}{7} = 200$

   When $y = 0$, $5x = 1400$

   $x = \frac{1400}{5} = 280$

   (x,y)(280,200)

3. $5x + 4y = 1000$

   When $x = 0$, $4y = 1000$

   $y = \frac{1000}{4} = 250$

   When $y = 0$, $5x = 1000$

   $x = \frac{1000}{5} = 200$

   (x,y)(200,250)

4. $20x = 1200$

   $x = \frac{1200}{20} = 60$

Graph the resulting straight line of each equation by locating two distinct points.

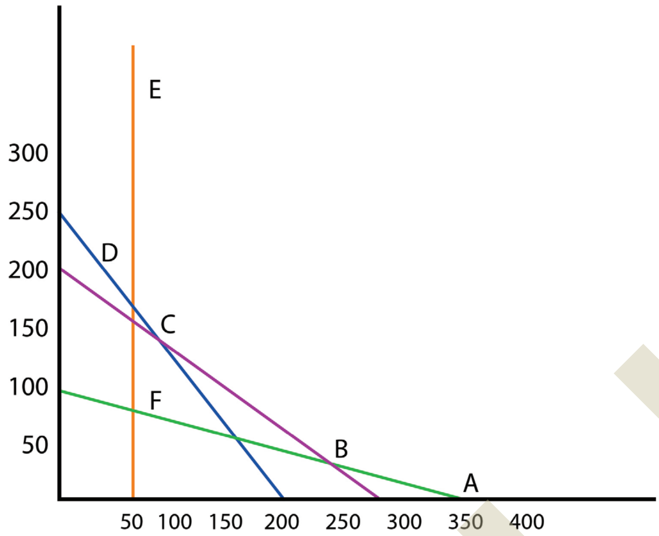


Fig 5.2.4 Feasible Solution

Determination of the Optimum Solution

The number of solution points in the feasible space ABC and D. The value of Z at each point on the feasible space is computed, and the solution with the minimum cost is the feasible point in the solution space.

Point	X	y	Z=9x +16y
A	360	0	Z=9*360=3240
B	**243.9**	25.8	Z=9*243.9+16*25.8=2607.90
C	93.4	133.3	Z=9*93.4+16*133.3=2973.4
D	60	175	Z=9*60+16*175=3340

The values of x and y associated with point B are determined by solving equations associated with lines (1) and (2).

$20x + 90y = 7200$ …..................(1)

$5x + 7y = 1400$ …..................(2)

Multiplying equation (2) with 4. $20x+28y=5600$

Subtracting equation (1) $20x+90y = 7200$

$62y = 1600$

$$y = \frac{1600}{62} = 25.8$$

Substituting y=25.8 **in equation (2)** $5x+7y=1400$

$5x + 7*25.8=1400$

$5x = 1291.4$

$x = \frac{1291.4}{5} = 243.9$

Therefore the minimum input required for producing A and B is at point B.

5.2.4 Duality

Linear programming problem can be analysed in two equivalent ways. Associated with every linear programming problem there exists another linear programming problem, based on the same data and have equivalent solution. For every maximisation problem, there is an associated minimisation problem that gives the same solution, and every minimisation problem has an equivalent maximisation problem. Here the linear programming problems come in pairs. The basic problem whose solution is attempted in Linear programming is called "Primal problem". To each primal problem there is a dual problem which provides additional information to the decision makers. The two problems are closely related in the sense that the optimal solution to one problem automatically provides optimal solution to the other. As such the solution to primal problem can be obtained by solving the dual. The nature of dual problem depends on the primal problem. eg;- if the primal is maximisation the dual is minimisation.

Before solving for the duality, the original linear programming problem is to be formulated in its standard form. If the primal is a maximisation problem then all constraints associated with the objective function must have '≤' restrictions with the resource availability unless a particular constraint is unrestricted (mostly represented by 'equal to' restriction). In minimisation case the constraints associated with the objective function must have '≥' sign.

Characteristics

- If the primal is a maximisation problem, then the dual is a minimisation problem and vice versa.
- The objective function co-efficients in the primal become constants on the right hand side of dual constraints.
- The right hand side constants of the constraints in the primal become the co-efficients of the objective function in the dual.
- Columns of constraint co-efficients in the primal become rows of constraint co-efficients in the dual.
- The row elements in the primal become column elements in the dual.
- The number of constraints in primal is equivalent to number of variables in the dual.
- The inequality signs of constraints in primal is reversed in case of dual,

such that in the primal problem inequality constraint is '≤' then in the dual problem the sign of inequality becomes '≥'

- If in the primal the variable is unrestricted the constraint in the dual becomes an equation.
- If the primal constraint is '≤' then the related dual variable is non-negative.
- If the primal constraint has = sign, then the associated dual variable is unrestricted in sign.
- The dual of the dual is primal.

Example 1

The concept of duality can be well understood with the help of problem given below:

Convert the following Primal to dual

Maximise $Z=2x + 3y$

Sub to $1x + 3y \leq 10$

$2x + 4y \leq 12$

x and y are ≥ 0

Solution

Since the objective function is maximisation type and the structural constraints are ≤ the dual of objective function is minimisation type and its associated structural constraints are ≥. The basis variables of the primal is x and y, different names are to be assigned to these variables say b and c. Now the objective function co-efficients in the primal (2 and 3) becomes constants on the right hand side of the dual constraints. The right hand side constants of the constraints of the primal become co-efficients of the objective function of the dual. Next the row elements of the constraints in primal becomes the column elements of constraints in the dual.

The duality can be applied to original linear programming problem as:

Minimise $G = 10b + 12c$

Sub to $1b + 2c \geq 2$

$3b + 4c \geq 3$

And b and c are ≥ 0

Example 2

Convert the following Primal to dual

Maximise $Z= 3a + 2b$

Sub to $3a + 1b \leq 10$

$2a + 6b \geq 12$

Both and b are ≥ 0

Solution

Here the structural constraints in the problem have got both $\geq$ and $\leq$ sign, since the objective function is maximisation all constraints should be of $\leq$ type. In order to convert $2a+6b\geq12$ to $\leq$ type multiply the constraint by -1. After this process, write the dual as stated in the above example.

Maximise $Z= 3a+2b$

Sub to $3a + 1b \leq 10$

$2a + 6b \geq 12$

Multiplying $2a+ 6b \geq 12$ with -1 we get the constraint as

$-2a-6b \leq -12$

The dual is

Minimise $Z=10x-12y$

Sub to $3x-2y \geq 3$

$1x-6y \geq 2$

Both x any y are ≥ 0

Example 3

Convert the following Primal to dual

Maximise $Z=2x+3y$

Sub to $1x + 3y \geq 10$

$2x + 4y =12$

x and y are ≥ 0

Solution

Here one of the constraints has equal to sign and the other constraint has $\geq$ sign. Since the primal is maximisation the constraint should be of $\leq$ type. So convert $1x+3y\geq10$ to $\leq$ type by multiplying it with -1. In case of constraint with an equation, the equal to sign can be removed by writing two versions i.e, constraint with $\leq$ type and $\geq$ type. Further the constraint with $\geq$ should be converted to $\leq$ type following the above said process. Finally write the dual.

Primal can be written as

Maximise $Z=2x+3y$

Sub to $1x + 3y \geq 10$

$2x + 4y \leq 12$

$2x + 4y \geq 12$

x and y are ≥ 0

This can be written as

Maximise $Z=2x+3y$

Sub to $-1x - 3y \leq -10$

$-2x - 4y \leq -12$

$2x + 4y \leq 12$

x and y are ≥ 0

The dual can be written as

Minimise $Z =-10a -12b + 12c$

Sub to $-1a-2b+2c \geq 2$

$-3a - 4b+4c \geq 3$

Where a,b and c are ≥ 0

Example 4

Convert the following Primal to dual

Minimise $Z = 5x +11y+3z$

Sub to $1x + 3y + 1z \geq 10$

$2x+1y-3z = 8$

x,y and z are ≥ 0

Solution

The primal can be written as

Minimise $Z=5x+11y+3z$

Sub to $1x+3y+1z \geq 10$

$2x+1y-3z \geq 8$

$2x+1y-3z \leq 8$

x,y and z are ≥ 0

This can be written as

Minimise Z = 5x+11y+3z

Sub to 1x+3y+1z ≥10

2x+1y-3z ≥8

-2x-1y+3z ≥-8

x,y and z are ≥0

The dual can be written as

Maximise Z=10a+8b-8c

Sub to 1a+2b-2c ≥5

3a+1b-1c ≥11

1a-3b+3c ≥3

a,b and c are ≥0

5.2.5Applications of Linear Programming in Economics

Linear Programming has wide range of applications in various fields of business/ government/industrial/agricultural and social sector. Areas where linear programming can be applied are given here as under:

1. Defense

Since the Second World War, linear programming has been applied in defense sector with the aim of obtaining maximum gains with minimum efforts. It helps in selecting the weapon and minimising the aviation gasoline used.

2. Product Mix

The Linear programming problem helps in determining the optimum product mix that can be produced by an industry using the available resources which maximises profit, revenue, sales and minimises the cost of production. For example paint and chemical industries use linear programming to determine their product mix. Linear Programming helps in identifying the advertising mix that maximise their exposure subject to constraints. Moreover they play a role in distributing their consumer products all over the world.

3. The Diet Problem

Linear programming can be applied in solving the diet problem. It helps in finding the cheapest combination of foods that help in meeting the nutritional requirements.

4. Transportation Problem

In order to determine the shortest possible route from origin to the destination LP can be applied. It is applicable under vehicle routing system. Thus it helps in minimising the cost of transportation.

5. Agriculture

With the rising population and scarcity of land every country is facing the problem of allocating the scarce land to different crops, given the climatic conditions and resource availability. It is used for identifying the crop rotation mix, fertilizer mix and allocating limited resources such as land, labour and capital in order to maximise revenue

6. Portfolio Optimisation

Many investment companies are applying linear programming to take decisions regarding allocation of assets and thus minimise the risk in portfolio investment which is subject to certain return.

Recap

- The graphical method solution in linear programming is used when there are only two decision variables
- The inequalities are measured as equations. Since the problem deals with two decision variables, it is easy to draw straight lines as the relationship is linear
- The feasible region in graphical method is defined by the constraints. If the constraint is 'less than equal to' then the feasible area lies towards the origin. On the other hand if constraints have 'greater than equal to' sign then the feasible zone lies towards the origin
- A resource allocation problem in which the firm decides to produce the optimum output mix which maximises its profit subject to certain constraints is a maximisation problem in linear programming
- A resource allocation problem in which the firm decides to produce the optimum input mix which minimises the cost of production subject to certain constraints is a minimisation problem
- The basic problem whose solution is attempted in linear programming is called "Primal problem"
- The nature of dual problem depends on the primal problem. eg, if the primal is maximisation the dual is minimisation
- The dual of the dual is primal

Objective Questions

1. When is graphical method of solving linear programming applied?
2. What is a maximisation problem in linear programming?
3. What is a minimisation problem in linear programming?
4. Where does the feasible region lie in a linear programming with greater than or equal to constraint?
5. What is the dual of dual problem?
6. What happens to the dual variable if the primal constraint is originally in equation form?

Answers

1. The graphical method solution in linear programming is used when there are only two decision variables.

2. A resource allocation problem in which the firm decides to produce the optimum output mix which maximises its profit subject to certain constraints is a maximisation problem in linear programming.

3. A resource allocation problem in which the firm decides to produce the optimum input mix which minimises the cost of production subject to certain constraints is a minimisation problem.

4. The feasible region in a linear programming with greater than or equal to constraint lies outside the constraint line.

5. The dual of dual problem yields the original primal.

6. If the primal constraint has = sign, then the associated dual variable is unrestricted in sign.

Assignments

1. Explain step by step how a linear programming problem is solved by graphical method?

2. What do you mean by duality? State its characteristics.

3. Solve the following maxima linear programming problem by graphical method.

   Max $Z= 5x+3y$

   Sub to $3x+5y \leq 15$

   $5x+2y \leq 10$

   x and y are ≥ 0

4. Solve the following minima linear programming problem by graphical method.

   Minimise $Z=15x+12y$

   Sub to $1x+2y \geq 12$

$2x - 4y \geq 5$

x and y are ≥ 0

5. Find the dual of the following LPP?

   a)Minimise $Z=9x+16y$

   Subject to $20x + 90y \geq 7200$

   $5x + 7y \geq 1400$

   $5x + 4y \geq 1000$

   $20x \geq 1200$

   $x,y \geq 0$

   b) Maximise $Z=5x_1+4x_2$

   Subject to

   $6x_1+4x_2 \leq 24$

   $x_1+2x_2 \leq 6$

   $-x_1+x_2 \leq 1$

   $x_1, x_2 \geq 0$

6. Discuss briefly the applications of linear programming in Economics?

References

1. Murthy P Rama(2007) *Operations Research,* New age International Publishers New Delhi
2. Hamdy A Taha(2017) *Operations Research-An introduction,* Pearson Education Limited, England
3. Mike Rosser(2003) *Basic Mathematics for Economists,* Routledge London
4. Teresa Bradley and Paul Patton(2002) *Essential Mathematics for Economics and Business,* John Wiley & Sons Ltd, England
5. Dowling, Edward T, *Introduction to Mathematical Economics,* Schaum's Outline Series, McGrawHill.2nd Edition

6 BLOCK

Input-Output Analysis

UNIT 1

Input-Output Analysis

Learning Outcomes

After completing this unit, the learner will be able to:

- to understand the input output analysis
- to familiarise in solving two sector and three sector economy
- to get a basic idea of Leontief matrix

Prerequisites

The various sectors of an economy are interrelated. For example the output of agricultural sector flows as inputs or raw materials to manufacturing sector. An industrial sector may play the role of both a consumer of output and supplier of inputs. So each sector depends on every other sector for producing its output. Thus each industry sells part of its output as raw materials or inputs to other sectors and the remaining as output for final consumption. The input output analysis model is used to describe such interdependent relationship between different sectors of the economy. In this unit we provide an introduction to input output analysis and Leontief matrix.

Keywords

Input-Output Table, Technical Coefficients, Leontief Matrix, Gross Output

Discussion

6.1.1 Input Output analysis

In a modern economy the production of one good requires many other goods as intermediate goods in the production process. For example car industry uses steel, rubber, glass, plastic and other components produced by other industries. The total demand for product X will be the summation of intermediate demand for the product plus the final demand for the product arising from consumers, investors the government, exporters and the ultimate users. Input output analysis is a forecasting and planning technique developed by Wassily Leontief a Soviet American economist which has wide applications. It provides a descriptive set of social accounting. It records purchases to and sales from different sectors. It is an economic model that studies the mutual interdependence between various industrial sectors of the economy; therefore it is also known as inter industry analysis. The interdependence between individual sectors is mathematically described with the help of a set of linear equations and the resultant matrix representations. Here the output of each industry is viewed as a final consumption commodity and as a factor of production. It shows how the output of one sector serves as input for another sector. Thus it shows how industries are linked together by supplying inputs for the output of an economy. The model is of two types - open model and closed model. Under open model the total output is consumed by industries and ultimate consumers whereas in closed model the output is consumed by industries. There are three stages in input output analysis modeling. They are:

i. Input output Table

ii. Input output coefficients

iii. Leontief matrix

Input-Output table

Input-output analysis can be illustrated with the help of a table. Table 6.1.1 depicts a simplified Input Output table. Here a_{ij} shows the rupee value of output of industry i which is consumed by or sold to industry j. x_i represents the rupee value of the total output of industry i. The rows of the table describe the total output produced by the sector whereas the columns denote the composition of inputs required by the concerned industry to produce the necessary output. The column final demand constitutes the demand for final consumption by consumers , government, investors, exporters and by ultimate users. The additional column labeled total output corresponds to the sum of inter-industry use plus final demand.The additional rows give the value added to the inputs such as labour and depreciation of capital. The sum of each row gives the various factor payments and the sum of each column gives total value added to the inputs. Moreover the totals of columns and of rows have to match.

Table 6.1.1 Input - Output Table

Input	Output	Final dd	Total Output
I	a_{11} a_{12} .... a_{1n}	d_1	X_1
II	a_{21} a_{22} .... a_{2n}	d_2	X_2
.			
.			
.			
n	a_{n1} a_{n2} .... a_{nn}	d_n	X_n

Where

$$a_{11} + a_{12} + \ldots\ldots\ a_{1n} + d_1 = X_1$$

$$a_{21} + a_{22} + \ldots\ldots\ a_{2n} + d_2 = X_2$$

$$a_{n1} + a_{n2} + \ldots\ldots\ a_{nn} + d_n = X_n$$

The summation of intermediate and final demand gives the total output as represented by the above equations.

Technical coefficients

Technical coefficient shows the number of units of any industry's output needed to produce one unit of another industry's output i,e it shows the amount of raw materials needed by an industry from any other industry to produce a certain product.

Let a_{ij} be the input needed from i^{th} industry to j^{th} industry. Let x_i be the total output of i^{th} industry then the technical coefficient is:

$b_{ij} = a_{ij} / x_{i.}$

where a_{ij} represents the rupee value of the output of industry i consumed by industry j

x_i represents the rupee value of the total output of industry i.

Using the above formulae we can construct the technical coefficient matrix. Therefore technical coefficient is obtained by dividing the input of the desired sector by the total output of the same sector.

Technical coefficient matrix

A table showing the technical coefficients is called technical coefficient matrix or technology and the matrix showing the coefficients along with final demand and the total outputs is called technological matrix. In matrix form this can be expressed as Here A is called the matrix of technical coefficients.

$$A = \begin{bmatrix} b_{11} & b_{12} & b_{1n} \\ b_{21} & b_{22} & b_{2n} \\ b_{n1} & b_{n2} & b_{nn} \end{bmatrix}$$

Where

$$b_{11} = \frac{a_{11}}{x_1} \quad b_{12} = \frac{a_{12}}{x_2} \quad b_{1n} = \frac{a_{1n}}{x_n}$$

$$b_{21} = \frac{a_{21}}{x_1} \quad b_{22} = \frac{a_{22}}{x_2} \quad b_{2n} = \frac{a_{2n}}{x_n}$$

$$b_{n1} = \frac{a_{n1}}{x_{1n}} \quad b_{n2} = \frac{a_{n2}}{x_2} \quad b_{nn} = \frac{a_{nn}}{x_n}$$

The equations 1 take the form

$$b_{11}x_1 + b_{12}x_2 + \ldots\ldots b_{1n}x_n + d_1 = X_1$$

$$b_{21}x_1 + b_{22}x_2 + \ldots\ldots b_{2n}x_n + d_2 = X_2$$

$$b_{n1}x_1 + b_{n2}x_2 + \ldots\ldots b_{nn}x_n + d_n = X_n$$

The above equation can be rearranged as

$$(1-b_{11})x_1 - b_{12}x_2 + \ldots\ldots -b_{1n}x_n = d_1$$

$$-b_{21}x_1 + (1-b_{22})x_2 + \ldots\ldots -b_{2n}x_n = d_2$$

$$-b_{n1}x_1 - b_{n2}x_2 + \ldots\ldots (1-b_{nn})x_n = d_n$$

The matrix form of the above equation is

$$\begin{bmatrix} 1-b_{11} & -b_{12} & -b_{1n} \\ -b_{21} & 1-b_{22} & -b_{2n} \\ -b_{n1} & -b_{n2} & 1-b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_n \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} & b_{1n} \\ b_{21} & b_{22} & b_{2n} \\ b_{n1} & b_{n2} & b_{nn} \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_n \end{bmatrix}$$

$$(I - A)X = D$$

Where $A = \begin{bmatrix} b_{11} & b_{12} \ldots\ldots & b_{1n} \\ b_{21} & b_{22} \ldots\ldots & b_{2n} \\ b_{n1} & b_{n2} \ldots\ldots & b_{nn} \end{bmatrix}$ $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_n \end{bmatrix}$

$X = (I - A)^{-1} D$

Leontief matrix

Leonteif input output model delves in the following matter, what level of output should be an industry produce, in order to meet the total input requirements and demand of the economy. If it follows this principle, then its output level must satisfy the following equation:

X=AX+B

B=AX-X

B=X(I-A)

Let A be the technical coefficient matrix, I be the unit matrix. Then I-A matrix is called Leontiff matrix.

Solution to Leonteif Model problems

Let A be the technical coefficient matrix, I be the unit matrix. Let F be the matrix showing final demands and G be the total output.

Steps

1. Find (I-A) matrix
2. Find the inverse of I-A i.e. $(I\text{-}A)^{-1}$
3. Multiply $(I\text{-}A)^{-1}$ with D which gives the matrix of gross output. i.e.

   $X=(I\text{-}A)^{-1}*D$ is the solution to Leontief model problems.
4. D=(I-A)X.

Therefore to estimate Gross output, multiply $(I\text{-}A)^{-1}$ with D and to estimate final demand multiply (I-A) with X.

Example 1

Given the Leontiff input and output table

	I	II	III	Final demand	Total Output
Sector I	3	5	7	16	
Sector II	8	12	16	16	
Sector III	5	3	6	20	

Write down the gross output of each sector and form the technical coefficients.

Solution:

Gross output of sector I = 3 + 5 + 7 + 16 = 31

Gross output of sector I = 8 + 12 + 16 + 16 = 52

Gross output of sector I = 5 + 3 + 6 + 20 = 34

Technical Coefficient Matrix

$$A = \begin{bmatrix} \frac{3}{31} & \frac{5}{52} & \frac{7}{34} \\ \frac{8}{31} & \frac{12}{52} & \frac{16}{34} \\ \frac{5}{31} & \frac{3}{52} & \frac{6}{34} \end{bmatrix}$$

Example 2

Write down the technical coefficient matrix for the following input output table.

	1	2	Final demand	Total Output
Sector I	150	500	350	1000
Sector II	300	400	1300	2000
Primary Input	350	500	150	1000

Technical Coefficients are

	Sector I	Sector II
Sector I	$\frac{150}{1000} = 0.15$	$\frac{5000}{2000} = 0.25$
Sector II	$\frac{3000}{1000} = 0.3$	$\frac{400}{2000} = 0.2$
Primary Input	$\frac{350}{1000} = 0.35$	$\frac{500}{2000} = 0.25$

Technical coefficient matrix

$$A = \begin{bmatrix} 0.15 & 0.25 \\ 0.3 & 0.2 \\ 0.35 & 0.25 \end{bmatrix}$$

Example 3

The technological coefficient matrix of 2 sectors is given as $A = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$. If final demand of the two sectors are 600 and 1500 respectively. Find the gross output of the two sectors.

Solution

$A = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$

So $(I - A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix} = \begin{bmatrix} .85 & -.25 \\ -.20 & .95 \end{bmatrix}$

$|I - A| = (.85 \times .95) - (-.20 \times -.25)$

$= .8075 - 0.05 = 0.7575$

$L = (I - A)^{-1}$

Minor Co-factor

$a_{11} = .95$ $(-1)^2 \times .95 = .95$

$a_{12} = -.20$ $(-1)^3 \times -.20 = .20$

$a_{21} = -.25$ $(-1)^3 \times -25 = .25$

$a_{22} = .85$ $(-1)^1 \times .85 = -.85$

$C = \begin{bmatrix} .95 & .20 \\ .25 & .85 \end{bmatrix}$ $C^1 = \begin{bmatrix} .95 & .25 \\ .20 & .85 \end{bmatrix}$

$\frac{1}{0.7575} \begin{bmatrix} .95 & .25 \\ .20 & .85 \end{bmatrix}$

$L = \begin{bmatrix} 1.2541 & 0.3300 \\ 0.2640 & 1.1221 \end{bmatrix}$

$X = (I - A)^{-1} D$

$X = \begin{bmatrix} 1.2541 & 0.3300 \\ 0.2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$

$= \begin{bmatrix} 1.2541 \times 600 & 0.3300 \times 1500 \\ 0.2640 \times 600 & 1.1221 \times 1500 \end{bmatrix}$

$= \begin{bmatrix} 1247.46 \\ 1841.55 \end{bmatrix}$

Gross output of 2 sectors is $\begin{bmatrix} 1247.46 \\ 1841.55 \end{bmatrix}$

Example 4

In a 3 sector economy the output from 3 sectors could be distributed as follows

	Input				
	Agriculture	Industry	Services	Other demand	Total Output
Agriculture	150	225	125	100	600
Output Industry	210	250	140	300	900
from services	170	0	30	100	300
Other Inputs	70	425	5		
	600	900	300		

If the final demands from each sector are changed to 500 from agriculture, 550 from industry, 300 from financial services. Calculate the total output from each sector.

Solution

Technical coefficient matrix

$$A = \begin{bmatrix} \frac{150}{600} & \frac{225}{900} & \frac{125}{300} \\ \frac{210}{600} & \frac{250}{900} & \frac{140}{300} \\ \frac{170}{600} & \frac{0}{900} & \frac{30}{300} \end{bmatrix}$$

$$A = \begin{bmatrix} 0.25 & 0.25 & 0.42 \\ 0.35 & 0.27 & 0.46 \\ 0.28 & 0 & 0.1 \end{bmatrix}$$

Step 2: Get the inverse of matrix (I – A), since the inverse matrix is required in the equation $X = (I - A)^{-1} d$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.25 & 0.25 & 0.42 \\ 0.35 & 0.27 & 0.46 \\ 0.28 & 0 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{bmatrix}$$

$$|I - A| = 0.75 \begin{bmatrix} 0.72 & -0.47 \\ 0.00 & 0.90 \end{bmatrix} - (-0.25) \begin{bmatrix} -0.35 & -0.47 \\ 0.28 & 0.90 \end{bmatrix} + (-.42) \begin{bmatrix} -0.35 & 0.72 \\ -0.28 & 0.00 \end{bmatrix}$$

$$= 0.2896$$

Step 3

To calculate the increase of (I – A), use the cofactor method.

Cofactor = minor (±1)

$$0.75(a_{11}) = (-1)^2 \begin{vmatrix} 0.72 & -0.47 \\ 0.00 & 0.90 \end{vmatrix} = 0.648$$

$$-0.25(a_{12}) = (-1)^3 \begin{vmatrix} -0.35 & -0.47 \\ -0.28 & 0.90 \end{vmatrix} = 0.4466$$

$$-0.42(a_{13}) = (-1)^4 \begin{vmatrix} -0.35 & 0.72 \\ -0.28 & 0.00 \end{vmatrix} = 0.2016$$

$$-0.35(a_{21}) = (-1)^3 \begin{vmatrix} -0.25 & -0.42 \\ 0.00 & 0.90 \end{vmatrix} = 0.225$$

$$0.75(a_{11}) = (-1)^2 \begin{vmatrix} 0.75 & -0.42 \\ -0.28 & 0.90 \end{vmatrix} = 0.5574$$

$$a_{23}(-0.47) = (-1)^5 \begin{vmatrix} 0.75 & -0.25 \\ 0.28 & 0.00 \end{vmatrix} = 0.070$$

$$a_{31}(-0.28) = (-1)^4 \begin{vmatrix} -0.25 & -0.42 \\ 0.72 & -0.47 \end{vmatrix} = 0.4199$$

$$a_{32}(0.00) = (-1)^5 \begin{vmatrix} 0.75 & -0.42 \\ -0.35 & -0.47 \end{vmatrix} = 0.4995$$

$$a_{33}(0.90) = (-1)^6 \begin{vmatrix} 0.75 & -0.25 \\ -0.35 & 0.72 \end{vmatrix} = 0.4525$$

The inverse of (I-A) = $C^T \, |I - A|$

$$= \frac{1}{0.289678} \begin{bmatrix} 0.648 & 0.4466 & 0.2016 \\ 0.225 & 0.5574 & 0.070 \\ 0.4199 & 0.4995 & 0.4525 \end{bmatrix}$$

$$= \frac{1}{0.289678} \begin{bmatrix} 0.648 & 0.225 & 0.4199 \\ 0.4466 & 0.5574 & 0.4995 \\ 0.2016 & 0.070 & 0.4525 \end{bmatrix}$$

Step 4 Finally, state the column of new external demands d, solve for X.

$$X = (I - A)^{-1} d$$

$$= \frac{1}{0.289678}\begin{bmatrix} 0.648 & 0.225 & 0.4199 \\ 0.4466 & 0.5574 & 0.4995 \\ 0.2016 & 0.070 & 0.4525 \end{bmatrix}\begin{bmatrix} 500 \\ 550 \\ 300 \end{bmatrix}$$

$$= \begin{bmatrix} 1981.015 \\ 2346.225 \\ 949.735 \end{bmatrix}$$

Recap

- Input-output analysis is a planning and forecasting technique that studies the mutual interdependence between various industrial sectors of the economy
- Input output table a_{ij} shows the rupee value of output of industry i which is consumed by or sold to industry j
- Technical coefficient shows the number of units of any industry's output needed to produce one unit of another industry's output
- I-A matrix is called Leontiff matrix, where I represents unit matrix and A represents technical coefficient matrix

Objective Questions

1. Who put forward the concept of Input-Output analysis?
2. What is an input output analysis?
3. What is an input-output table?
4. What is a Leontief matrix?
5. What is a technical coefficient mtrix?

Answers

1. Wassiley Leontief

2. Input-output analysis is a planning and forecasting technique that studies the mutual interdependence between various industrial sectors of the economy

3. Input output table. a_{ij} shows the rupee value of output of industry I which is consumed by or sold to industry j.

4. I-A matrix is called Leontiff matrix, where I represents unit matrix and A represents technical coefficient matrix.

5. Technical coefficient shows the number of units of any industry's output needed to produce one unit of another industry's output.

Assignments

1. What is input-output analysis?

2. Write down the stages of input-output analysis.

3. The input-output table for a two sector economy is given as follows

	Input to A	Input to B	Final demand
Output A	250	70	50
Output B	200	380	100

a) Find the technical coefficient matrix
b) Calculate gross output from each sector when final demand of sector A increases from 50 to 100 and sector B from 100 to 200.

References

1. Miernyk H William(2020 *The Elements of Input-Output Analysis*, Regional Research Institute West Virginia University.

2. Teresa Bradley and Paul Patton(2002) *Essential Mathematics for Economics and Business* John Wiley & Sons Ltd, England.

3. Chiang C Alpha *Fundamental Methods of Mathematical Economics* Mc Graw-Hill, Inc 2nd edition.

4. Dowling, Edward T. *Introduction to Mathematical Economics, Schaum's Outline Series, McGrawHill 2nd/3rd editions.*

UNIT 2

Viability conditions for Input-output models

Learning Outcomes

After completing this unit, the learner will be able to:

- to understand viability condition in input output analyses
- to familiarise the Hawkin Simon condition
- to get a basic idea of application of input output analysis

Prerequisites

In the previous unit we discussed in detail the input-output analysis and familiaried in solving two sector and three sector economy problems. In this unit we discuss the feasibility and viability condition of input output analyses and test whether the input output system is viable with the aid of Hawkins Simon condition. This part focuses on a fundamental set of mathematical conditions for input-output models. The part also delves into the application of input-output analyses.

Keywords

Viability, Hawkins-Simon Condition

Discussion

6.2.1 Viability of the System

If an economy is to move on steady growth path, then it is necessary that each industry satisfies the viability criterion. An industry is considered viable if its own input coefficient is less than 1. In case of violation of this condition only then the relevant diagonal element is less than the sum of the elements of its row.

6.2.2 Hawkins –Simon Condition

The Hawkins –Simon condition which ensures viability of the system is attributed to David Hawkins and Herbert A Simon.

If A represents the technological matrix then an input-output system has feasible solution only if

- The determinant of matrix, I-A is always positive or the solution values should remain non negative.
- The leading diagonal elements of the matrix I-A are positive

More precisely this is the condition for I-A under which input output system (I-A) X=D has a solution where I represents identity matrix A indicates the Leontief matrix.

The Hawkins-Simon condition ensures that for the system to remain viable the system should not generate negative output. These conditions constitute the properties of Leontief input output model.

Let us discuss it with the help of an example:

Given the following inputs coefficient matrix. State Hawkins Simon condition for viability of the system.

$$A = \begin{bmatrix} 0.40 & 0.25 \\ 0.45 & 0.35 \end{bmatrix} \quad D = \begin{bmatrix} 80 \\ 100 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.40 & 0.25 \\ 0.45 & 0.35 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.25 \\ -0.45 & 0.65 \end{bmatrix}$$

$$= (0.6 \times 0.65) - (-0.25 \times -0.45)$$

$$= 0.39 - 0.11 = 0.28$$

Here $|I - A| > 0$ and the diagonal matrix values are positive. Therefore the input coefficient matrix satisfies Hawkins – Simon conditions.

6.2.3 Applications of Input-Output Analysis

Input-output analysis is recognised as a major achievement in the field of Economics for the following reasons:-

- It explains general equilibrium in its simplest form
- It is useful for national accounting as it makes possible detailed studies of monetary inflows.
- It is useful in the field of economic planning as it is possible to arrive at optional combinations to maximise production.
- It can be used for studying trade cycles and uncontrolled fluctuations in the economic field.
- It makes possible to discover the interrelations and interdependencies of firms and industries.

6.2.4 Game Theory

Game theory and input-output analysis are both economic tools used to study complex systems, but they differ in their focus. Game theory is concerned with the strategic decision-making of competing agents, analysing how individuals or firms interact in situations where outcomes depend on each other's choices. Input-output analysis focuses on the relationships between different sectors of an economy, following how the output of one sector serves as an input for another. Basically, game theory explores "who" is making decisions and how their strategies influence each other, while input-output analysis examines "how" various industries are interconnected within the broader economy.

Application of Input-Output Analysis in Game Theory:

1. Modeling Strategic Interactions in a Networked Economy: In a networked or interconnected economy, where different industries or sectors depend on each other (as described by input-output tables), game theory can model the strategic decisions of firms or sectors. For example, if one sector (say, oil production) reduces its output or changes its prices, the effect on other sectors (such as transportation or manufacturing) can be examined. Game theory could be applied to determine how firms or sectors should adjust their strategies to optimize their payoffs, taking into account the interdependencies highlighted by the input-output model.

2. Cooperative Game Theory in Supply Chains: Input-output analysis can be used to map out the supply chain relationships and dependencies between different firms. In this context, cooperative game theory might be used to determine how firms in a supply chain can collaborate to maximize total output or profit, given the input-output structure. For instance, firms can negotiate shared profits or decide on joint investments to improve efficiency across the chain.

3. Non-Cooperative Games with Interdependent Industries: In a non-cooperative game, each player (or firm) tries to maximize its own profit or utility, potentially at the expense of others. The input-output framework helps identify how decisions made by one firm in a particular industry affect the revenues or costs of other firms in related industries. For example, if a firm in one sector increases its production, it could lead to either an increase or decrease in demand for goods in other sectors, which can be modeled using game theory to predict strategic decisions like pricing or output levels.

4. Environmental and Resource Management Games: In sectors like energy, natural resources, or environmental policy, input-output analysis is often used to understand the flow of resources. Game theory can then be applied to model the strategic behavior of multiple countries or companies that are involved in resource extraction or environmental conservation. The input-output model could illustrate how the use or depletion of resources in one country impacts other countries, and game theory could help in analyzing negotiation or conflict scenarios.

5. Industrial Organization and Market Power: Input-output analysis can also be useful for understanding the structure of industries and the potential for market power. Firms in interconnected sectors may use game theory to assess the effects of actions like price-setting, mergers, or joint ventures. Input-output relationships help to understand the market structure and interdependencies, which can be factored into game-theoretic models of competitive strategy.

The application of input-output analysis in game theory enables a more comprehensive understanding of the interactions between economic agents in interdependent systems. By incorporating these interrelationships into strategic models, policymakers and firms can better predict the consequences of their decisions and work toward optimal outcomes, whether in competitive or cooperative contexts. The integration of these tools is especially valuable in industries where the actions of one player can significantly impact the outcomes of others.

Objective Questions

1. How many conditions are there under Hawkins-Simon test for the viability of an input-output analysis?
2. What are the conditions of Hawkins-Simon test?

Answers

1. 2
2. The determinant of matrix, I-A is always positive and the leading diagonal elements of the matrix I-A are positive

Assignments

1. The technological coefficient matrix of two sectors P and Q

$$A = \begin{bmatrix} .1 & .3 \\ 0 & .2 \end{bmatrix}$$

a. State the Hawkins – Simon conditions for viability of the system.
b. If the demand of two sectors are 10 and 20 respectively. Find grossoutput of two sectors.

References

1. Miernyk H William(2020 *The Elements of Input-Output Analysis*, Regional Research Institute West Virginia University.
2. Teresa Bradley and Paul Patton(2002) *Essential Mathematics for Economics and Business* John Wiley & Sons Ltd, England.
3. Chiang C Alpha *Fundamental methods of Mathematical Economics* Mc Graw-Hill, Inc 2nd Edition.
4. Dowling, Edward T. *Introduction to Mathematical Economics*, Schaum's Outline Series, McGrawHill. 2nd/3rd editions.

സർവ്വകലാശാലാഗീതം

വിദ്യയാൽ സ്വതന്ത്രരാകണം
വിശ്വപൗരരായി മാറണം
ഗ്രഹപ്രസാദമായ് വിളങ്ങണം
ഗുരുപ്രകാശമേ നയിക്കണേ

കൂരിരുട്ടിൽ നിന്നു ഞങ്ങളെ
സൂര്യവീഥിയിൽ തെളിക്കണം
സ്നേഹദീപ്തിയായ് വിളങ്ങണം
നീതിവൈജയന്തി പാറണം

ശാസ്ത്രവ്യാപ്തിയെന്നുമേകണം
ജാതിഭേദമാകെ മാറണം
ബോധരശ്മിയിൽ തിളങ്ങുവാൻ
ജ്ഞാനകേന്ദ്രമേ ജ്വലിക്കണേ

കുരീപ്പുഴ ശ്രീകുമാർ

SREENARAYANAGURU OPEN UNIVERSITY

Regional Centres

Kozhikode
Govt. Arts and Science College
Meenchantha, Kozhikode,
Kerala, Pin: 673002
Ph: 04952920228
email: rckdirector@sgou.ac.in

Thalassery
Govt. Brennen College
Dharmadam, Thalassery,
Kannur, Pin: 670106
Ph: 04902990494
email: rctdirector@sgou.ac.in

Tripunithura
Govt. College
Tripunithura, Ernakulam,
Kerala, Pin: 682301
Ph: 04842927436
email: rcedirector@sgou.ac.in

Pattambi
Sree Neelakanta Govt. Sanskrit College
Pattambi, Palakkad,
Kerala, Pin: 679303
Ph: 04662912009
email: rcpdirector@sgou.ac.in

Mathematical Economics

COURSE CODE: B21EC01DE

ISBN 978-81-984025-8-5

Sreenarayanaguru Open University
Kollam, Kerala Pin- 691601, email: info@sgou.ac.in, www.sgou.ac.in Ph: +91 474 2966841