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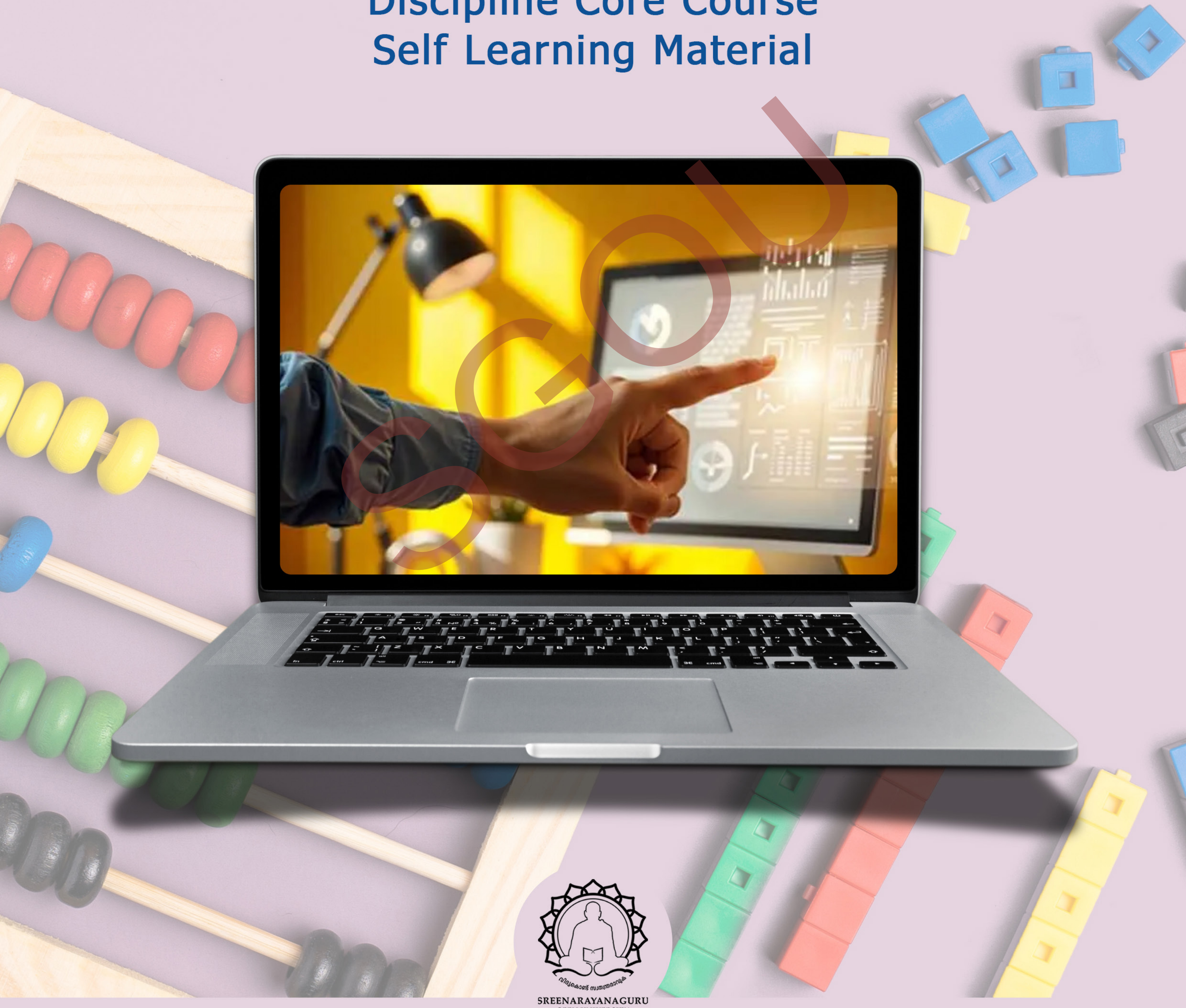
Mathematical Foundation for Computer Applications

COURSE CODE: M25CA02DC

Master of Computer Applications

Discipline Core Course

Self Learning Material



SREENARAYANAGURU
OPEN UNIVERSITY

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The State University for Education, Training and Research in Blended Format, Kerala

SREENARAYANAGURU OPEN UNIVERSITY

Vision

To increase access of potential learners of all categories to higher education, research and training, and ensure equity through delivery of high quality processes and outcomes fostering inclusive educational empowerment for social advancement.

Mission

To be benchmarked as a model for conservation and dissemination of knowledge and skill on blended and virtual mode in education, training and research for normal, continuing, and adult learners.

Pathway

Access and Quality define Equity.

Mathematical Foundation for Computer Applications

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Semester - I

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Postgraduate Programme
Master of Computer Applications
Self Learning Material

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MATHEMATICAL FOUNDATION FOR COMPUTER APPLICATIONS

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SREENARAYANAGURU
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MESSAGE FROM VICE CHANCELLOR

Dear Learner,

It gives me immense pleasure and a deep sense of pride to warmly welcome you to Sreenarayanaguru Open University—a vibrant and progressive institution committed to transforming lives through inclusive, flexible, and high-quality education.

Established in September 2020 as a forward-looking initiative of the Government of Kerala, the University stands as a beacon of opportunity for learners seeking to advance their academic and professional aspirations through the open and distance learning mode. Guided by our foundational principle that "access and quality define equity," we are steadfast in our mission to democratize education while upholding uncompromising academic standards.

Our university is inspired by the timeless vision and philosophy of Sree Narayana Guru, whose ideals of knowledge, equality, and social transformation continue to guide our academic journey. His enduring legacy instils in us the responsibility to create an educational environment that empowers individuals, nurtures critical thinking, and contributes meaningfully to society.

Understanding the dynamic needs of contemporary learners, we have adopted a robust and learner-centric blended learning model, seamlessly integrating Self-Learning Materials, Academic Counselling, and Advanced Digital Learning Platforms. This holistic approach ensures flexibility without sacrificing academic depth, enabling you to learn at your own pace while remaining meaningfully connected to a vibrant academic ecosystem.

The Master of Computer Applications (MCA) programme you are embarking upon is carefully designed to position you at the forefront of the digital revolution. It uniquely blends strong theoretical foundations with practical, industry-oriented competencies. The curriculum emphasizes algorithmic thinking, system design, programming, database management, networking, and emerging technologies such as artificial intelligence, data science, and cloud computing.
What sets this programme apart is its forward-thinking design, which offers:

- Opportunities for skill enhancement aligned with industry needs
- Multidisciplinary learning pathways for broader intellectual development
- Exposure to innovative and emerging technology domains
- Multiple specialization options tailored to evolving career landscapes
- A strong focus on employability, entrepreneurship, and global career readiness

Our Self-Learning Materials are meticulously developed by experts, enriched with contemporary case studies, real-world applications, and practical insights to ensure clarity, engagement, and relevance. We are committed to equipping you not just with knowledge, but with the confidence and competence to excel in a competitive global environment.

At Sreenarayanaguru Open University, you are never alone in your learning journey. Our dedicated learner support system is designed to provide continuous academic guidance, timely assistance, and effective grievance redressal. We encourage you to actively engage with us, share your concerns, and make the most of the resources and support available to you.

As you begin this important phase of your academic journey, I urge you to embrace learning with curiosity, discipline, and determination. The world of technology is ever-evolving, and your willingness to adapt, innovate, and grow will define your success.

Remember, this is not just a programme—it is a pathway to transforming your future.
I wish you a fulfilling learning experience and a successful, inspiring career ahead.

Warm regards,

Prof. (Dr.) Jagathy Raj V. P.
Vice Chancellor

Contents

BLOCK 1

Linear Algebra

1

UNIT

Matrices

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand methods for solving system of linear equations
- determine eigen values and eigen vectors of a matrix
- apply vector space concept to find linear independence
- form a clear concept of orthogonality of vectors

Background

Linear algebra is the study of the algebraic properties of matrices and linear transformations .It is one of the most important branches of modern mathematics concerned with vectors and matrices. It provides a powerful and compact notation that can be used in geometry, calculus, differential equations, among others. Its applications in Machine Learning enhances its requirement in various fields including Computer Science. Matrices are the fundamental objects in linear algebra.

Matrices play multiple roles in mathematics. They represent a set of numbers arranged in rows and columns, in that sense they can compactly represent data with a tabular form. They themselves are an object of study like numbers and sets. They have rules for addition and multiplication. They have special operations and properties. They represent linear transformations. So they behave like functions and many hidden characteristics of transformations are revealed through their study. They represent a system of linear equations. So solution of a system is obtained from their properties. Dimensions of large data set can be efficiently reduced by applying matrix properties.

Solving a system of linear equations is a basic problem in linear algebra. Assuming that basic operations of matrices are familiar, how such a system can be solved using matrices is discussed. In several applications, this system may contain a large number of equations in large number of variables. Then Gaussian elimination method provides a widely used algorithm for solution.

One of the important characteristics of many data set is their representation through matrices. The properties of these data sets can be identified from different measures that can be determined from the matrices. The idea of eigen values and eigen vectors are two most important topics here. They become important when practical problems are re-written in terms of matrices as in image processing. Then it is seen that many crucial properties are from the properties of matrix operations.

Linear algebra supplies a geometric meaning for many concepts and orthogonality is such a concept. Vector spaces is an abstract concept that generalises many familiar objects. Norms and inner products are abstractions of basic operations but in a general context. Linear independence and basis are important in many applications. They are helpful in understanding many applied problems .Linear algebra is also a study of linear maps on finite dimensional vector spaces.

Keywords

Matrices, Echelon forms, Gaussian elimination, Eigen values and eigen vectors, diagonalisation, Vector space, norms, linear independence, orthogonality

Discussion

1.1.1. Matrices

A Matrix of numbers is an arrangement of numbers in rows and columns. It is one of the most important object now used in mathematics. It can represent an arrangement of numbers and also it itself is an object of study.

1.1.2 Representation and Order

A Matrix is represented by A,B,C,… If a matrix has m rows and n columns, its dimension is mxn. The numbers in matrix A are called elements and each element is denoted by a_{ij} where i denotes the row number and j denotes the column number. If matrix A has order 3x4, then A can be represented by $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Order of a matrix is mxn means it has m rows and n columns

1.1.3 Types of matrices

Matrices can be classified based on order and nature of its elements.

1. Square Matrix

A matrix is said to be a square matrix if it has same number of rows and columns. If its order is nxn, sometimes it is said to be a matrix of order n. Elements a_{ii} are called diagonal elements.

For example, $A=\begin{bmatrix} 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{bmatrix}$ is is a square matrix of order 3.

Here $a_{11}=3$, $a_{22}=4$ and $a_{33}=5$ are the diagonal elements.

2. Diagonal Matrix

A square matrix in which all elements except diagonal elements are zeros is called a diagonal matrix. For example, $A=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. If all diagonal elements are unity, it is called identity matrix denoted by I.

3. Triangular Matrix

A square matrix in which $a_{ij}=0$ for i >j , i.e., all elements below the diagonal are zeros, is called an upper triangular matrix . For example, $A=\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}$ is upper triangular.

A square matrix in which $a_{ij}=0$ for i <j , i.e., all elements above the diagonal are zeros, is called a lower triangular matrix . For example, $A=\begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 8 & 3 & 7 \end{bmatrix}$ is lower triangular.

4. Row and Column vectors

If a matrix has order 1xn, it is called a row vector and if the order is nx1 it is called a column vector.

For example, A=[1 3 4 5] is a row vector and $A=\begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}$ is a column vector.

Each row of matrix A is a row vector and each column is a column vector.

5. Zero matrix

If a matrix has all elements zeros, then the matrix is called zero matrix denoted by O.

1.1.4 Matrix operations

Similar to various operations of numbers, there are different operations defined on matrices. But there are conditions for doing these operations.

1. Equality of matrices

Two matrices are said to be equal if (1) they have the same order and (2) their elements in the corresponding positions are equal

Matrix operations are similar to operations in numbers. Unlike numbers, conditions apply for matrix operations.

2. Scalar multiplication

If k is any number (called scalar) and A is a matrix of order mxn, then kA is called scalar multiple of A which is a matrix of the same order. Each element of A is multiplied by k and is given in the corresponding position of kA. For example, if

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 9 & 9 & 9 \\ 5 & 6 & 7 \end{bmatrix}, then\ 8A = \begin{bmatrix} 16 & 24 & 32 \\ 72 & 72 & 72 \\ 40 & 48 & 56 \end{bmatrix}$$

Properties of scalar multiplication

1. (c k) A=c (kA)
2. 1A=A

3. Addition of matrices

If A and B are matrices of the same order, their sum denoted by A+B has the same order and the elements in A+B is obtained by adding elements in corresponding positions of A and B.

For example , if

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 20 & 30 & 50 \\ 13 & 15 & 17 \\ 22 & 11 & 33 \end{bmatrix}, then$$

$$A+B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 20 & 30 & 50 \\ 13 & 15 & 17 \\ 22 & 11 & 33 \end{bmatrix} = \begin{bmatrix} 21 & 33 & 55 \\ 15 & 19 & 23 \\ 23 & 12 & 34 \end{bmatrix}$$

Subtraction of A and B is A-B and is done in a similar way

Properties of matrix addition

1. Commutative: A+B=B+A
2. Associative: A+(B+C)=(A+B)+C
3. Zero matrix is the identity: A+O=A+O+A
4. –A is the additive inverse of A: A + -A=O= -A + A
5. If k is a scalar, k(A+B)=kA+kB
6. If k_1 and k_2 are scalars, $(k_1 + k_2)$ A= k_1A+ k_2A

> Matrix addition has all the properties of addition of numbers. When multiplication is defined, its properties are not similar to operations in numbers.

4. Matrix multiplication

Suppose A is a matrix of order mxn and B is a matrix of order nxp, then the product AB is defined. Its order is m x p. If the i^{th} row of A is $\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$ and j^{th} column of B is $\begin{bmatrix} b_{1j} \\ b_{2j} \\ . \\ . \\ . \\ b_{nj} \end{bmatrix}$ then i^{th} row, j^{th} column element of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$. So, to get an element in AB a row in A is multiplied by a column in B.

For example, suppose $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \\ 5 & 2 \end{bmatrix}$.

Then AB is defined and AB= $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 31 & 17 \\ 7 & 15 \end{bmatrix}$

It is clear that BA is not defined.

Properties of matrix multiplication

1. Not commutative: AB need not be equal to BA.
2. Associative: A(BC)=(AB)C
3. Identity matrix I is the multiplicative identity: A I= A= I A
4. If k is a scalar, k(AB)=(kA)B

5. A (B+C)= AB+ AC and (A +B) C=A C+B C

5. Transpose of a Matrix

Suppose order of matrix A is m x n. Its transpose is a matrix A^T or A' order n x m . Every row in A is changed to a column in A^T.

For example, suppose $A=\begin{bmatrix}1&3&0&7\\2&4&8&4\\2&0&2&5\end{bmatrix}, then\ A^T=\begin{bmatrix}1&2&2\\3&4&0\\0&8&2\\7&4&5\end{bmatrix}$

Transpose of a matrix is an operation defined for matrices only. It has several properties that are useful in further study.

6. Symmetric matrix

A square matrix A is called symmetric if $A^T=A$.

For example, $A=\begin{bmatrix}1&3&0\\3&6&4\\0&4&10\end{bmatrix}, then\ A^T=\begin{bmatrix}1&3&0\\3&6&4\\0&4&10\end{bmatrix}$

Properties of transpose

1. $(A^T)^T=A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$

7. Inverse of a Matrix

Suppose A is a square matrix of order m x m. Then matrix B is called inverse of A if AB=BA=I. Inverse of A is usually denoted by A^{-1}.

For example, if $A=\begin{bmatrix}2&1\\2&3\end{bmatrix}, then\ A^{-1}=\begin{bmatrix}3/4&-1/4\\-2/4&2/4\end{bmatrix}$

Inverse of a matrix can be used to find properties of a matrix .A matrix and its inverse are closely related as we see in the future lessons.

Properties of inverse

$(A^{-1})^{-1}=A$

$(AB)^{-1} = B^{-1} A^{-1}$

Illustration.1.1.1

$$A=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} \textit{ find } \text{A-2B}$$

Solution:

$$2B=\begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}, \text{ A-2B}=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}-\begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$

Illustration.1.1.2

$$\text{Compute : A-5I if } A=\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

Solution

$$A=\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}, \text{ I}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 5I=\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A-5I=\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}-\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}=\begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$$

Illustration.1.1.3

$$\textit{If } A=\begin{bmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{bmatrix}, B=\begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{bmatrix} \textit{ and } \text{C}=\begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

find 3A+4B-2C

Solution

$$3A=\begin{bmatrix} 6 & -15 & 3 \\ 9 & 0 & -12 \end{bmatrix}, 4B=\begin{bmatrix} 4 & -8 & -12 \\ 0 & -4 & 20 \end{bmatrix} \textit{ and } 2\text{C}=\begin{bmatrix} 0 & 2 & -4 \\ 2 & -2 & -2 \end{bmatrix}$$

$$So,\ 3A+4B-2C=\begin{bmatrix} 6 & -15 & 3 \\ 9 & 0 & -12 \end{bmatrix}+\begin{bmatrix} 4 & -8 & -12 \\ 0 & -4 & 20 \end{bmatrix}-\begin{bmatrix} 0 & 2 & -4 \\ 2 & -2 & -2 \end{bmatrix}$$

$$=\begin{bmatrix} 10 & -25 & -5 \\ 7 & -2 & 10 \end{bmatrix}$$

Illustration.1.1.4

Compute AB and BA where $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

Order of A is 3 x2 and order of B is 2x2, so AB is possible but BA is not.

Row1Column1 element of AB is $\begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \end{bmatrix}$

Row1column2 element of AB=$\begin{bmatrix} -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$

Similarly finding the other rows, AB = $\begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

Illustration.1.1.5

Compute AB and BA where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

Order of A is 3 x2 and order of B is 2x3, so AB has order 3 X3 and BA has order 2 X 2.

$$AB = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -21 \\ 10 & -3 \end{bmatrix}$$

Illustration.1.1.6

$A=\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ *find* A^2 and A^3

Solution

If $A=\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

$A^2 = A.A=\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}=\begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$

$A^3 = A^2 A=\begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}=\begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$

Illustration.1.1.7

If $A=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix}$ *find* AA'

Solution

$A=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix}$, $A'=\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 5 \end{bmatrix}$

then, $AA'=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 5 \end{bmatrix}=\begin{bmatrix} 21 & 24 \\ 24 & 30 \end{bmatrix}$

Illustration 1.1.8

Show that if

$A=\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ *, then* $B=\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ is the inverse of A

Solution

$AB=\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$BA=\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So B is the inverse of A.

Summarized Overview

Order of a matrix is an important quantity in the study of matrices. Operations on matrices are defined depending on the order also. Addition of matrices has properties similar to addition of numbers. But multiplication of matrices gives an example where multiplication is not commutative. Distributive property combines addition and multiplication. Multiplication has most other properties similar to that of multiplication of numbers. Transpose is an operation peculiar to matrices. Inverse of matrix is important to determine the nature of a matrix. It has several applications in various fields in science and engineering.

In appearance many of the properties of the matrices seem very simple and easy to understand. But as the properties are investigated more, it is found that they have deep relationship with other objects of study like numbers. The operations on matrices play crucial role in many applications.

Assignments

1. $A=\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ *and* $B=\begin{bmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{bmatrix}$ find 3A-4B

2. If $A=\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ show that B= $\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$ is the inverse of A.

3. *If* A= $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 5 & 3 & 2 \end{bmatrix}$ find A^2

4. Use an example of 2 x 2 matrices to show that the formula

5. $(A+B)^2 = A^2 + 2AB + B^2$ need not be valid for matrices.

6. By trial and error find examples of 2 x 2 matrices such that $A^2 = -I$

7. Verify whether true by an example that if AB=B then A=I

8. Find all matrices $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *that* satisfy $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$?

Suggested Reading

1. Gilbert Strang – ***Introduction to Linear Algebra***, 5th ed., Wellesley-Cambridge Press, 2016.
2. David C. Lay, Steven R. Lay, Judi J. McDonald – ***Linear Algebra and Its Applications***, 5th ed., Pearson, 2015

Reference

1. 1. K.Hoffman,R.Kunze –*Linear Algebra*, 2nd ed., Pearson, 1971
2. Fundamentals of Matrix Algebra, Gregory Hartman, third Edition

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

2
UNIT

Systems of Linear Equations

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand the different types of solutions for a system of equations
- understand and apply echelon form of a matrix
- understand and apply Gaussian elimination method
- understand rank and apply it for testing consistency

Background

System of linear equations appear very commonly in applied problems and in modelling real-life problems. For instance in discussing recurrence equations or in analysing an electrical circuits, we come across such a sytem system. Because of their universal appearance, an efficient method for solution is very essential. Gaussian elimination method is an algorithmic way to solve a system of linear equations. It is simple to apply and easy to understand. But it does not require the nature of solutions directly. Consistency of a system of equations describe the existence and nature of its solutions. To test consistency, the equations are compactly expressed in matrix form. Echelon form of a matrix carries all information present in the original matrix but in a simpler form. Transforming a matrix into its echelon form requires elementary transformations of rows. Then it provides an easy way to determine consistency of a system of equations. Rank of a matrix is a single number that captures many properties of a given matrix especially in a discussion of consistency it has a major role. This unit discusses these concepts.

Systems of equations are themselves interesting. At the same time, they arise in analysing data sets. For instance, in determining eigen vectors and in dimension reduction problems, solving a system becomes important. The nature of solutions greatly affect the end results which can be seen while solving a system of homogeneous equations.

Keywords

Gaussian Elimination Method, Elementary Row Transformations of a Matrix, Echelon form of a Matrix, Rank of a Matrix

Discussion

1.2.1 Systems of Linear Equations

Linear equations appear everywhere in Science, Engineering and other fields. For instance, a linear model is described by a system of linear equations. A system of linear equations contain more than one equation in the same variables.

A linear equation in variables $x_1, x_2, \ldots, x_n$ has a general form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ where a_i and b are real or complex numbers. Number of variables in most applications is very large.

System of linear equations have several applications. So their representation and solution method are very important.

A system of m linear equations in n variables is represented as

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

This can be conveniently represented in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ & \ldots & \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

Or AX=b where A is called coefficient matrix. X and b are column vectors.

The matrix in which A is attached with an additional column b is called augmented matrix denoted by $[Ab]$ or [A|b].

For example, the system $\begin{matrix} 4x_1+3x_2+7x_3-x_4=12 \\ 9x_1+x_2-2x_3+4x_4=10 \\ x_1+6x_2+x_3+5x_4=24 \end{matrix}$ has an augmented matrix $[Ab]=$

$\begin{bmatrix} 4 & 3 & 7 & -1 & 12 \\ 9 & 1 & -2 & 4 & 10 \\ 1 & 6 & 1 & 5 & 24 \end{bmatrix}$. It encloses all the essential information that is required from a system of equations.

A solution of the system is a set of numbers $x_1, x_2, \ldots, x_n$ that satisfies all the m equations. A solution vector X is a vector all whose components is a solution of the system.

Three cases are possible while solving a system of linear equations:

- the system has a unique solution
- the system has infinitely many solutions
- the system has no solution.

In the first two cases, the system has a solution and the system is said to be consistent. In the third case, it is called inconsistent.

When there are only two variables, the nature of the solution can be clearly understood geometrically. The equations are represented as lines and whenever the lines

(1) intersect at a single point, system has a single solution

(2) are coinciding, the system has infinitely many solutions

(3) are parallel, the system has no solution.

So the question is when the system is consistent, if consistent, what the solution is and what the geometrical meaning of solution is.

A linear system of equation is consistent when it has at least one solution. Before trying to solve a linear system, it must be guaranteed that the system is consistent.

1.2.2 Row reduction and echelon forms

A given matrix can be converted to a simpler matrix by a set of row transformations. They are called elementary row transformations given by

1. interchanging two rows Ri and Rj denoted by Ri ↔ Rj
2. multiplying a row Ri by a no-zero constant k denoted by Ri →k Ri
3. adding or subtracting a constant k multiple of a row Ri to another row Rj denoted by Ri → Ri -k Rj.

In terms of the given equations these operations are obvious. When a finite number of the above row operations are applied on a given system, the system is modified and the two are called row-equivalent systems. It can be proved that row-equivalent systems have the set of solutions.

Row echelon form of a matrix

Suppose elementary row transformations are applied on a matrix and the resulting matrix satisfies the following three conditions:

1. all zero rows are below the non-zero rows
2. first non-zero element in a row from the left (called leading entry) is always in a column to the right of the leading entry in the row above it.

   i.e. The number of zeros before the first non-zero element in each row will be in an increasing order.
3. all elements below the leading entry in a column will be zeros.

Converting a linear system of equations into matrix form helps us in many ways. Augmented matrix can be transformed to echelon form to determine the solution easily.

For example, $\begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 4 & 7 & 9 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form but

$\begin{bmatrix} 3 & -1 & 2 & 5 \\ 0 & 4 & 7 & 9 \\ 0 & 6 & 0 & -5 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ is not.

A matrix is in reduced row echelon form or simply reduced echelon form if

1. it is in echelon form
2. the leading entry in each row is 1
3. the column containing leading entry 1 should have all other elements zeros

For instance, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form. If a matrix is in echelon form it is called echelon matrix.

Now, the question is there any essential difference between echelon matrix and reduced echelon matrix. For a nonzero matrix its echelon matrix need not be unique but reduced echelon matrix is unique. So we call an echelon matrix U of A as an echelon form of A while a reduced echelon matrix U is called the reduced echelon form of A.

Since reduced echelon form is unique, the leading entries in all echelon forms are in the same position and the position of 1 in reduced echelon form is called pivot position. A column containing pivot position is called a pivot column.

As seen above row equivalent systems have the same set of solutions, so transforming augmented matrix into an echelon matrix will make the solution process simple.

Echelon matrix and reduced echelon matrix are different but for doing problems any of them can be used. Number of pivots gives rank of a matrix which is important.

Illustration 1.2.1

Determine if the following matrices are in echelon form. Are they in reduced echelon form?

(a) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

(a) is in echelon form but not in reduced echelon form as all elements other than leading entry are not zero in columns 2 and 3.

(b) is not in echelon form as zero row is not below non zero rows.

(c) is in reduced echelon form as each leading entry is 1, column containing this 1 has all other elements zeros.

Illustration 1.2.2

Find echelon form of the matrix A= $\begin{bmatrix} 3 & 3 & 1 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

Solution

Consider A= $\begin{bmatrix} 3 & 3 & 1 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - \frac{1}{3}R_1$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 0 & -4 & -1 \\ 0 & -2 & 5/3 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_2$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 0 & -4 & -1 \\ 0 & 0 & 13/6 \end{bmatrix}$$

This is in echelon form.

1.2.3 Gaussian elimination

There are different methods for solving a system of linear equations. Gaussian elimination method is one among them. We start with augmented matrix and apply elementary row transformations to convert into an echelon matrix. Then rewriting the equation from this last matrix, back substitution (where values of the variables are substituted from the last equation) is done to get a solution. We classify the nature of the solution after re-writing the equations from the echelon matrix as :

1. if number of equations is still equal to number of variables, a unique solution will be obtained.
2. if number of equations m is less than the number of variables n, infinitely many solutions are possible, choose n-m variables arbitrarily.
3. if any row gives an absurd equation, the system has no solution.

For example, suppose $\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$ is the last matrix after applying elementary row transformations to the augmented matrix, then rewriting this into equations, we get,

$3x_1 + 2x_2 + x_3 = 3$

$0x_1 - x_2 + x_3 = -6$. Last equation is absurd, so the system has no solution.

$0x_1 + 0x_2 + 0x_3 = 12$

Illustration 1.2.3

Solve by Gauss elimination method:

$$x - 2y + 3z = 9, \quad -x + 3y = -4, \quad 2x - 5y + 5z = 17$$

Solution

Augmented matrix is $\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$

Step 1: Row1 contains the leading entry. column 1 contains this leading entry, so elements in row2 and row 3 in column 1 must be made zero. So apply elementary row transformations on these rows.

Step 2: Row 2 contains leading entry in column 2, so the element below it in column 2 is made zero.

Step 1

$R_2 \to R_2 + R_1,$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

Step 2

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Now rewriting rows into equations, starting from R3,

2z=4 implies z=2.

From R2, y+3z=5 implies y=5-3z=5-6=-1

From R1, x-2y+3z=9 implies x=9+2y-3z=1

This is back substitution. So solution is

$$x = 1, y = -1, z = 2$$

Illustration 1.2.4

Solve by Gaussian elimination method:

$$x_1 - x_2 + 2x_3 = 4, \quad x_1 + x_3 = 6,$$
$$2x_1 - 3x_2 + 5x_3 = 4, \quad 3x_1 + 2x_2 - x_3 = 1$$

Solution

Augmented matrix is $\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}$

Step 1: Leading entry 1 in R1 is in column 1. So other elements below this 1 in R2, R3 and R4 are made zeros.

Step 2: leading entry in R2 is 1 and is in column 2, so all entries below it in column 2 are made zeros

Step 1

$$R_2 \to R_2 - R_1,$$
$$R_3 \to R_3 - 2R_1$$
$$R_4 \to R_4 - 3R_1$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 3 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix}$$

Step 2

$$R_3 \to R_3 + R_2$$
$$R_4 \to R_4 + 5R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 9 \end{bmatrix} \qquad R_4 \to R_4 - 5R_2 \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & -21 \end{bmatrix}$$

Step 3:

Replace $\boldsymbol{R_3}$ and $\boldsymbol{R_4}$

$$R_4 \to R_4 - 3/5R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Back substitution, starting from R4, 0=-2 , which is absurd. So the system has no solution.

Illustration 1.2.5

Solve by Gaussian elimination method:

2x+z =3, x-y-z =1, 3x-y =4

Solution

Augmented matrix is $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}$

Step 1: Leading entry in R1 is 2 in column 1, so all elements below it in column1 are made zeros.

Step 2: Leading entry in R2 is -1 in column 2, so entry below it in R3 is made zero.

Step 1

$$R_2 \to R_2 - 1/2R_1,$$
$$R_3 \to R_3 - 3/2R_1$$
$$\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & -3/2 & -1/2 \\ 0 & -1 & -3/2 & -1/2 \end{bmatrix}$$

Step 2

$$R_3 \to R_3 - R_2$$
$$\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form. So by back substitution number of equations 2 is now less than number of unknowns 3, so choose one variable to have arbitrary value, say, z h= k. From R2, -y -3/2 z = - 1/2 implies y=1/2 -3/2 k and from R1, x= 3/2 – ½ k .

So solution is : x= 3/2 -1/2 k, y= ½- 3/2 k, z=k

1.2.4 Rank of a matrix and consistency of a system of equations

A general method for solving linear system of equations has to determine first whether the system is consistent and then find solution, if that exists. Rank of a matrix is a number that identifies whether the system is consistent.

> Rank of a matrix can be conveniently determined using echelon form of a matrix. It is defined for any matrix, whether square or not. It provides a simple criterion to determine the nature of solution

Suppose an m x n matrix is given. Transform that into echelon form using elementary row transformations. The number of non-zero rows in the echelon matrix is called rank of the matrix.

For example, matrix in the echelon form $\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ has 3 non-zero rows, so rank=3.

Consider a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

This system has a coefficient matrix A and an augmented matrix [Ab]. Its augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} \ldots & a_{2n} & b_2 \\ & \ldots & & \\ a_{m1} & a_{m2} \ldots & a_{mn} & b_m \end{bmatrix}.$$

Step 1: Transform [Ab] into an echelon matrix

Step 2: Find rank of A, then drop the last column and the rank of the resulting matrix gives rank of A. Three cases are possible:

1. if rank of A = rank of [Ab], then the system is consistent.

2. If rank of A = rank of [Ab] = n, the number of variables, then the solution is unique. If rank of A = rank of [Ab] = r, not n, then the system has infinitely many solution. Such a solution is described using the number of arbitrary variables that is chosen and this is equal to n-r.

3. If rank of [Ab] is not equal to rank of A, the system is not consistent.

Homogeneous system of linear equations

Consider a system of linear equations where all the constant terms are equal to zero. This system is called a homogeneous system of equations. Otherwise the the system is called nonhomogeneous. In general such a system is represented by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$$

In matrix form this system is AX= O.

In this case, augmented matrix has no relevance, so only coefficient matrix is considered. A solution in which all variables are zeros is called a trivial solution, otherwise the solution is called nontrivial. A homogeneous system is always consistent since it has trivial solution. So the only question in solving homogeneous system is to find out whether it has a trivial solution or not. Modifying the above discussion on consistency, it can be found:

1.If rank of A=n, the system has only trivial solution.

2. If rank of A=r is less than n, the system has non trivial solution containing n-r arbitrary variables.

Illustration 1.2.6

Solve: $x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 2$

$$-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 = 6$$
$$2x_1 + 4x_2 - 3x_3 + 2x_4 = 3$$
$$-3x_1 - 6x_2 + 2x_3 + 3x_5 = 9$$

Solution

This is a non homogeneous system of equations. Consider its augmented matrix:

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

Transform it into echelon form using elementary row transformations:

Step 1

$$R_2 \to R_2 + R_1,$$
$$R_3 \to R_3 - 2R_1$$
$$R_4 \to R_4 + 3R_1$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

Step 2

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

Step 3

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{bmatrix}$$

Step 4

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 3 & 3 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 6

$R_1 \to R_1 - R_3$

$R_2 \to R_2 + 1/2R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7

$R_2 \to (-1)R_2$

$R_3 \to (1/4)R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the matrix is in reduced echelon form. (steps can be stopped in step 5, this is to show the reduced echelon form).

Rank of [Ab]= 3, rank of A=3 so the system is consistent.

Number of variables=5, so system has infinitely many solutions. Choose 5-3=2 variables arbitrarily.

Now rewriting equations,

R1$\longrightarrow$ $x_1+2x_2-x_5 = -5$

R2$\longrightarrow$ $x_3 = -3$

R3$\longrightarrow$ $x_4+x_5 = 2$

Now it is clear that we can choose only from x_1, x_2 and x_5. Choosing x_5=k, , x_2 = p, we get ,

x_1= -5-2p+k, x_2 = p, x_3 = -3, x_4 =2-k, x_5=k as the solution.

Illustration 1.2.7

Solve: $x_1 + 2x_2 + x_3 = 3$

$2x_1 + 3x_2 + 2x_3 = 5$

$3x_1 - 5x_2 + 5x_3 = 2$

$3x_1 + 9x_2 - x_3 = 4$

Solution

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \\ 3 & 9 & -1 & 4 \end{bmatrix}$$

Step 1

$R_2 \to R_2 - 2R_1,$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix}$$

Step 2

$R_2 \to (-1)R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -11 & 2 & -7 \\ 0 & 3 & -4 & -5 \end{bmatrix}$$

Step 3

$R_3 \to R_3 + 11R_2,$
$R_4 \to R_4 - 3R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -8 \end{bmatrix}$$

Step 4

$R_3 \to (1/2)R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & -8 \end{bmatrix}$$

Step 5

$R_4 \to R_4 + 4R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So rank of [Ab]=3=rank of A ,so the system is consistent and number of variables also is 3. So the system has a unique solution.

Rewriting the equations, solution is $x_1 = -1$, $x_2 = 1$, $x_3 = 2$

Illustration 1.2.8

Solve : $3x_1 + 2x_2 + x_3 = 3, \quad 2x_1 + x_2 + x_3 = 0,$

$6x_1 + 2x_2 + 4x_3 = 6$

Solution

The augmented matrix of the system is

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

Step 1

$$R_2 \to R_2 - 2/3R_1,$$
$$R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

Step 2

$$R_2 \to (-3)R_2$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & -6 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

Step 3

$$R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

rank of [Ab]= 3, rank of A=2

So the system is inconsistent.

Illustration 1.2.9

Solve : $x_1 + 2x_2 + 2x_3 + 3x_4 = 0,$

$2x_1 + 4x_2 + x_3 + 3x_4 = 0,$

$3x_1 + 6x_2 + x_3 + 4x_4 = 0$

Solution

This is a homogeneous system, so only coefficient matrix will be considered.

The coefficient matrix of the system is

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix}$$

Step 1

$R_2 \to R_2 - 2R_1,$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{bmatrix}$$

Step 2

$R_2 \to (-1/3)R_2$
$R_3 \to (-1/5)R_3$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Step 3

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ Rank of A=2 < number of variables

So the system has non trivial, i.e., infinitely many solution. Now choose 4-2=2 variables arbitrarily.

Choosing x_4=k, x_2=p, from R2, $x_3 + x_4$ =0 which implies,

x_3 =-k. From R1, x_1 = -2p-k

So solution is x_1 =-2p-k, x_2=p, x_3 =-k, x_4=k

Illustration 1.2.10

Solve : $x_1 -3x_2 -8x_3 = 0,$

$3x_1 + x_2 = 0,\ 2x_1 + 5x_2 + 6x_3 = 0$

Solution

This is a homogeneous system, so only coefficient matrix will be considered.

The coefficient matrix of the system is

$$\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$$

Step 1

$R_2 \to R_2 - 3R_1,$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix}$$

Step 2

$R_3 \to R_3 - (11/10)R_2$

$$\begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 0 & -44/10 \end{bmatrix}$$

Rank of A = 3, number of variables = 3, so the system has only trivial solution, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

Summarized Overview

One of the major applications of matrices is in solving a system of linear equations. Solving such a system has become very important as it is required in various fields including computer science. Basic question is how to determine whether a given system is consistent and if consistent which method can be applied to find the solution. Two methods are suggested, Gaussian elimination method and rank of a matrix method. In both, a system of linear equations is represented in matrix form. Then the matrix is transformed to a simpler form using echelon forms. From this form solution can be determined without any confusion regarding their existence.

Solving non homogeneous and homogeneous systems poses different challenges. In most applications, the number of equations and number of variables are very large. So representing them in matrix form and in transforming these matrices to some simpler forms are serious problems. Row transformations and echelon forms will be always required in linear algebra.

Assignments

1. Solve the system of equations using Gauss elimination method:

   $3\,x_1 + x_2 - 9x_3 = 2,\ x_1 + x_2 - 5\,x_3 = 0,\ 2x_1 + x_2 - 7x_3 = 1$
2. Solve: $x_1 + 2x_2 + 2x_3 + x_4 = 4,\ 3x_1 + 7x_2 + 7x_3 + 3x_4 = 13,\ 2x_1 + 5x_2 + 5x_3 + 2\,x_4 = 9$
3. Find the solution: $3x_1 + x_2 - 9x_3 = 0,\ x_1 + x_2 - 5x_3 = 0,\ 2x_1 + x_2 - 7x_3 = 0$
4. Find the values of λ and μ in the following system of equations such that the system has all cases possible in solving a system: $x_1 + x_2 + x_3 = 6,\ x_1 + 2x_2 + 3\,x_3 = 10,\ x_1 + 2x_2 + \lambda\,x_3 = \mu$
5. Solve explicitly: $4x - 6y = -11,\ -3x + 8y = 10$
6. If **A** is the coefficient matrix for a homogeneous system consisting of four equations in eight unknowns and if there are five free variables, what is *rank* (A)**?**
7. Suppose that A is the coefficient matrix for a homogeneous system of four equations in six unknowns and suppose that A has at least one nonzero row.

   (a) Determine the fewest number of free variables that are possible.

   (b) Determine the maximum number of free variables that are possible.

8. Solve by Gaussian elimination method and show in parallel how the row transformations affect the equations: $x_1 - 2x_2 - x_3 + 3x_4 = 0$, $-2x_1 + 4x_2 + 5x_3 - 5x_4 = 3$, $3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$.
9. Give examples of a 4 x 3 matrix which is in echelon form but not in row reduced echelon form.

Suggested Reading

1. Gilbert Strang – *Introduction to Linear Algebra*, 5th ed., Wellesley-Cambridge Press, 2016. 1971.

Reference

1. David C. Lay, Steven R. Lay, Judi J. McDonald –*Linear Algebra and Its Applications*, 5th ed., Pearson, 2015.
2. K. Hoffman, R. Kunze – *Linear Algebra*, 2nd ed., Pearson
3. Linear Algebra Done Right Second Edition, Sheldon Axler, Springer,1997

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

3 UNIT Determinants and Eigen Concepts

Learning Outcomes

After completion of this unit, the learner will be able to:

- evaluate determinant of a matrix and apply their properties
- understand the role of determinants in matrix properties
- understand the concept of eigen values and eigen vectors
- apply eigen values and eigen vectors in problems
- understand diagonalisation and its significance in eigen concepts

Background

Matrix is usually considered as an arrangement of numbers in rows and columns. In several situations, a single number like rank can represent some of their properties. Determinant is a number that is calculated from a matrix which reveals some of the not so obvious properties of the matrices. Earlier it was widely used but now it is not that much discussed mainly because of several new concepts developed in matrix theory. Determinants give a formula for pivots in echelon form and also measures how much dependency is there between the constants of the equation and the solution while we solve a system of linear equations using the inverse of a matrix. They possess many interesting properties that are helpful in many computations. Determinants lead to eigen values in a simple way which opens a door for wide range of applications of matrices.

The concept of eigen values and eigen vectors appear not only in matrices but in other areas like differential equations. That is why sometimes they are called characteristic values and vectors. They describe the properties of a system which is represented using a matrix. This is related to the concept of directions and give important insights in the geometry of matrices. In the theory of linear operators, they play a significant role. Their study helps to develop algorithms for dimensionality reduction and to characterise system dynamics. Corners and edges of images are detected in computer vision through this analysis. Given a dataset, the relevance and significance of data points in the set can be identified using a study of eigen values and eigen vectors which is used in many algorithms. Their importance comes from the definition itself, they express how much compression or stretching happens to data points during transformation.

Diagonal matrices are the simplest type of matrices which have several simple properties. These properties are helpful in computations. Diagonalising a matrix has several advantages and it reveals the structure of the matrix making it possible to detect eigen values and eigen vectors easily.

Keywords

Determinants, Eigen values, eigen vectors, characteristic equation, diagonalisation

Discussion

1.3.1 Determinants

Determinant is defined for a square matrix. It is a single number. It can also be used to represent many formulas like the area of a triangle in a compact way.It gives an idea about some properties of the associated matrix. Eralier methods for solving a system of linear equations involve determinants. There are different methods for evaluating the determinant of a matrix. Determinant computed in different ways.Their properties are hrelpful for evaluation of higher order determinats.

Matrix consists of a set of numbers in rows and columns. Determinant is a single number computed from a square matrix.

Computation of determinant

Determinant of a matrix A is denoted by $|A|$ or by by detA.

Consider a 1 x 1 matrix A = [a_{11}], its determinant is a_{11}.

Consider a 2 x2 matrix A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Its determinant is defined by $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

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To define determinant for higher order matrices, cofactor of an element is defined.

Consider a 3 x 3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

For a_{11}, define minor as a matrix $M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

which is from $\begin{bmatrix} \bullet & \boxed{a_{12} \quad a_{13}} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{bmatrix}$ and define cofactor as

$C_{11} = (-1)^{1+1} | M_{11} | = a_{22} a_{33} - a_{23} a_{32}$

Sign of the cofactor is determined by the sum of row number and column number, if the sum is even positive sign and if odd, negative sign is assigned.

For a_{12}, define minor as a matrix $M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$

which is from $\begin{bmatrix} \boxed{a_{11}} & \bullet & \boxed{a_{13}} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{bmatrix}$ and define cofactor as

$C_{12} = (-1)^{1+2} | M_{12} | = -(a_{21} a_{33} - a_{23} a_{31})$

Similarly for every element cofactors are defined.

Now determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} + a_{12}M_{12} + a_{13}M_{13}$$

Any row can be used to define matrix, for instance, using third row,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31}M_{31} + a_{32}M_{32} + a_{33}M_{33}$$

This definition can be extended to higher order determinants, select any row (column), multiply each element with its cofactor and addup.

Properties of determinants

1. If any two rows (columns) are interchanged in matrix, its determinant will change sign.
2. If any two rows (columns) are identical, the value of the determinant is zero.
3. If a row (column) is multiplied by a constant k, the determinant will get multiplied by k.
4. If multiple of any row(column) is added to another row (column) determinant will change.
5. If A is a square matrix, $|A|=|A^T|$.
6. If A and B are square matrices, $|AB|= |A||B|$. This shows that $|AB| =|BA|$.
7. If $|A|$ is non zero, then A is said to be invertible and inverse of A exists.
8. If A is a matrix of order n x n and if $|A|$ is non zero, then rank of A is n and if $|A|$ is zero, rank is less than n.

Illustration 1.3.1

Evaluate determinant of $A=\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

$$|A|=\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}=3\times 5--1\times 2=17$$

Illustration 1.3.2

Evaluate determinant of $A=\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$$|A|=\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}=1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix}-5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}+0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}=-2$$

It can be seen that third row contains more zeros, so if that row is used for expansion, calculations can be reduced as below:

$$|A|=\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}=0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}-(-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}+0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}=-2$$

Illustration 1.3.3

Evaluate determinant of A= $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 20$$

Illustration 1.3.4

Evaluate determinant of A= $\begin{bmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Fourth row of A has maximum number of zeros, so expanding using that row,

$$|A| = \begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= -0\begin{vmatrix} 5 & -8 & 4 \\ -2 & 3 & -7 \\ 0 & 1 & 5 \end{vmatrix} + 0\begin{vmatrix} 3 & -8 & 4 \\ 0 & 3 & -7 \\ 0 & 1 & 5 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 & 4 \\ 0 & -2 & -7 \\ 0 & 0 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 5 & -8 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 5 & -8 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 2\left\{0\begin{vmatrix} 5 & -8 \\ -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 3 & -8 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix}\right\} \text{using R3}$$

$$= -12$$

Note: here column 1 can also be used to start with.

Illustration 1.3.5

Use determinant to find if A is invertible: A= $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1 \neq 0$$

So A is invertible.

Note: Also the rank of A is 3.

Illustration 1.3.6

If $A=\begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ show that even though AB is not BA,

det (AB)=det (BA)

$$AB=\begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix},\ BA=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}=\begin{bmatrix} 33 & 10 \\ 12 & 5 \end{bmatrix}$$

$so,\ AB \neq BA$

$$|AB|=\begin{vmatrix} 25 & 20 \\ 14 & 13 \end{vmatrix}=325-280=45,\ |BA|=\begin{vmatrix} 33 & 10 \\ 12 & 5 \end{vmatrix}=165-120=45$$

Thus, det (AB)= det (BA)

1.3.2 Eigen Values and Eigen Vectors

The concept of eigen values and eigen vectors is a general one appearing in matrices, linear transformations, differential equations etc. For matrices, they together describe many of the important properties. It is also giving a geometrical meaning for multiplication of a matrix and a vector. This has several applications in computer science.

Suppose A is a square matrix of order n x n, let X be a column vector of order n x 1. Then if AX=λ X for some number λ , then X is called an eigen vector of A and λ is called an eigen value of A.

To determine them, we rewrite the definition as

AX - λX=O which implies $(A\text{-}\lambda I)X = O$. This is a homogeneous system of equations. So the system has a non trivial solution if and only if λ is an eigen value. Then X is its eigen vector.

The homogeneous system $(A\text{-}\lambda I)X = O$ has non trivial solution when rank of coefficient matrix $(A-\lambda I)$ is less than n which is guaranteed when $|A-\lambda I|=0$. So, eigen values are determined by solving $|A-\lambda I|=0$. This is a polynomial equation of degree n in λ called characteristic equation of A. Once λ is obtained, eigen vectors for each λ can be obtained from the homogeneous system.

The set of all solutions of $(A\text{-}\lambda I)X = O$ is called eigen space corresponding to λ .

Properties of eigen vectors and eigen values

1. Any non zero multiple of an eigen vector is also an eigen vector for the same eigen value.
2. If λ is an eigen value of A, then $1/\lambda$ is an eigen value of inverse of A
3. If λ is an eigen value of A, then $\lambda^2, \lambda^3, \ldots$ are the eigen values of $A^2, A^3, \ldots$ respectively
4. Sum of the eigen values is sum of the diagonal elements of the matrix(called trace of the matrix).
5. Product of the eigen values is the determinant of the matrix.
6. For diagonal and triangular matrices, eigen values are always the diagonal elements.

Note: Using the above properties, characteristic equation of a 3 x 3 matrix can be written as $\lambda^3 - \begin{pmatrix} \textit{sum} \text{ of} \\ \text{diagonal elements} \end{pmatrix} \lambda^2 + \begin{pmatrix} \textit{sum} \text{ of} \\ \text{minors of diagonal elements} \end{pmatrix} \lambda - \det A = 0$

Illustration 1.3.7

Determine eigen values and eigen vectors of $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Solution

To determine eigen values

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$$

$$\text{consider } |A-\lambda I| = O \Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = O \Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$$

$$\lambda^2 - 10\lambda + 24 = 0 \Rightarrow \lambda = 4, 6$$

To determine eigen vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector for λ

Consider $(A-\lambda I)X=0 \Rightarrow$ coefficient matrix$=\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

Suppose $\lambda = 4 \Rightarrow$ coefficient matrix$=\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}$

$R_2 \to R_2 - (1/2)R_1$

$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow rank = 1 < 2, so$ nontrivial solution

choose $x_2 = k$

$R_1 \Rightarrow 4x_1 - 4x_2 = 0 \Rightarrow x_1 = k, x_2 = k$

$So\ X = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Suppose\ \lambda = 6 \Rightarrow$ coefficient matrix$= \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow rank = 1 < 2, so$ nontrivial solution

choose $x_2 = k$

$R_1 \Rightarrow 2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2k, x_2 = k$

$So\ X = \begin{bmatrix} 2k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So eigen vectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for 4 and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for 6

Note: since any multiple of an eigen vector is also an eigen vector, only one simple vector need be given.

Illustration 1.3.8

Find eigen values and eigen vectors of $\begin{bmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{bmatrix}$

Solution

To find the eigen values

Let $A = \begin{bmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -3-\lambda & 1 & -3 \\ 20 & 3-\lambda & 10 \\ 2 & -2 & 4-\lambda \end{bmatrix}$

consider $|A - \lambda I| = O \Rightarrow \begin{vmatrix} -3-\lambda & 1 & -3 \\ 20 & 3-\lambda & 10 \\ 2 & -2 & 4-\lambda \end{vmatrix} = O$

OR trace=sum of diagonal elents=4

Minors of diagonal elements:

$$M_{11} = \begin{vmatrix} 3 & 10 \\ -2 & 4 \end{vmatrix} = 32, M_{22} = \begin{vmatrix} -3 & -3 \\ 2 & 4 \end{vmatrix} = -6, M_{33} = \begin{vmatrix} -3 & 1 \\ 20 & 3 \end{vmatrix} = -29$$

their sum =-3

$$|A| = \begin{vmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{vmatrix} = -96 - 60 + 138 = -18$$

Characteristic equation is

$$\lambda^3 - \begin{pmatrix} \textit{sum} \text{ of} \\ \text{diagonal elements} \end{pmatrix} \lambda^2 + \begin{pmatrix} \textit{sum} \text{ of} \\ \text{minors of diagonal elements} \end{pmatrix} \lambda - \det A = 0$$

$i.e., \lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0$

One solution is found by trial and error: $\lambda = 3$

So $\lambda - 3$ is afcator of the polynomial,dividing it by the factor, we get,

$\lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0 \Rightarrow (\lambda - 3)(\lambda - 3)(\lambda + 2) = 0$

$Thus, \lambda = 3, \lambda = 3, \lambda = -2$

To find the eigen vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector for λ

Consider $(A\text{-}\lambda I)X=0 \Rightarrow$ coefficient matrix= $\begin{bmatrix} -3-\lambda & 1 & -3 \\ 20 & 3-\lambda & 10 \\ 2 & -2 & 4-\lambda \end{bmatrix}$

Suppose $\lambda = 3 \Rightarrow$ coefficient matrix= $\begin{bmatrix} -6 & 1 & -3 \\ 20 & 0 & 10 \\ 2 & -2 & 1 \end{bmatrix}$

$R_1 \to \frac{-1}{6} R_1 \qquad R_2 \to R_2 - 20R_1 \qquad R_2 \to \frac{6}{20} R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & \frac{-1}{6} & \frac{3}{6} \\ 20 & 0 & 10 \\ 2 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{-1}{6} & \frac{3}{6} \\ 0 & \frac{20}{6} & 0 \\ 0 & -\frac{10}{6} & 0 \end{bmatrix}, \begin{bmatrix} 1 & \frac{-1}{6} & \frac{3}{6} \\ 0 & 1 & 0 \\ 0 & -\frac{10}{6} & 0 \end{bmatrix}$$

$R_3 \to R_3 + \frac{10}{6} R_2$

$$\begin{bmatrix} 1 & \frac{-1}{6} & \frac{3}{6} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ so rank=2<3}$$

Choose 1 varibale, x_3 =k, then, x_2 =0, x_1 =(-1/2)k

$$\text{So } X = k\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} or\ k\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$Suppose\ \lambda = -2 \Rightarrow \text{coefficient matrix=}\begin{bmatrix} -1 & 1 & -3 \\ 20 & 5 & 10 \\ 2 & -2 & 6 \end{bmatrix}$$

$$R_2 \to R_2 + 20R_1$$
$$R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} -1 & 1 & -3 \\ 0 & 25 & -50 \\ 0 & 0 & 0 \end{bmatrix} \text{ so rank=2<3, so choose } x_3 = k, then$$

$$x_2 = 2k,\ x_1 = -k,\ X = k\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

So eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ for 3 and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ for -2

Note: here an eigen value 3 occurs twice, number of repetitions of an eigen value is called algebraic multiplicity. So 3 has algebraic multiplicity 2. For a single eigen value, sometimes, eigen vectors which are not multiples of each other may be obtained, they are said to be linearly independent eigen vectors, number of such eigen vectors is called geometric multiplicity.

Illustration 1.3.9

Find eigen values and eigen vectors of $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Solution

Let A=$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

To find eigen values

$$A-\lambda I=\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$\text{consider } |A-\lambda I| = O \Rightarrow \begin{vmatrix} (3-\lambda) & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = O$$

$$i.e., (3-\lambda)(3-\lambda)(5-\lambda)=0 \Rightarrow \lambda=3, \lambda=3, \lambda=5$$

To find eigen vectors

Let $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector for λ

Consider $(A-\lambda I)X=0 \Rightarrow$ coefficient matrix$=\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$

Suppose $\lambda=3 \Rightarrow$ coefficient matrix$=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $rank=2<3$

Choose 1 variable, here $x_2=0$, $x_3=0$, $x_1=k$

$$X=k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

Suppose $\lambda=5 \Rightarrow$ coefficient matrix$=\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $rank=2<3$

$x_1=0,\ x_2=0,\ x_3=k,\ X = k\begin{bmatrix}0\\0\\1\end{bmatrix}$

So eigen vectors are $\begin{bmatrix}1\\0\\0\end{bmatrix}$ for 3 and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ for 5

Illustration 1.3.10

Find eigen values and eigen vectors of $\begin{bmatrix}-3 & -7 & -5\\2 & 4 & 3\\1 & 2 & 2\end{bmatrix}$

Solution

Let $A=\begin{bmatrix}-3 & -7 & -5\\2 & 4 & 3\\1 & 2 & 2\end{bmatrix}$

To find eigen values

$$A-\lambda I=\begin{bmatrix}-3-\lambda & -7 & -5\\2 & 4-\lambda & 3\\1 & 2 & 2-\lambda\end{bmatrix}$$

$$\text{consider } |A-\lambda I| = 0 \Rightarrow \begin{vmatrix}-3-\lambda & -7 & -5\\2 & 4-\lambda & 3\\1 & 2 & 2-\lambda\end{vmatrix} = 0$$

$$i.e., \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Rightarrow \lambda = 1, \lambda = 1, \lambda = 1$$

$\lambda = 1$ has algebraic multiplicity 3

To find eigen vectors

Let $X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be the eigen vector for λ

coefficient matrix=$A-\lambda I=\begin{bmatrix}-3-\lambda & -7 & -5\\2 & 4-\lambda & 3\\1 & 2 & 2-\lambda\end{bmatrix}$

Suppose $\lambda = 1 \Rightarrow$ coefficient matrix=$\begin{bmatrix}-4 & -7 & -5\\2 & 3 & 3\\1 & 2 & 1\end{bmatrix}$

$$R_1 \leftrightarrow R_3 \qquad R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 + 4R_1 \qquad R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2<3, so choose 1 variable,

x_1=-3k, x_2 =k, x_3 =k, $X = k\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

So eigen vector is $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ for 1.

1.3.2 Diagonalisation of a matrix

Eigen vectors corresponding to an eigen value are multiples of a single eigen vector. Sometimes we get more than one eigen vectors for an eigen value which are not multiples of each other called linearly independent eigen vectors. Then we can find a matrix P such that

$P^{-1}AP=D$ where D is a diagonal matrix called spectral matrix of A. The diagonal elements of D are called spectral values of A. Now A is said to be diagonalisable.

When two matrices are such that $P^{-1}AP=B$, then A and B are said to be similar matrices. They have the same set of eigen values.

For distinct eigen values, the eigen vectors are always linearly independent. So if eigen values are distinct, the matrix can be always diagonalised. When eigen values repeat, the number of linearly independent eigen vectors must be equal to the order of the matrix to diagonalise the matrix.

Columns of Matrix P will be the eigen vectors of A and diagonal elements of D are eigen values of A in the same order of their eigen vectors in P.

Illustration 1.3.11

If possible diagonalise $\begin{bmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{bmatrix}$

Solution

Let A=$\begin{bmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{bmatrix}$

To find eigen values

$$A-\lambda I=\begin{bmatrix} 1-\lambda & -4 & -4 \\ 8 & -11-\lambda & -8 \\ -8 & 8 & 5-\lambda \end{bmatrix}$$

$$\text{consider } |A-\lambda I| = O \Rightarrow \begin{vmatrix} 1-\lambda & -4 & -4 \\ 8 & -11-\lambda & -8 \\ -8 & 8 & 5-\lambda \end{vmatrix} = O$$

i.e., $\lambda^3 + 5\lambda^2 + 3\lambda - 9 = 0 \Rightarrow (\lambda - 1)(\lambda + 3)^2 = 0.$

so $\lambda = 1$ is asimple eigen value and $\lambda = -3$ has algebraic multiplicity 2

To find eigen vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector for λ

$$\text{coefficient matrix}=A-\lambda I=\begin{bmatrix} 1-\lambda & -4 & -4 \\ 8 & -11-\lambda & -8 \\ -8 & 8 & 5-\lambda \end{bmatrix}$$

$$\textit{Suppose } \lambda = 1 \Rightarrow \text{coefficient matrix}=\begin{bmatrix} 0 & -4 & -4 \\ 8 & -12 & -8 \\ -8 & 8 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $\quad$ $R_3 \rightarrow R_3 + R_1$ $\quad$ $R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 8 & -12 & -8 \\ 0 & -4 & -4 \\ -8 & 8 & 4 \end{bmatrix} \quad \begin{bmatrix} 8 & -12 & -8 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{bmatrix} \quad \begin{bmatrix} 8 & -12 & -8 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

So rank=2<3, choose 1 variable, $x_1=-(1/2)k$, $x_2=-k$, $x_3=k$,

$$X = k\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & -4 & -4 \\ 8 & -11-\lambda & -8 \\ -8 & 8 & 5-\lambda \end{bmatrix}$$

$$Suppose\ \lambda = -3 \Rightarrow \text{coefficient matrix} = \begin{bmatrix} 4 & -4 & -4 \\ 8 & -8 & -8 \\ -8 & 8 & 8 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} 4 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so rank=1<3, choose 2 varibales}$$

x_1= k+p, x_2 =k, x_3 =p

$$X = \begin{bmatrix} k+p \\ k \\ p \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} + \begin{bmatrix} p \\ 0 \\ p \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + p\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, So eigen vectors are $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ for 1 and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ for -3.

Thus there are 2 linearly independent eigen vectors for eigen value -3.

So A is diagonalisable.

Matrix P= $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Illustration 1.3.12

If possible diagonalise $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

Solution

Let A= $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

To find eigen values

$$A-\lambda I=\begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}$$

consider $|A-\lambda I| = O \Rightarrow \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = O$

i.e., $(\lambda -1)(\lambda -2)^2 = 0.$

so λ =1 is asimple eigen value and λ=2 has algebraic multiplicity 2

To find eigen vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector for λ

coefficient matrix=$A-\lambda I=\begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}$

Suppose $\lambda = 1 \Rightarrow$ coefficient matrix=$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}$

$R_2 \to R_2 + \frac{1}{4} R_1$

$R_3 \to R_3 - \frac{3}{4} R_1$ $\qquad R_3 \to R_3 - R_2$

$$\begin{bmatrix} 4 & -6 & -6 \\ 0 & \frac{6}{4} & \frac{2}{4} \\ 0 & \frac{6}{4} & \frac{2}{4} \end{bmatrix} \qquad \begin{bmatrix} 4 & -6 & -6 \\ 0 & \frac{6}{4} & \frac{2}{4} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank=2<3}$$

Choose 1 variable, $x_1 = k$, $x_2 = -k/3$, $x_3 = k$

$$X = k\begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix} \quad Or\ X = k\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

$$\text{coefficient matrix}=A-\lambda I=\begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}$$

$$Suppose\ \lambda=2 \Rightarrow \text{coefficient matrix}=\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}$$

$$R_2 \to R_2+\frac{1}{3}R_1$$

$$R_3 \to R_3-R_1$$

$$\begin{bmatrix} 3 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank}=1<3\text{,choose 2 variables}$$

$x_1=2k+2p$, $x_2=k$, $x_3=p$

$$X=\begin{bmatrix} 2k+2p \\ k \\ p \end{bmatrix}=\begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix}+\begin{bmatrix} 2p \\ 0 \\ p \end{bmatrix}=k\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}+p\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}+p\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ Thus, So eigen vectors are $\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$ for 1 and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ for 2.

Thus there are 2 linearly independent eigen vectors for eigen value 2. So A is diagonalisable.

Matrix $P=\begin{bmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $P^{-1}AP=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Illustration 1.3.13

If possible diagonalise $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Solution

Let $A=\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

To find eigen values

$$A-\lambda I=\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$\text{consider } |A-\lambda I| = O \Rightarrow \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = O$$

i.e., $-\lambda^3-3\lambda^2+4=0 \Rightarrow -(\lambda-1)(\lambda+2)^2=0.$

so $\lambda=1$ is asimple eigen value and $\lambda=-2$ has algebraic multiplicity 2

To find eigen vectors

Let $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector for λ

$$\text{coefficient matrix}=A-\lambda I=\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$Suppose\ \lambda=1 \Rightarrow \text{coefficient matrix}=\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2+4R_1$

$R_3 \to R_3-3R_1$ $\qquad R_3 \to R_3+R_2$

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 9 & 9 \\ 0 & -9 & -9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 3 \\ 0 & 9 & 9 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank}=2<3,\text{choose 1 variable}$$

$x_1=k,\ x_2=-k,\ x_3=k$, so $X=k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\text{coefficient matrix}=A-\lambda I=\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$Suppose\ \lambda=-2 \Rightarrow \text{coefficient matrix}=\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 3/4R_1$

$$\begin{bmatrix} 4 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 9/4 \end{bmatrix}$$ rank=2<3,choose 1 variable

x_1=-k, $x_2 = $ k, x_3 =0, so $X = k\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector for 1 and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ for -2, thus only 2 eigen vectors, so cannot be diagonalised.

Illustration 1.3.14

If possible diagonalise $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution

Let $A=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

To find eigen values

$$A-\lambda I=\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

consider $|A-\lambda I| = O \Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = O$

i.e., λ^3-6λ^2+11λ − 6=0 $\Rightarrow$ (λ -1)(λ-2)(λ-3) = 0.
so λ =1,2,3 are asimple eigen values.

To find eigen vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector for λ

coefficient matrix=A-λI=$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$

$Suppose\ \lambda = 1 \Rightarrow$ coefficient matrix=$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Eigen vector is $X = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Similarly, the eigen vectors for the other values can be determined. Thus, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ for eigen value 1, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for eigen value 2 and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ for eigen value 3. Thus, there are 3 distinct eigen values, the matrix is diagonalisable with P=$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and D=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Summarized Overview

Determinant is a number associated with a square matrix. Properties of determinats make them easy to calculate. Their applications can found in various fields Determinant can be computed in different ways. Computation of eigen values and eigen vectors require determinants. They describe several properties of square matrices. Number of eigen values and eigen vectors provides hints to the properties of matrices. They form the basis of several applications in computer science. Diagonalisation is the process of converting a square matrix into a diagonal matrix without loosing the eigen values. Conditions for this should be understood before undertaking this process.

Assignments

1. Evaluate the determinant: $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -17 \end{vmatrix}$

2. Find if the matrix is invertible: $\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$

3. Solve using determinants: $2v_1 - v_2 + 3v_3 = 5, -4v_1 - 3v_2 - 2v_3 = 8, 3v_1 + v_2 - v_3 = 4$

4. Show that if A is invertible, det $A^{-1}=1/\det A$

5. If U is a square matrix such that$UU^T=I$, find the value of detU

6. Find the eigen values and eigen vectors of A, A^2 and A^{-1}: $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. If diagonalisable, do it for A=$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

Suggested Reading

1. Gilbert Strang – ***Introduction to Linear Algebra***, 5th ed., Wellesley-Cambridge Press, 2016.

Reference

1. David C. Lay, Steven R. Lay, Judi J. McDonald – *Linear Algebra and Its Applications*, 5th ed., Pearson, 2015.
2. K. Hoffman, R. Kunze – *Linear Algebra*, 2nd ed., Pearson, 1971
3. Linear Algebra Done Right Second Edition, Sheldon Axler, Springer,1997 .

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

4
UNIT

Vector spaces and Norms

Learning Outcomes

After completion of this unit, the learner will be able to:

- discuss vectors in a general sense
- identify vector spaces
- understand linear independence
- understand norms
- understand orthogonal vectors

Background

A set of objects and a linear combination of these objects appear frequently in mathematics. In matrices, linear combinations of rows play a significant role. In calculus, linear combinations of functions are everywhere. In vectors, linear combinations are used. Some define linear algebra as the study of a set and a linear combination appropriately defined on the set. When the properties of R^n are discussed, several similarities are found with other sets usually arise in problems. Moreover it has a geometric structure. The concept of vector spaces combines these ideas and it is described using a set of properties. Vectors have a general definition. High dimensional vector spaces are used to represent data spaces.

A basic idea in vector spaces is that of linear independence which is very important to understand the foundations of several sets of objects. It enables to define clearly the building blocks of spaces in mathematics. It is also closely related to the rank of a matrix.

Distance always gives an idea how individual objects are situated within a set of objects. Depending on the particular set, definition of distance needs modification. Magnitude of an object can be described using a generalised concept of distance. This is done in the context of vector spaces.

Orthogonality is a geometrical concept and it neatly describes some properties. This is generalised using vector spaces and its applications are many. It is a compact way of saying some essential characteristics of a set of data points. It plays a crucial role in dimensionality reduction. Independence of system components in software design is explained through orthogonality.

Keywords

Vector spaces, linear combinations, linear independence, norms, orthogonality

Discussion

1.4.1 Vector space and subspace

Properties of R^n are generally known. An abstraction of these useful properties is seen to be quite useful

The set of real numbers or complex numbers is usually called a field denoted by F or K. Elements of the field are called scalars. a,b,c…or α, β,… are used to denote scalars.

A vector space generalises n dimensional real or complex space. Its definition is given through a set of properties of its elements.

Vector space:

Let K be a given field and let V be a nonempty set with rules of two operations called addition and scalar multiplication which assigns to any u,v in V a sum u+v in V and to any u in V, k in K, a product ku in V. Then V is called a Vector Space over K and the elements of V are called vectors if the following axioms are held:

Suppose $u,v,w \in V$ are any 3 vectors and $a,b \in K$ are scalars.

$A1$: $(u+v)+w = u+(v+w)$

$A2$: there is a vector denoted by $0 \in V$ for which $u+0=u$

0 is called zero vector.

$A3$: for each $u \in V$, there is a vector denoted by $-u \in V$ such that $u+(-u)=0$.

$A4: u + v = v + u$

$A5: a(u + v) = au + av$

$A6: (a + b)u = au + bu$

$A7: (a\ b)u = a(bu)$

$A8:$ for the unit element $1 \in K$, $1u = u$

Thus,

A vector space (linear space) consists of

1. A field F of scalars
2. A set V of objects called vectors
3. A rule called vector addition which associates with each pair of elements in V an element in V and
4. A rule called scalar multiplication which associates for each vector in V and for each scalar in K, a vector in V.
5. A vector space over the field R is called real vector space.

Illustration 1.4.1

On R^N one can define the operations of addition and scalar multiplication:

When elements are considered vectors, It is often more convenient to write as rows for R^N.

Vector addition:

$x + y = (x_1 \dots x_N) + (y_1 \dots y_N) = (x_1 + y_1 \dots x_N + y_N)$

Scalar multiplication:

$a\,x = a \cdot (x_1 \dots x_N) = (a\,x_1, , a\,x_N)$

Then R^N is a vector space over the field R, called real vector space.

Illustration 1.4.2

If m and n are positive integers, let A, B and C be m x n matrices. Consider V as the set of all m x n matrices.

Define vector addition as usual matrix addition and scalar multiplication as usual scalar multiplication of matrices. Then,

Suppose $a, b \in K$ are scalars.

$A1: (A+B)+C = A+(B+C)$

$A2:$ there is a vector zero matrix denoted by $O \in V$ for which $A+O = A$

0 is called zero vector.

$A3$: for each $A \in$ V, there is a vector denoted by -A $\in$ V such
that $A+(-A)=0$.

$A4$: $A + B = B + A$

$A5$: $a(A + B) = aA + aB$

$A6$: $(a + b)u = au + bu$

$A7$: $(a\ b)A = a(bA)$

$A8$: for the unit element $1 \in K$, $1A = A$

This vector space is sometimes denoted by $R^{m \times n}$.

Illustration 1.4.3

Consider polynomials in a variable x,

$p(x)= c_0 + c_1x + c_2x^2+\ldots+c_n x^n$. Let V be the set of all such polynomials, define usual addition of polynomials

$$p(x)+q(x)= c_0 + c_1x + c_2x^2+\ldots+c_n x^n+$$
$$d_0 + d_1x + d_2x^2+\ldots+d_n x^n$$
$$=(c_0+ d_0) +(c_1+ d_1)x +\ldots++(c_n + d_n)x^n.$$

Scalar multiplication is defined as for any a in K,

$$a\, p(x) = a\,(c_0 + c_1x + c_2x^2+\ldots+c_n x^n)$$
$$= ac_0 + ac_1x + ac_2x^2+\ldots+ac_n x^n$$

p(x) + q(x) is again a polynomial in V and ap(x) is also a polynomial in V.

Suppose a,b $\in$ K are scalars.

$A1$: $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$

$A2$: there is a vector zero polynomial denoted by $0 \in$ V for which
$p(x) + 0 = p(x)$ and 0 is called zero vector.

$A3$: for each $p(x) \in$ V, there is a vector denoted by $-p(x) \in$ V such
that $p(x)+(-p(x))=0$.

$A4$: $p(x) + q(x) = q(x) + p(x)$

$A5$: $a(p(x) + q(x)) = ap(x) + aq(x)$

$A6$: $(a + b)p(x) = ap(x) + bp(x)$

$A7$: $(a\ b)p(x) = a(bp(x))$

$A8$: for the unit element $1 \in K$, $1p(x) = p(x)$

Illustration 1.4.4

Let V be the set of all functions from a nonempty X into a field K. For any functions f,g in V and for any scalar a, define vector addition as (f + g) (x)=f(x) +g(x) and (a f)(x)=a f(x). Then,

Suppose a,b $\in$ K are scalars.

$A1:\ (f(x)+g(x))+h(x)=f(x)+(g(x)+h(x))$

$A2:$ there is a vector zero function denoted by $0\in$ V for which $f(x)+0=f(x)$ and 0 is called zero vector.

$A3$: for each $f(x)\in$ V,there is a vector denoted by $-f(x)\in$ V such that $f(x)+(-f(x))=0$.

$A4$: $f(x)+g(x)=g(x)+f(x)$

$A5$: $a(f(x)+g(x))=af(x)+ag(x)$

$A6:(a+b)f(x)=af(x)+bf(x)$

$A7:(a\,b)f(x)=a(bf(x))$

$A8:$ for the unit element $1\in K,\ 1f(x)=f(x)$

Illustration 1.4.5

Let V be the set of ordered pairs of real numbers,

Find if V is a vector space under the following operations: (a, b) +(c, d)=(a + c, b + d) and k (a, b)= (k a , b).

Suppose $\alpha,\beta\in$ K are scalars.

Let $u=(a_1,b_1),\ v=(a_2,b_2),\ w=(a_3,b_3)$

$A1:\ (u+v)+w=u+(v+w)$

$A2:$ there is a zero vector denoted by O=(0,0) $\in$ V for which $u+O=u$

$A3$: for each $u=(a_1,b_1)\in$ V,there is a vector denoted by $-u=(-a_1,-b_1)\in$ V such that $u+(-u)=0$.

$A4$: $u+v=v+u$

$$A5:\ \alpha(u+v)=\alpha\{(a_1,b_1)+(a_2,b_2)\}=\alpha(a_1+a_2,b_1+b_2)$$
$$=(\alpha a_1+\alpha a_2,b_1+b_2)$$
$$\alpha u+\alpha v=\alpha(a_1,b_1)+\alpha(a_2,b_2)=(\alpha a_1,b_1)+(\alpha a_2,b_2)$$
$$=(\alpha a_1+\alpha a_2,b_1+b_2)$$

$$A6:(\alpha+\beta)u=(\alpha+\beta)(a_1,b_1)=((\alpha+\beta)a_1,b_1)$$
$$\alpha u+\beta u=\alpha(a_1,b_1)+\beta(a_1,b_1)=(\alpha a_1,b_1)+(\beta a_1,b_1)$$
$$=((\alpha+\beta)a_1,2b_1)$$
$$(\alpha+\beta)u\neq\alpha u+\beta u$$

So V is not a vector space.

Subspace of a vector space

Suppose V is a vector space over a field K. Let W be a subset of V. Then W is called a subspace of V if W itself is a vector space over K.

A non-empty subset W is a subspace of V if and only if it is closed under vector addition and scalar multiplication, i.e. if u , v are in V and a in K, then (i) u +v is in V and (ii) au is in V.

This can be rewritten as : W is a subspace of V if and only if o vector is in V and au + b v is in V for u, v in V and a, b in K.

An example is the set of all polynomials over field F is a subspace of the vector space of all functions from F into F.

Illustration 1.4.6

Consider a homogeneous system of m in n variables equations given by AX=O. Any solution of it X may be considered as a vector in R^n. Let W be the set of all solutions of this system.

Let X_1, X_2 be any two solutions of the system.

O is already in W. Also, $AX_1 = O,\ AX_2 = O$.

$Now,\ A(\alpha X_1 + \beta X_2) = A(\alpha X_1) + A(\beta X_2) = \alpha AX_1 + \beta AX_2 = O$

for any $\alpha, \beta \in K$.

$Thus,\ \alpha X_1 + \beta X_2 \in W$.

So W is a subspace of V called solution space.

Illustration.1.4.6

$Let\ W = \{(0, a_2, \ldots a_n) / a_i \in R^n\}$. Then W is a subspace of R^n.

O is already in W. Let $u = (0, a_2, \ldots a_n), v = (0, b_2, \ldots b_n)$

Now, for any$\alpha, \beta \in K$

$\alpha u + \beta v = \alpha(0, a_2, \ldots a_n) + \beta(0, b_2, \ldots b_n) = (0, \alpha a_2 + \beta b_2, \ldots \alpha a_n + \beta b_n)$.

$Thus,\ \alpha u + \beta v \in W$.

Illustration.1.4.7

Consider the vector space of all n x n s matrices. Let W be the subset of symmetric matrices.

Let $A, B \in V$, then $A^T = A$, $B^T = B$.

O is already in W.

$$Now, \text{ for any } \alpha,\beta \in K, \ [\alpha A+\beta B]^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T$$
$$= \alpha A + \beta B$$

Thus, $\alpha A + \beta B \in W$.

So W is a subspace V.

1.4.2 Linear Independence of vectors

Let V be a vector space over a field K. Let $u_1, u_2,\ldots u_n$ be vectors in V and $a_1, a_2, \ldots, a_n$ be in K . Then $a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$ is called a linear combination of $u_1, u_2,\ldots u_n$. The set of all linear combinations of vectors in V is a subspace of V.

The vectors $u_1, u_2,\ldots u_n$ is said to be linearly independent if $a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = 0$ implies $a_1 = 0, a_2 = 0, \ldots, a_n = 0$. Otherwise they are linearly dependent .A set of vectors is linearly independent (dependent) if the vectors are linearly independent (dependent).

For example, the nonzero rows of a matrix in echelon form are linearly independent.

Properties from linear independence:

1. If two vectors are linearly dependent then one of them is a multiple of the other.
2. The nonzero vectors $u_1, u_2,\ldots u_n$ are linearly dependent if and only if one of them is a linear combination of the others.
3. If any subset of a set of vectors is linearly dependent, that set is linearly dependent.
4. Any subset of a linearly independent set is linearly independent.
5. Any set which contains zero vector is linearly dependent.

Definition: A set of vectors $\{ u_1, u_2,\ldots u_n \}$ is said to span a vector space V if every vector v in V is a linear combination of elements of the set,

i.e., $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$. This set is called a spanning set of V.

A linearly independent spanning set is called basis of the vector space.

Number of elements in the basis is called dimension of the vector space.

For example, $\{(1,0), (0,1)\}$ is a basis for R^2 and dimension of R^2 is 2.

It can be proved that if given more than n vectors in an n dimensional vector space, the vectors are linearly dependent.

Illustration.1.4.8

Determine if linearly dependent.

(i) u=(3,4) and v=(1,-3)

(ii) $u=(2,-3), v=(6,-9)$

(iii) $u=\begin{pmatrix}1 & 2-3\\ 6 & -5 \quad 4\end{pmatrix}, v=\begin{pmatrix}6 & -5 & 4\\ 1 & 2 & -3\end{pmatrix}$

(iv) $u=1-3t+2t^2-3t^3,\ \ v=-3+9t-6t^2+9t^3$

Solution

i. u and v are not multiples of each other, so linearly independent.

ii. v=3u, so they are linearly dependent

iii. u and v are not multiples, so linearly independent

iv. v=-3u, so they are linearly dependent

Illustration.1.4.9

Determine if linearly dependent.

i. u=(1,-2,1), v=(2,1,-1), w=(7,-4,1)

ii. $u=(1,2,-3), v=(1,-3,2), w=(2,-1,5)$

iii. $u=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, v=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, w=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$

iv. $u=1+5t-3t^2+t^3,\ \ v=2+8t-t^2+t^3,\ \ w=5+9t-4t^2+2t^3$

Solution

(i) u=(1,-2,1), v=(2,1,-1), w=(7,-4,1)

consider x(1,-2,1)+y(2,1,-1)+z(7,-4,1) =(0,0,0)

i.e., (x+2y+7z, -2x+y-4z, x-y+z)= (0,0,0)

so, x+2y+7z=0, -2x+y-4z=0, x-y+z=0

This is a homogeneous sytem, so coefficient matrix

$$A=\begin{bmatrix}1 & 2 & 7\\ -2 & 1 & -4\\ 1 & -1 & 1\end{bmatrix}$$

$R_2 \to R_2+2R_1$

$R_3 \to R_3-R_1$ $\qquad R_3 \to R_3+3/5R_2$

$$\begin{bmatrix}1 & 2 & 1\\ 0 & 5 & 10\\ 0 & -3 & -6\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 8\\ 0 & 0 & 0\end{bmatrix}$$

So rank=2, so vectors are dependent.

(ii) $u=(1,2,-3), v=(1,-3,2), w=(2,-1,5)$

Another method is just write the given vectors as rows in a matrix and transform to echelon form to find the rank without writing homogeneous system.

(i) $u=(1,2,-3), v=(1,-3,2), w=(2,-1,5)$

Consider $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{bmatrix}$ so

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$ $\qquad$ $R_3 \rightarrow R_3 - R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 8 \\ 0 & -5 & 11 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 8 \\ 0 & 0 & 3 \end{bmatrix}$

So rank=3, the vectors are linearly independent.

(iii) $u=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, v=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(i) $u=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, v=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

consider $x\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+y\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+z\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

i.e., $\begin{pmatrix} x+y+z & x+z \\ x & x+y \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so, x+y+z=0, x+z=0, x,=0, x+y=0

Thus x = 0, y = 0, z = 0 implying the vectors are linearly independent.

iv) $u=1+5t-3t^2+t^3,\ v=2+8t-t^2+t^3,\ w=5+9t-4t^2+2t^3$

$(i) u = 1+5t-3t^2+t^3,\ v = 2+8t-t^2+t^3, w = 5+9t-4t^2+2t^3$

consider $x(1+5t-3t^2+t^3)+y(2+8t-t^2+t^3)+z(5+9t-4t^2+2t^3)$

i.e., $(x+2y+z)+(5x+8y+9z)t+(-3x-y-4z)t^2+(x+y+2z)t^3 = 0+0t+0t^2+0t^3$

so, $x+2y+z=0, 5x+8y+9z=0, -3x-y-4z=0, x+y+2z=0$

This is a homogeneous system, so coefficient matrix

$$A=\begin{bmatrix} 1 & 2 & 1 \\ 5 & 8 & 9 \\ -3 & -1 & -4 \\ 1 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$

$R_3 \to R_3 + 3R_1$

$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 9 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus the system has only trivial solution, so the vectors are linearly independent.

1.4.3 Vector norms and distances

Inner Product

Consider a vector space V over field R. An inner product on V is a function that assigns to each pair of vectors a scalar $\langle u,v \rangle$ called inner product of u and v such that

1. $\langle u, u \rangle \geq 0$ *and* $\langle u, u \rangle = 0$ *if and only if* $u = 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

Then V is called an inner product space.

For example, *in* R^n, $u=(a_1, a_2, \ldots a_n)$, $v=(b_1, b_2, \ldots b_n)$

$\langle u,v \rangle = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$ is an inner product.

As another example if V is the vector space of m x n matrices over real numbers,

$\langle A,B \rangle = trace\ of\ (B^T A)$ where trace is the sum of diagonal elements.

Norm of a vector

Suppose u and v are vectors in a vector space V, the norm of u denoted by $\|u\| = [\langle u,u \rangle]^{1/2}$

For example in *in* R^n, $u=(a_1, a_2, \ldots a_n)$, $\|u\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.

Distance between two vectors

The nonnegative real number $d(u,v)=\|u-v\|$ is called the distance between vectors u and v.

Different inner products can be defined in a vector space. For instance , $in\ R^2, \mathrm{u}=(\mathrm{a}_1, a_2), \mathrm{v}=(\mathrm{b}_1, b_2)$ are two vectors. The usual inner product is $\langle \mathrm{u,v} \rangle = \mathrm{a}_1\mathrm{b}_1 + a_2 b_2$. Another one is $\langle \mathrm{u,v} \rangle = \mathrm{a}_1\mathrm{a}_2 - \mathrm{a}_1\mathrm{b}_2 - \mathrm{a}_2\mathrm{b}_1 + 3b_1 b_2$. So different norms and hence different distances can be defined. Two common norms are :

1.L^1 norm: if $\mathrm{u}=(\mathrm{a}_1, a_2, ..., a_n)$, $\|u\| = |\mathrm{a}_1| + |\mathrm{a}_2| + ... + |\mathrm{a}_\mathrm{n}|$

2.L^2 norm: if $\mathrm{u}=(\mathrm{a}_1, a_2, ..., a_n), \|u\| = \sqrt{\mathrm{a}_1{}^2 + a_2{}^2 + ... + a_n{}^2}$

Second one is called Euclidean norm and the first one as Manhattan norm.

Distance can be defined in a vector space in different ways depending on the application It is defined using norm of a vector.

Illustration.1.4.10

Determine the L1 and L2 norms: u= (3, 4)

Solution

Norm by usual norm : $\|u\| = \sqrt{\langle \mathrm{u,u} \rangle} = \sqrt{\mathrm{a}_1{}^2 + \mathrm{b}_1{}^2}$

$u=(3,4), so, \|u\| = \sqrt{3^2+4^2} = 5$

L^1 norm: if $\mathrm{u}=(3,4)$, $\|u\| = |3| + |4| = 7$

Illustration.1.4.11

Determine the L1 and L2 norms for u=(1,-2,2,0)

Solution

Norm by L2 norm : $for\ \mathrm{u}=(\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4) \|u\| = \sqrt{\langle \mathrm{u,u} \rangle} = \sqrt{\mathrm{a}_1{}^2 + \mathrm{a}_2{}^2 + \mathrm{a}_3{}^2 + \mathrm{a}_4{}^2}$

$u=(1,-2,2,0), so, \|u\| = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = 3$

Norm by L1 norm $\|u\| = |1| + |-2| + |2| + |0| = 5$

Illustration.1.4.12

Find the distance between (i) u=(7,1) and v=(3,2)

(ii) u=(0,-5,2) and v=(-4,-1,8)

Solution

(i) u=(7,1) and v=(3,2), $u-v=(4,-1), so,$

(1) $\|u-v\|=\sqrt{4^2+(-1)^2}=\sqrt{17}, \quad d(u,v)=\|u-v\|=\sqrt{17}$

(2) by L2 norm: $\|u-v\|=|4|+|-1|=5, \quad d(u,v)=\|u-v\|=5$

(ii) (1) u=(0,-5,2) and v=(-4,-1,8), $u-v=(4,-4,-6)$,

$so, \|u-v\|=\sqrt{4^2+(-4)^2+(-6)^2}=\sqrt{68}, \quad d(u,v)=\|u-v\|=\sqrt{68}$

(2) by L2 norm: $\|u-v\|=|4|+|-4|+|-6|=14, \quad d(u,v)=\|u-v\|=14$

1.4.4 Orthogonal vectors and projection

Two vectors u and v are said to be orthogonal if $\langle u,v\rangle=0$.

The orthogonal projection of vector y onto u is defined as $\frac{\langle y,u\rangle}{\langle u,u\rangle}u$

> Orthogonal vectors have inner product equal to zero. They have several applications. The orthogonal projection is based on this.

Illustration.1.4.13

Find which are orthogonal: (i) u=(8,-5) and v=(-2,-3)

(ii) u=(3,2,-5,0) and v=(-4,1,-2,6)

Solution

(i) u=(8,-5) and v=(-2,-3)

$\langle u,v\rangle = 8\times-2+-5\times-3=-1\neq 0, so$ not orthogonal

(ii) u=(3,2,-5,0) and v=(-4,1,-2,6)

$\langle u,v\rangle = 3\times-4+2\times1+-5\times-2+0\times6=0, so$ orthogonal

Illustration.1.4.14

Find the orthogonal projection of :y=(7,6) onto v=(4,2)

Solution

Orthogonal projection of y=(7,6) onto u =(4,2) is

$$\frac{\langle y,u\rangle}{\langle u,u\rangle}u=\frac{28+12}{4^2+2^2}(4,2)=(8,4)$$

Summarized Overview

Vector spaces generalises the basic structure of 2 dimensional or 3 dimensional spaces. In fact every vector space of finite dimension can be identified to be n dimensional R space for some n. The magnitude and direction of a vector can be brought into a general setting. Vectors are generalised to be a set of ordered set of numbers in computer science. Other generalisations are given in other branches of study. How these are combined and how the property of linear independence becomes important is explained using examples.

Once vectors have a general definition, the concept of orthogonality is also generalised. Its geometrical significance provides several applications especially in computations. It minimises coupling and overlapping of modules, data structures etc.so that testing and changing can be done independently. Norms define distances and magnitudes in a vector space making easy to define similarities and disparities in a data set. This is explained through examples.

Once orthogonality and projection are defined, underlying properties of a data set can be extracted based on these concepts. This is clear when we define matrix factorisation. Norms and orthogonality make many problems easier for computation.

Assignments

1. Determine if linearly dependent.

   i. u=(1,-3,7), v=(2,0,-6), w=(3,1,-1),t=(2,4,-5)

   ii. $u=(2,-3,7), v=(0,0,0), w=(3,-1,4)$

2. Check linear independence

   i. $u=\begin{pmatrix}3 & 1\\ 1 & 2\end{pmatrix}, v=\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}, w=\begin{pmatrix}1 & -5\\ -4 & 0\end{pmatrix}$

   ii. $u=3-2t+4t^2+t^3,\ \ v=4-t+6t^2+t^3,\ w=7-8t+8t^2+3t^3$

3. Find the distance between (0, -5,2) and (-4,-1,8)
4. Find if orthogonal: u= (3,-7,4,0) and v=(1,-8,15,-7)

5. Compute L1 and L2 norms for (4, 1,3,5)
6. Find the L1 and L2 distance between (3,1,4) and (-1,2,-3)
7. If (3,k,7) is orthogonal to (2,1,5) find the value of k.
8. Find the orthogonal projection of y=(3,1,1) onto (2,2,2).
9. Compare L1 and L2 distance between (1,3,1,4) and -4,3,-2,1)
10. Find another norm for R^n other than L1 and L2 norms and compare the three norms for (1,2,3,-1).

Suggested Reading

1. David C. Lay, Steven R. Lay, Judi J. McDonald – *Linear Algebra and Its Applications*, 5th ed., Pearson, 2015

Reference

1. K. Hoffman, R. Kunze – *Linear Algebra*, 2nd ed., Pearson, 1971.
2. Schaum's Outline Series Linear algebra: Theory and Problems; McGrawHill

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

BLOCK 2

Discrete Mathematics and Numerical Methods

1 UNIT Set Theory and Logic

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand sets and their operations
- find applications of sets
- develop concepts in mathematical logic
- apply truth tables to find logical proof

Background

Discrete mathematics provides the back ground for developing many modern mathematical concepts and their applications. Many of modern mathematical topics are described in discrete mathematical structures. Sets form a clear and concise representation for vague concepts. They are the building blocks for much of the theory that are discussed in discrete mathematics. Their operations and properties play a significant role in applications. In analysis of algorithms, database systems and machine learning, sets form an integral part of description.

Reasoning is important in verifying the truth of a statement in mathematics but it expression is vague in natural language. In mathematical logic, reasoning is represented in symbols. Its concern is about the relationship between statements rather than the content of a statement. Starting from a set of premises, valid conclusions are required to be drawn. A set of rules is formulated in symbols for the purpose. This provides a precise language for reasoning. So truth of a mathematical statement can be verified objectively. Truth tables form the first part of such a framework. From that experience, deductive reasoning is developed. The whole theory has applications in proof theory , recursion theory etc. Proofs help to show that the algorithms are true and also to decide suitable algorithms for a particular task.

Keywords

Sets, subsets, union, intersection, truth table, logical operators, predicates

Discussion

2.1.1 Set

Most modern mathematics and much of computer science have sets as the underlying concept. It enables to formulate many a problem precisely and thus helps to find solutions for them. This is used to define relations, functions and then proofs. Set is the fundamental block on which all other discrete structures are built.

A set can be considered a well-defined collection of objects, these objects are called elements or members. Well-defined means given an object it must be possible to identify whether it is an element of the set. Sets are denoted by A, B ,C,… and elements by a,b,c,… If a is an element of the set. An element a in A is denoted by a ∈ A. If a is not an element of A, it is denoted by notation $a \notin A$. A set A with elements a,b,c,… is represented by A={ a,b,c,…}.For example, the set of even positive numbers is represented by B={2,4,6,…}.This is called roster form. Another way of representation is using set builder form where a common description is available to the elements. For example, if A is the set of all positive multiples of 5, then

i. in roster form A= { 5,10,15,…}

ii. in set builder form A={x/x=5n where n is a natural number}.

Some common sets are:

N = {0, 1, 2, 3, . . .}, the set of **natural numbers**

Z = {. . . ,−2,−1, 0, 1, 2, . . .}, the set of **integers**

Z+ = {1, 2, 3, . . .}, the set of **positive integers**

Q = {*p*/*q* | *p* ∈ **Z**, *q* ∈ **Z**, and *q* _= 0}, the set of **rational numbers**

R, the set of **real numbers**

R+, the set of **positive real numbers**

C, the set of complex numbers.

When *a* and *b* are real numbers with

a < *b*, we write

$[a, b] = \{x \mid a \le x \le b\}$

$[a, b) = \{x \mid a \le x < b\}$

$(a, b] = \{x \mid a < x \le b\}$

$(a, b) = \{x \mid a < x < b\}$

Note that $[a, b]$ is called the closed interval from a to b and (a, b) is called the open interval from a to b.

Empty set

A set with no element is called empty set denoted by $\emptyset$ or by { }.

Singleton set

A set with only one element is called a singleton set.

Venn diagram

A pictorial way of representing a set is using the Venn diagram where a rectangle represents Universal set which all elements under consideration are elements and each set of discussion is represented as a circle inside the rectangle.

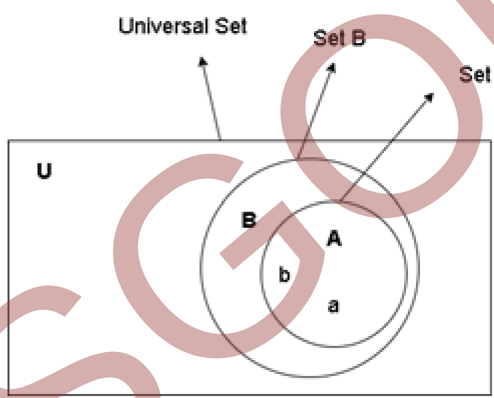


Fig 2.1.1 Venn Diagram

2.1.2 Subset

A is called a subset of B if every element of A is contained in B denoted by $A \subseteq B$. For example the set of natural numbers less than 5 is a subset of the set of all natural numbers. The above Venn diagram shows A as a subset of B. It is clear that $\phi \subseteq A$ and $A \subseteq A$ for any set A.

Cardinality of a set

The number of elements in set A is called cardinality of A denoted by n(A) or |A|. For example, the cardinality of set of letters in the word ARM is 3.

If cardinality is not finite it is called an infinite set. An example is the set of positive integers.

Power set

The set of all subsets of a set A is called its power set denoted by P(A).

Illustration 2.1.1

If A={ -1,0,3 } write its cardinality and its power set.

Solution:

Cardinality is 3.

P(S)={Φ,{-1},{0},{3},{-1,0},{-1,3},{0,3},A}.

Cardinality of P(S) is 8.

2.1.3 Operations on sets

Equality of sets

Two sets A and B are equal if they have the same elements denoted by A=B. To show that A and B are equal, it must be shown that $A \subseteq B$ and $B \subseteq A$.

Union of two sets

If A and B are two sets their union is a set denoted by $A \cup B$ containing all elements of A and B without repetition.

Intersection of sets

If A and B are two sets the intersection of A and B denoted by $A \cap B$ is the set of all elements both in A and B.

If intersection of two sets is empty, they are called disjoint sets.

Union and intersection can be extended to more than two sets in the same way as is done for two sets.

Cardinality of union of sets

It can be proved that $|A \cup B| = |A| + |B| - |A \cap B|$. This is generalised and this result is called the Inclusion-Exclusion principle.

Difference of sets

If A and B are two sets a difference B denoted by A-B is the set of all elements in A but not in B.

i.e., A-B={ $x \in A, x \notin B$ }. It is also called complement of B with respect to A. If U is the universal set, then U-A is called complement of A denoted by $\overline{A}$.

Symmetric difference

The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B and $A \oplus B = (A \cup B) - (A \cap B)$.

Sets form the building block of discrete mathematics and so their properties are important.

Illustration 2.1.2

Find which of the following sets are equal

1. A=The set of multiples of 2 between 1 and 11, B=the set of even numbers between 1 and 11.
2. A=the set of set of prime numbers between 1 and 11, B= the set of odd numbers between 1 and 11.

Solution:

In 1, A=B and in 2 A is not B called complement of A denoted by $\overline{A}$.

Illustration 2.1.3

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$ b) $A \cap B$ c) $A - B$ d) $B - A$.

Solution:

$A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

a) $A \cup B = \{0,1,2,3,4,5,6\}$

b) $A \cap B = \{3\}$ which is a singleton set

c) $A - B = \{1,2,4,5\}$

d) $B - A = \{\{0,6\}$

Illustration 2.1.4

Let A = {9, 20, 33, 41, 52} and B = {20, 33, 62,71}. Find

A $\oplus$ B.

Solution:

A = {9, 20, 33, 41, 52} and B = {20, 33, 62,71}

A $\cup$ B={9,20,33,41,52,62,71}

A $\cap$ B={20,33}

$A \oplus B = (A \cup B) - (A \cap B) = \{9,41,52,62,71\}$

Illustration 2.1.5

Show that if $A = \{x \mid x^2 + x - 6 = 0\}$ and $B = \{2,-3\}$, then A = B.

Solution:

Suppose x is in A, then x is a solution of $x^2 + x - 6 = 0$, so x=2, x=-3, thus x is in B.Thus, A $\subseteq$ B. Suppose x is in B, then x is in A. So B $\subseteq$ A. This shows A=B.

Illustration 4.1.5

Find if disjoint: A = {1, 4, 5}, B= {2, 6}}

Solution:

A and B have no element in common, so they are disjoint.

Properties of set operations

Some properties of set operations are given below.

1. Commutative Laws: For all sets A and B,

   (a) AUB = BUA and (b) $A \cap B = B \cap A$.

2. Associative Laws: For all sets A, B, and C,

   (a) (AUB)UC = AU(BUC) and

   (b) $(A\cap B)\cap C = A\cap(B\cap C)$.

3. Distributive Laws: For all sets, A, B, and C,

   (a) $AU(B\cap C) = (AUB)\cap(AUC)$ and

   (b) $A\cap(BUC) = (A\cap B)U(A\cap C)$

4. Identity Laws: For all sets A,

   (a) AUØ =A and (b) $A\cap U= A$

5. Complement Laws:

   (a) $AUA^c = U$ and (b) $A\cap A^c = Ø$

6. Double Complement Law: For all sets A,

   $(A^c)^c = A$.

7. Idempotent Laws: For all sets A,

   (a) AUA =A and (b) $A\cap A = A$

De Morgan's laws for sets:

$\overline{(A\cup B)} = \overline{A}\cap\ \overline{B}$, $\overline{(A\cap B)} = \overline{A}\cup\overline{B}$

Cartesian Products of sets

Let $A \times B$ denote the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Then $A \times B$

is the Cartesian product of A and B.

Illustration 2.1.6

If A = {1, 2,3} and B = {x, y, z} find $A \times B$ and

$B \times$ A

Solution:

A = {1, 2, 3}, B = {x, y, z}

$A \times B$= { (1,x),(1,y),(1,z),(2,x), (2,y), (2,z), (3,x),(3,y), (3,z)}

$B \times$ A={(x,1),(x,2),(x,3),(y,1),(y,2),(y,3),(z,1),(z,2),(z,3)}

2.1.4 Propositions

A proposition or statement is a sentence that can be decided as either true or false. For instance the sentence, "How old are you?', cannot be said true or false, so it is not a proposition. The sentence ' New Delhi is the capital of India " is true, so it is a proposition.

Propositions (statements) are denoted by variables p,q,r,… or by P,Q,R,… called propositional variables. Every proposition is either true where the truth value is said to be True denoted by T or false where truth value is False denoted by F the part of logic that discusses proposition is called propositional calculus or propositional logic.

Connectives

Now, different operations that can be done on propositions to frame new propositions are discussed. These logical operators are called connectives. Their truth values are completely described using a table which shows the possible truth values of the resulting proposition. This table is called truth table. In other words, each new proposition that is obtained is defined by its truth table.

Negation of a proposition

The negation of p, denoted by $\neg p$ (also denoted by p), is the statement "It is not the case that p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p. For example, if

p: raju is smart

$\neg p$: it is not the case that Raju is smart or simply, Raju is not smart.

Truth table for negation:

p	$\neg p$
T	F
F	T

Propositions can be combined using connectives to define new statements. They are defined by truth tables.

Illustration 2.1.7

Write the negation:

(i) P: It is not hot

(ii) P: This room is empty,

(iii) Q: $12+7<20$

(iv)P: Vandana's smartphone has at least 32GB of memory

(v) P: At least one student is selected

Solution:

(i) ~P: It is not true that it is hot

(ii) ~P: It is not true that this room is empty

(iii) ~Q: $12+7\geq 20$

(iv) ~P: Vandana's smartphone does not have at least 32GB of memory"

or even more simply as

"Vandana's smartphone has less than 32GB of memory

(v) ~P: No student is selected

Conjunction

If p and q are two statements, their conjunction denoted by $p \wedge q$ (read as p and q) is defined by the truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Illustration 2.1.8

i. P: Today is Friday Q: Raju is walking

ii. P: Smitha is smart, Q: Smitha is clever

iii. P:Rebecca's PC has more than 16 GB free hard disk space, Q: the processor in Rebecca's PC runs faster than 1 GHz.

Solution:

i. P∧Q : Today is Friday *and* Raju is walking

ii. P∧Q : Smitha is smart *and* Smitha is clever

iii. P∧Q : Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz.

Disjunction

If p and q are two statements, their disjunction denoted by P vQ (read as p or q), is defined by the truth table:

> Conjunction and disjunction are said to be duals of each other as their truth tables can be obtained from each other .

P	q	p vq
T	T	T
T	F	T
F	T	T
F	F	F

Illustration 2.1.9

Write the disjunction:

1. P: I will borrow the book, Q: I will purchase it,
2. P: There is something wrong with the wiring, Q: There is something wrong with the bulb.
3. P: Students who have taken calculus can take this class

Q: Students who have taken computer science can take this class

Solution:

1. P v Q : I will borrow the book or I will purchase it
2. P v Q : There is something wrong with the wiring or with the bulb
3. P v Q :Students who have taken calculus or computer science can take this class

Conditional

If p and q are two propositions, the conditional statement is denoted by $p \rightarrow q$ (read as: if p then q) is defined by the truth table:

P	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

P is called antecedent (hypothesis) and Q is called consequent(conclusion).

> Conditional is the most frequently used connective and it has several versions. Biconditional usually appears as a 'necessary and sufficient condition' form.

Illustration 2.1.10

1. P: Anjana has a degree, Q: Anjana will find a job

2. P: Today is Sunny, Q: We shall go for a picnic

3.P: you get 100% on the final,Q: you will get an A

Solution:

1. $P \to Q$: If Anjana has a degree, then she will find a job.

2. $P \to Q$: If today is Sunny, then we shall go for a picnic

3. $P \to Q$: If you get 100% on the final, then you will get an A

Following are some other ways to express this conditional statement:

"if p, then q" "p implies q

"p only if q" "p is sufficient for q"

"a sufficient condition for q is p" "q if p"

"a necessary condition for p is q" "q follows from p"

Converse

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements.

The proposition $q \to p$ is called the converse of $p \to q$.

Inverse

The proposition $\neg p \to \neg q$ is called the inverse of $p \to q$

Contrapositive

The contrapositive of $p \rightarrow q$ is the proposition

$\neg q \rightarrow \neg p$.

Of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

Equivalent statements

When propositions are formed by connectives, resulting statements are called compound statements. They are said to be equivalent when they have the same truth table for the same assignment of truth values. For example, a conditional and its contrapositive are equivalent.

Compound statements contain connectives. Same statement can be expressed in different ways, they are said to be equivalent

Illustration 2.1.11

What are the contrapositive, the converse, and the inverse of the conditional statement

"if it is raining, then the home team wins"

Solution:

The contrapositive of this conditional statement is

"If the home team does not win, then it is not raining."

The converse is

"If the home team wins, then it is raining."

The inverse is

"If it is not raining, then the home team does not win"

Biconditional

If p and q are two statements, their biconditional denoted by $p \leftrightarrow q$ (read: p if and only if q or p iff q) is defined by the truth table:

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It is also expressed as : p is necessary and sufficient for q.

Illustration 2.1.12

1.P: Ashok gets the job, Q: Ashok passes the test

2.P: Life exists in Mars Q: Water is present in mars

3.P: You can take the flight, Q: You buy a ticket

Solution

1 .Ashok gets the job iff Ashok passes the test

2. $P \leftrightarrow Q$: Life exists in Mars if and only if water is present in Mars

3. $p \leftrightarrow q$: You can take the flight if and only if you buy a ticket

Illustration 2.1.13

Construct truth table:

1.$(p \vee q) \rightarrow (p \wedge q)$ 2. $(p \rightarrow q) \rightarrow (q \rightarrow p)$

Solution

1.

p	q	pvq	p∧q	(pv q) →(p ∧q)
T	F	T	F	F
T	T	T	T	T
F	T	T	F	F
F	F	F	F	T

2.

p	q	p→q	q→p	(p→q)→(q→p)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Illustration 2.1.14

Show that following formulas are equivalent:

1. $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$

2. $P \leftrightarrow Q$ and $(P \rightarrow Q) \wedge (Q \rightarrow P)$

Solution

1.

P	Q	PvQ	¬(PvQ)	¬P	¬Q	¬P∧¬Q
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Look at the columns corresponding to both the formulas, they are identical, so equivalent.

2.

P	Q	$P \leftrightarrow Q$	P→Q	Q→P	(P →Q) ∧ (Q→P)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Look at the columns corresponding to both the formulas, they are identical, so equivalent

2.1.5 Predicates

In propositional logic, we are considering statement formulas but we find (i) not all statements can be expressed as statement formulas (e.g., 'some students are boys' where quantification is done) and the common features of component statements are not considered (e.g. 'John is a bachelor and Sajith is a bachelor'). So we require a generalization to consider these aspects also. For that we consider predicates in a statement and the logic based on predicates is called predicate logic.

Predicate calculus form an important part of mathematical logic. They help to frame statements that commonly appear in practical use.

Consider the statement: John is a bachelor

In English language we say John is the subject and is a bachelor is its predicate.

Propositional function

'is beautiful", 'is taller than', are predicates but they do not form statements. Suppose ' x is beautiful' then it is not a statement but statements can be formed from it.

The statement *P(x)* is also said to be the value of the propositional function *P* at *x*. Once a value has been assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$ and P is also called an n-place predicate or a n-ary predicate.The set from which $(x_1, x_2, \ldots, x_n)$ is chosen is called domain of discourse. For example, if $R(x, y, z)$ denote the statement`'$x+y=z$.", then R(1,3,5): 1+3=5, is a statement.

Quantifiers

From predicates, statements can be formed using predicate functions. Another important method is quantifying predicate functions. The words *all*, *some*,*many*, *none*, and *few* are considered quantifiers in English. Two types of quantification are commonly discussed in predicates, one is universal quantification, which tells us that a predicate is true for every element under consideration, and the other is, existential quantification, which tells us that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the predicate calculus.

Universal Quantifier

Let P be a propositional function with domain of discourse D. The statement for every x, $P(x)$ is said to be a universally quantified statement written as $\forall\ xP(x)$ where $\forall$ means "for every" called a universal quantifier.

The statement is true if $P(x)$ is true for every x in D. The statement is false if $P(x)$ is false for at least one x in D.

For example, for domain of discourse, the set of natural numbers $\forall x\ P(x)$: x+2=3 is false where as

$\forall x\ P(x)$: $x^2+1 > 1$ is always true .

Let P be a propositional function with domain of discourse D.

Universal and existential quantifiers generate statements that are expressed in natural languages containing some, few etc.

Existential Quantifier

The statement there exists x, $P(x)$ denoted by $\exists xP(x)$. is said to be an existentially quantified statement where he symbol $\exists$ means "there exists." called an existential quantifier.

The statement $\exists xP(x)$ is true if $P(x)$ is true for at least one x in D. The statement is false if $P(x)$ is false for every x in D.

For example, for the set of natural numbers as the domain of discourse, $\exists x\ (P(x)$: x is the solution of x+3=5) is true but $\exists x\ (P(x)$: x is the solution of $x^2+5=1$) is false

Illustration 2.1.15

What is the truth value of $\forall xP(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution

The statement $\forall xP(x)$ is the same as the conjunction P(1) ∧ P(2) ∧ P(3) ∧ P(4), because the domain consists of the integers 1, 2, 3, and 4. Because *P(4)*, which is the statement "42 < 10," is false, it follows that $\forall xP(x)$ is f Illustration 2.1.16

Express using quantifiers: All that glitters is not gold

Solution

Interpreting that All that glitters is not gold means

Some object that glitters is not gold, define

$P(x)$: "x glitters" and $Q(x)$: "x is gold,"

Then the given statement is : $\exists x(P(x) \wedge \neg Q(x))$.

Illustration 2.1.17

What is the truth value of $\exists xP(x)$, where *P(x)* is the statement "$x2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution

Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists xP(x)$ is the same as the disjunction P(1) ∨ P(2) ∨ P(3) ∨ P(4).

Because *P(4)*, which is the statement "42 > 10," is true, it follows that $\exists xP(x)$ is true.

Binding Variables

When a quantifier is used on the variable *x*, we say that this occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free. A propositional function becomes a statement when either all variables are bound or the variable has a particular value. Universal quantifiers, existential quantifiers, and value assignments are used to ensure this.

The part of a logical expression to which a quantifier is applied is called the scope of this quantifier. When a variable appears outside the scope of all quantifiers in the function, the variable is free. For example, in $\exists x(x + y+z = 10)$,only x is bound and other variables are free.

Logical Equivalence of propositions involving quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent. For example, $\forall x(P(x) \wedge Q(x))$ and $\forall xP(x) \wedge \forall xQ(x)$ are logically equivalent .

Negation of a quantified statement

Negation of quantified statements are done using the formulas:

- $\neg \forall x P(x) \equiv \exists x \neg P(x)$,negation is true when there is an *x* for which *P(x)* is false.
- $\neg \exists x Q(x) \equiv \forall x \neg Q(x)$, negation is true when for every *x*, *Q(x)* is false.

Illustration 2.1.18

What are the negations of the statements :

1.Some politicians are honest

2.All cars are electric cars

Solution

1.Let P(x) : x is a politician

Then, Some politicians are honest is $\exists x P(x)$, so negation is $\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$ ' no politician is honest'.

2. Let Q(x) : x is an electric car.

All cars are electric cars is $\forall x Q(x)$, negation is

$\neg \forall x Q(x)$ which is equivalent to $\exists x \neg Q(x)$,there is car which is not electric.

Illustration 2.1.19

Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution

Let *C(x)* : *x* has studied calculus

S(x): person *x* is in this class

So the given statement can be expressed as

$\forall x(S(x) \rightarrow C(x))$.

Illustration 2.1.20

Express the statements

1."Some student in this class has visited Mexico"

2. "Every student in this class has visited either

Canada or Mexico" using predicates and quantifiers.

Solution

1. Taking the given statement as : "There is a person *x* having the properties that *x* is a student in this class and *x* has visited Mexico."

Let M(x): x has visited Mexico.

$S(x)$: x is a student in this class

Then statement is : $\exists x(S(x) \wedge M(x))$

2. Taking the given statement as: for every x in this class, x has the property that x has visited Mexico or x has visited Canada. The statement can be expressed as

$$\forall x(S(x) \rightarrow (C(x) \vee M(x))).$$

Summarized Overview

Set is the basic building block of discrete mathematics. Sets provide an exact way of expressing concepts. Sets and their operations thus become important in discrete mathematics.

Mathematical logic provides the rules and methods for reasoning making it a formal topic of discussion. For algorithm design and data organisation, set and logic are required. Relational Database is based on set theory. Set and logic together provide fundamental ideas of many topics in modern mathematics. This makes concepts clear and reveals strong connections between them.

Propositional calculus and predicate logic concepts supplies rules for translating statements in natural language into formal mathematical statements. This gives precise expression and makes way for proofs of statements.

Assignments

1. If X is a finite set, what is $|X|$?
2. *Show that A is not a sub set of B if* $A = \{1, 2, 3\}$, $B = \{1, 2\}$
3. Show that $A \oplus B = (A - B) \cup (B - A)$.
4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find a) $A \cup B$. b) $A \cap B$. c) $A - B$. d) $B - A$.
5. Construct a truth table for each of these compound propositions.
   a) $p \rightarrow \neg q$ b) $\neg p \leftrightarrow q$

6. How many rows appear in a truth table for each of these compound propositions? a) $p \rightarrow \neg p$ b) $(p \vee \neg r) \wedge (q \vee \neg s)$

7. State the converse, contrapositive, and inverse of each of these conditional statements.

   a) If it snows today, I will ski tomorrow.

   b) I come to class whenever there is going to be a quiz.

8. Let $P(x)$ be the statement "x can speak Russian" and let

   $Q(x)$ be the statement "x knows the computer language

   C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

   a) There is a student at your school who can speak Russian and who knows C++.

   b) There is a student at your school who can speak Russian but who doesn't know C++.

   c) Every student at your school either can speak Russian or knows C++.

   d) No student at your school can speak Russian or knows C++.

Suggested Reading

1. Kenneth H. Rosen – *Discrete Mathematics and Its Applications*, 7th ed., McGraw-Hill, 2011.
2. C. L. Liu, D. P. Mohapatra – *Elements of Discrete Mathematics*, 3rd ed., McGraw-Hill, 2008.

Reference

1. Discrete and combinatorial Mathematics: An Applied Introduction –Ralph P Grimmaldy, Addison Wesly, 1994 .
2. DiscreteMathematics- Richard Johnsonbaugh,Eighth Edition, Pearson
3. Discrete and combinatorial Mathematics: An Applied Introduction –Ralph P Grimmaldy, Addison Wesly, 1994 .
4. DiscreteMathematics- Richard Johnsonbaugh,Eighth Edition, Pearson

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

2

UNIT

Graph Theory

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand basic concepts in graph theory
- recogonise different types of graphs
- understand Trees and their applications
- apply algorithms for graph traversals

Background

Graph theory originated for solving practical problems and its applications extend to nearly all branches of study. Its origin is relatively recent, in 1735, when Euler solved Konigsberg Bridge problem. Graphs are a general model for representing relationship between various points considered in a topic of study. It starts with a set of nodes and edges. They are discrete structures. Graphs are used to model road networks, social networks, competition between different species etc. They represent essential structure after discarding unnecessary aspects of a problem. That is why they can be used to represent all interconnected structures.

Different types of graphs like directed graph, weighted graph etc. are used for different applications but underlying structure is always described using nodes and edges. For detecting deadlock conditions in an operating system, graph theory suggests methods.

Problems in graph theory are not always easy. Finding special subgraphs in a large graph is actually a very difficult problem. Map coloring problems always posed a challenge to mathematicians. Finding the shortest path in a graph was always interesting not only because of the theoretical the theoretical importance but due to its several practical applications especially for large graphs like a highway stem or telecom network.

Keywords

Graphs, isomorphism, paths and circuits, Euler and Hamiltonian graphs, Trees, spanning trees

Discussion

2.2.1 Terminologies

Graphs are usually drawn by joining points. In graph theory, graph is a discrete structure with a simple description.

Graph

A graph G is a pair G=(V,E) where a nonempty set V consists of points called nodes or vertices and E consists of edges which are represented using an unordered pair of vertices. An edge is said to connect two vertices or incident on the vertices.

A graph with finite set of vertices and a finite set of edges is called a finite graph. Otherwise it is infinite graph. If there is an edge e between a pair of vertices (u,v), then e= (u,v).

Directed and undirected graphs

A directed graph (or digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices.

Adjacent vertices

If e=(u,v) is an edge, then u and v are said to be adjacent vertices.

When we depict a directed graph with a line drawing, we use an arrow pointing from u to v to indicate the direction of an edge that starts at u and ends at v. In a directed graph, if e=(u,v) is an edge, u is said to be adjacent to v and v is said to be adjacent from u.

Loop and Parallel edges

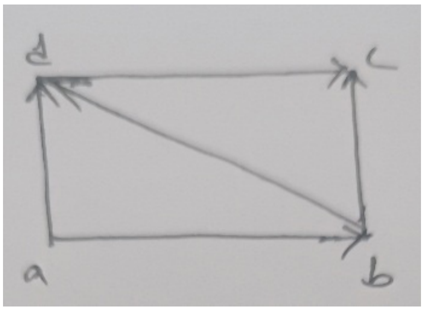
An edge that starts and ends at the same vertex is called a loop. If there are more than one edge between a pair of vertices, the edges are called parallel edges. If e= (u,v) is an edge, then u is called initial vertex and v is called terminal vertex.

Simple graph

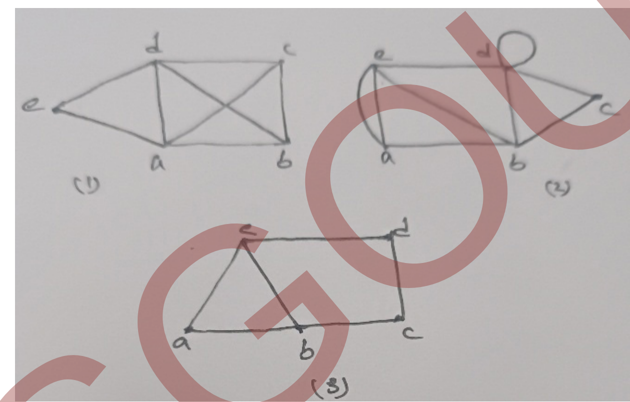
A graph that has no self loop or parallel edges is called a simple graph. i.e.,in a simple graph there is only one edge between every pair of vertices.

Degree of a vertex

In an undirected graph, degree of a vertex is the number of edges incident on it, counting a loop contributing a degree of two to that vertex.



This is a directed graph.



Graph 1 is an undirected graph where vertex a has degree 4, vertex b has degree 4 and vertex e has degree 2.

In graph , vertex a has degree 3 while vertex d has degree 5.

Graph 3 is a simple graph.

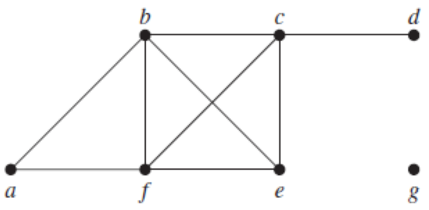
In-degree and Out-degree of a vertex

In a directed graph, the number of edges with v as terminal vertex is called in-degree of v denoted

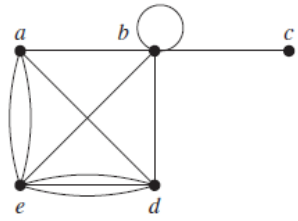
$\deg^{-}(v)$. The out-degree of vertex v is the number of edges with v as initial vertex denoted by $\deg^{+}(v)$.

Isolated and pendent vertices

A vertex of degree 0 is called isolated. A vertex having degree 1 is called a pendent vertex. A pendant vertex is adjacent to exactly one other vertex. For example in G, d is pendent and g is isolated. In H, there are parallel edges between a and e, between e and d.



G



H

The Handshaking Theorem

Suppose G=(V,E) is an undirected graph with m edges. Then sum of the degree of all vertices is twice the number of edges, i.e., $\sum_{v} d(v) = 2m$ where v is a vertex in G. Since each edge contributes a degree of 2 to the graph, the result is true.

Illustration 2.2.1

If a graph has 8 vertices each of degree 5, find the total number of edges in the graph.

Solution

Sum of degrees=40, so number edges=20

Theorem 2.2.1

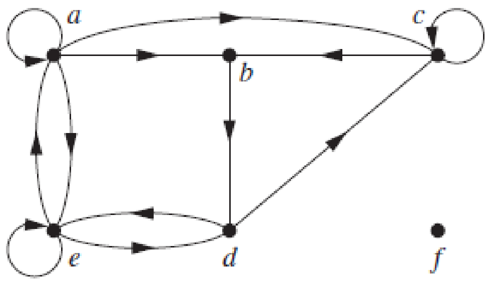
An undirected graph has an even number of vertices of odd degree.

Proof

Let G=(V,E) be the graph. Divide the set of vertices into two sets : V1: containing all vertices of even degree and V2: all vertices of odd degree. Then by the Handshaking Theorem, $\sum_{v} d(v) = 2m$. So $\sum_{V_1} d(v) + \sum_{V_2} d(v) = 2m$. First term on the left hand side is even and term on the right hand side is also even. So the reaming term must also be even. But in that sum, all terms are odd, so the number of terms, i.e., the number of odd vertices, must be even.

Illustration 2.2.2

Find the in-degree and out-degree of each vertex in the graph given:



G

Solution

Tn-degrees are

deg−*(a)* = 2, deg−*(b)* = 2, deg−*(c)* = 3, deg−*(d)* = 2,

deg−= 3 *(e)*, and deg−*(f)* = 0.

The out-degrees are deg+*(a)* = 4, deg+*(b)* = 1,

deg+*(c)* = 2,deg+*(d)* = 2, deg+*(e)* = 3, and deg+*(f)* = 0.

Theorem 2.2.2

Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v\in V} d^{-}(v) + \sum_{v\in V} d^{+}(v) = total \text{ number of edges.}$$

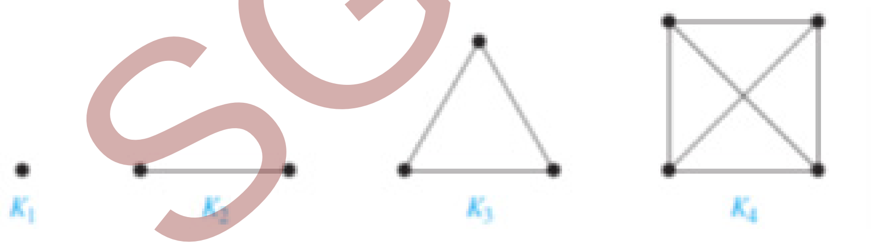
Proof

In a directed graph, each edge has an initial vetex and a terminal vertex, so total in-degree and total out-degree in graph are always equal which is the number of directed edges Hence the theorem.

Many properties of directed graphs are independent of whether the edges are directed or not. So it is convenient to discard the direction and consider the undirected graph. This undirected graph from a directed graph(digraph) is called underlying undirected graph. So the discussion is focussed on undirected graphs often.

Complete Graph

If in a simple graph, there is exactly one edge between every pair of distinct vertices it is called a complete graph on *n* vertices, denoted by *Kn*.

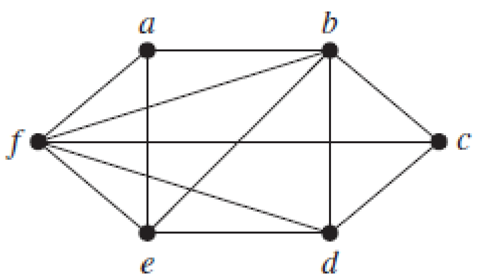


Bipatrite Graph

Suppose G=(V,E) is a simple graph. Suppose V can be divided into two disjoint sets V_1 and V_2 so that every edge in G has one vertex in V_1 and the other vertex in V_2. Then no edge connects vertices in V_1 or V_2.

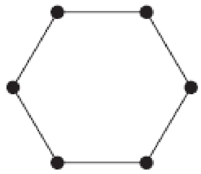
Illustration 2.2.3

Find if the following graph is bipartite:



Solution

No, the vertex set cannot be divided into disjoint subsets such that each edge has one vertex in one set and the other in the second set.



C_6

But the above graph is bipartite.

Sub graph

Given a graph G=(V,E), graph G_1=(V_1,E_1) is called a sub graph of G if $V_1 \subseteq V$ and every edge in E_1 is in E such that their initial and terminal vertices are in V_1.

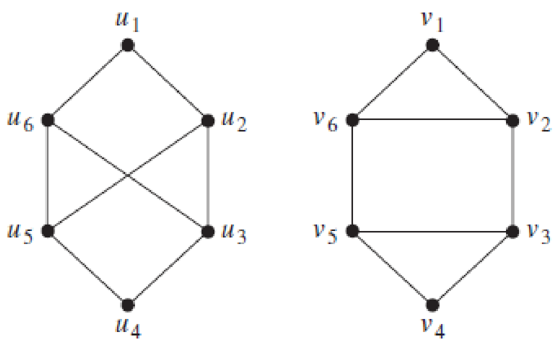
Isomorphic graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism .i.e., when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.

The number of vertices, the number of edges, and the number of vertices of each degree are all remain unchanged under isomorphism. If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic. But when they are all the same, it does not mean they are isomorphic, at present there is no universal method to determine if two graphs are isomorphic.

Illustration 2.2.4

Find if the following graphs are isomorphic:



G H

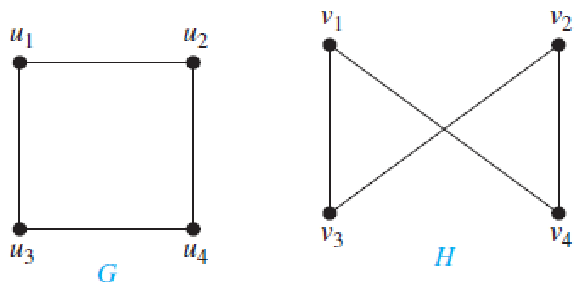
Solution

Both have 6 vertices and 8 edges. Number of vertices having degree 1 and 2 in both are the same. There is a one to one onto mapping that preserves incidence relationship. So the graphs are isomorphic.

Both G and H have five vertices and six edges. H has a vertex of degree one, e, whereas G has no vertices of degree one. So G and H are not isomorphic.

Illustration 2.2.5

Find if the following graphs are isomorphic:

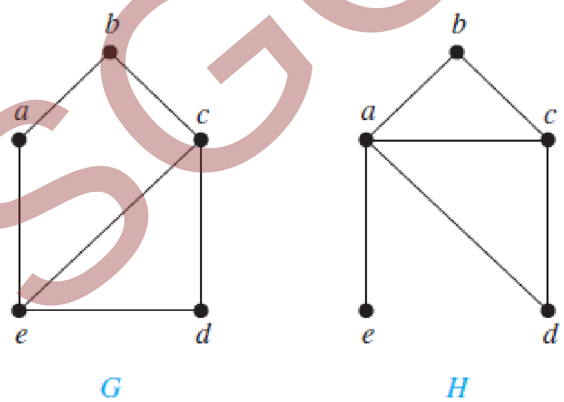


Solution

They have the same number of vertices and edges. Following is a correspondence between them:

u1-v1, u2-v2, u3-v3, u4-v4 and also foredges connecting them. Incidence relationship is also preserved with degree of vertices in the above correspondence.

But the following graphs are not isomorphic:



2.2.2 Connected Graphs

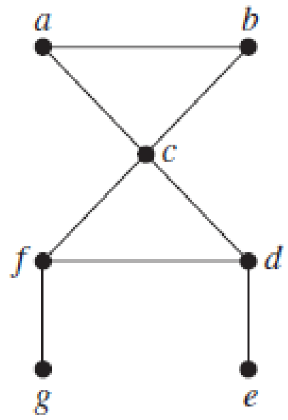
An important property of graphs is connectedness. There are paths and circuits in a graph that gives rise to many interesting problems.

Paths

Let v_0 and v_n be vertices in a graph. A path from v_0 to v_n of length n is an alternating sequence of n + 1 vertices and n edges beginning with vertex v_0 and ending with vertex v_n, $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n)$, in which edge e_i is incident on vertices v_{i-1} and v_i for i = 1, . . . , n. No vertex appears more than once. The number of edges in path is called length of the path.

Connected graph

Given any two vertices u and v in a graph if there is path from u to v, then the graph is said to be connected. If a graph is not connected, there will be two or more connected sub graphs which are called components.

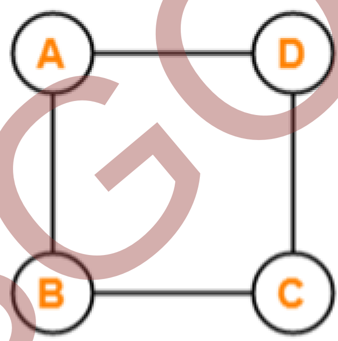


G_1

Graph G1 is connected.

Cycle (circuits)

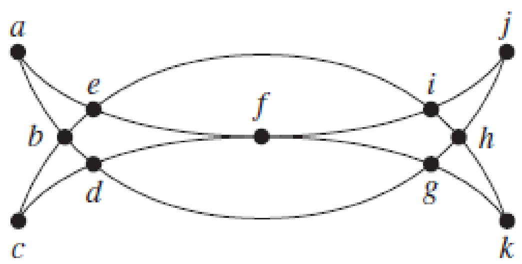
A cycle is a path that starts and ends with the same vertex.



Euler Graphs

A circuit that passes through every edge of a graph exactly once is called an Euler line and a graph with an Euler line is called an Euler graph. So Euler line contains all the edges of the graph. By this definition Euler graph is always connected.

It can be proved that if G is a connected graph and every vertex has even degree, then G is an Euler graph.



G

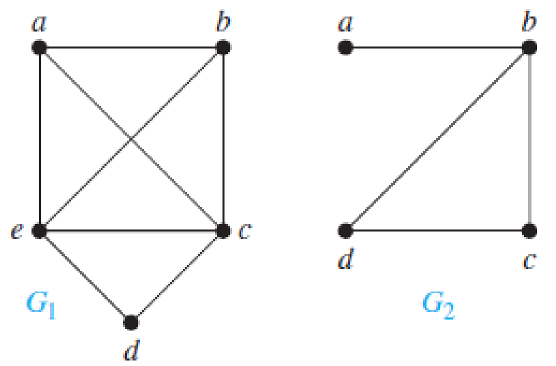
Graph G has an Euler line.

Hamiltonian circuit

If there is a closed path that traverses every vertex of the graph exactly once, it is called a Hamiltonian circuit. So it includes all the vertices of the graph.

Illustration 2.2.6

Find if the following has a Hamiltonian cycle:

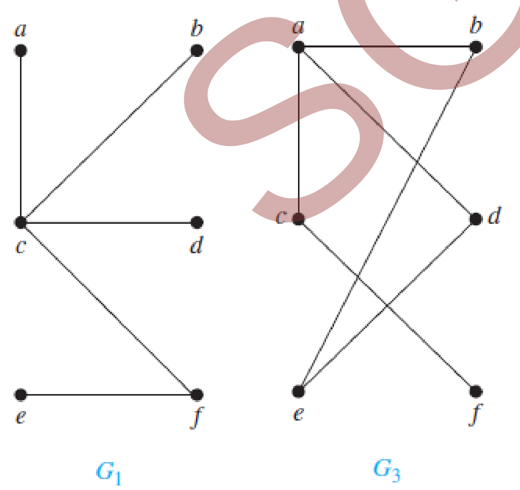


Solution

G_1 has but not for G_2 .

2.2.3 Trees

A tree is a connected graph without any circuits. By definition, tree must be a simple graph.



G_1 is a tree but G_3 is not.

Theorem 2.2.3

There is one and only one path between every pair of vertices in a tree.

Proof

Since a tree is connected there must be a path between every pair of vertices. But if there are is more than one path between any pair of vertices, that forms a circuit, which is not possible. Hence the theorem.

Theorem 2.2.4

If in a graph G there is exactly one path between every pair of vertices, G is a tree.

Proof

G is connected because there is one path between every pair of vertices. G has circuits when there is more than one path between any pair of vertices. So G is circuitless. Hence G is a tree.

Theorem 2.2.5

A tree with n vertices has n-1 edges.

Theorem 2.2.6

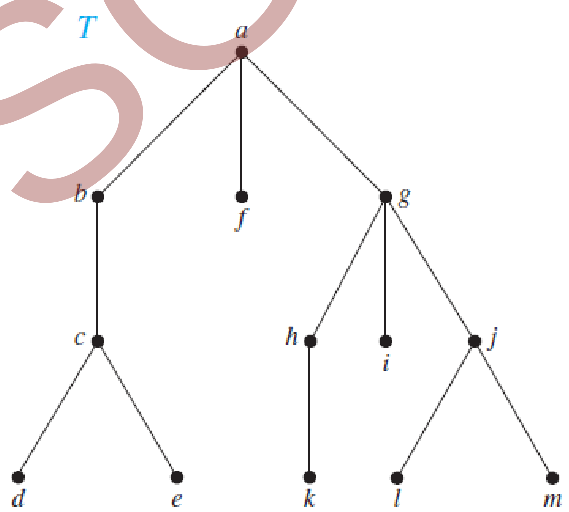
Any connected graph with n vertices and n-1 edges is a tree.

A tree is said to be minimally connected because deletion of any one edge from it will disconnect it.

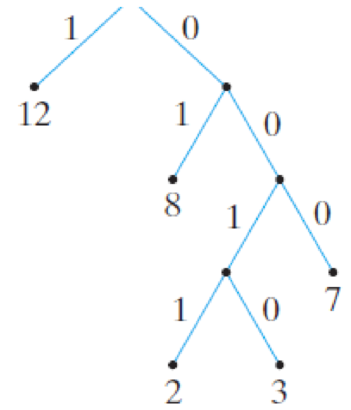
Rooted tree

A rooted tree is a tree in which one vertex is called root and all other edges are directed away from the root.

For example, the graph below is a rooted tree.

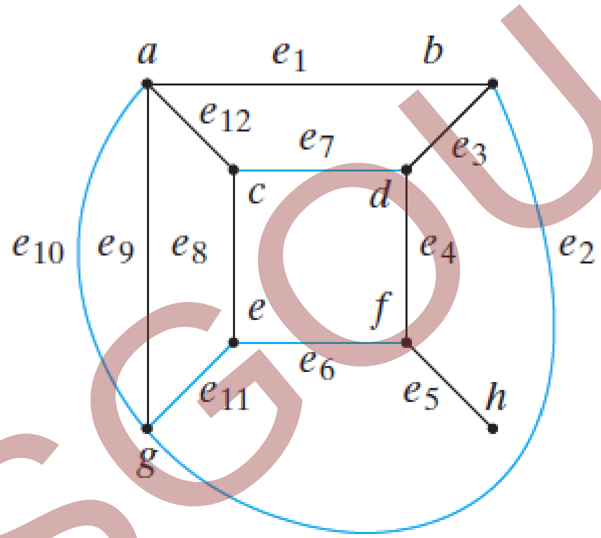


It can be proved that in any tree having at least two vertices, there are at least two pendant vertices. This is evident from the tree below:



Spanning tree

A subgraph T of a connected graph G is called a spanning tree if T has all the vertices of G. In general a connected graph can have a number of trees. Among all the trees of G, spanning tree is a maximal tree of G.



For the above graph, a spanning tree is given by the black edges.

Theorem 2.2.7

A graph G has a spanning tree if and only if G is connected.

Proof

Suppose G has a spanning tree. Then by definition G is connected.

Suppose G is connected. If it has no circuits, G itself is a spanning tree. If G has a circuit delete one edge from the circuit so that the graph remains connected. If it has no circuits now, it is a spanning tree. Otherwise continue till a spanning tree is obtained.

Finding spanning trees of a graph

One method for finding a spanning tree is Breadth First search method.

Suppose G is a connected graph with vertices ordered as $v_1, v_2, \ldots v_n$. Select the first vertex v_1 and label it the root. Now the tree contains only this vertex and no edges.

Now add all edges incident on v_1 together with all their vertices so that no circuit is formed in T .Repeat this to all the vertices added on the condition that no circuit is formed in T till all the vertices are over. Now T is a spanning tree.

Weighted Graph

A graph in which each edge has an associated number called its weight is called a weighted graph. Weight of a spanning tree is defined as the sum of all the edges (branches) of the spanning tree. A spanning tree with minimum weight in a weighted graph is called minimal spanning tree.

Kruskel's algorithm is one method for finding the minimal spanning tree.

Summarized Overview

Graph theory is branch of mathematics that is widely used in solving many practical problems. A rooted tree is used to represent folders in a computer system. Traveling salesman problem is related to the problem of finding a Hamiltonian cycle in a graph. Weighted graphs are used to model computer networks with weights representing communication costs or response time of the network. Euler paths and circuit is used in cases where one needs to cover exactly once every road in a transportation network, each connection in a utility network, each link in a communication network etc.

So formulating real life problems in graph theory provides a simple visualisation that may assist in deriving an efficient algorithm for the solution of the problem. Many problems posed as puzzles and riddles in their original form are not easily amenable to an answer but using graph theory their answers were obtained in a simple and clear manner.

Assignments

1. Express job assignment problem as a graph.
2. Find which of the complete graphs are bipartite examining graphs with vertices up to 6.
3. Write down traveling salesman problem and its graph theoretic version.
4. Check if it is possible to have graph with (a) 6 vertices each having degree 3 and (b) 6 vertices and 4 edges.
5. State a necessary and sufficient condition for a graph to have a spanning tree.
6. Explain Kruskal's algorithm and Dijkstra's algorithms.

Suggested Reading

1. Kenneth H. Rosen – *Discrete Mathematics and Its Applications*, 7th ed., McGraw-Hill, 2011.
2. C. L. Liu, D. P. Mohapatra – *Elements of Discrete Mathematics*, 3rd ed., McGraw-Hill, 2008.

Reference

1. Discrete and combinatorial Mathematics: An Applied Introduction –Ralph P Grimmaldy, Addison Wesly, 1994 .
2. DiscreteMathematics- Richard Johnsonbaugh,Eighth Edition, Pearson

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

3
UNIT

Numerical Methods

Learning Outcomes

After completion of this unit, the learner will be able to:

- compute solutions of linear and non linear equations
- understand and apply numerical differentiation
- understand and apply numerical integration
- understand various approximation techniques

Background

After a course in calculus and algebra, it is comfortable to see that most equations are solvable and solutions are in closed form. But when problems are modelled and attempts are made to solve them by the conventional methods, it becomes clear many problems are not amenable to standard methods. Then comes various numerical techniques to find solutions.

Numerical methods are required for solving algebraic equations whether linear or non linear, for computing derivatives and integrals and for solving differential equations. Several methods are available for each topic and so a careful choice must be made to select an appropriate technique depending on the nature of the problem and its applications. Here comes the importance of approximation methods and error criteria which help to clarify whether the selected technique is apt for that particular problem.

Keywords

Numerical solution of equations, numerical differentiation, numerical integration, approximation techniques.

Discussion

2.3.1 Numerical Solution of Equations

Numerical methods evolved after understanding the limitations of analytical methods. Though it is desirable to have exact solutions, often traditional methods fail when trying to draw inference about solutions of equations given a set of tabulated data pointsEven when the solution is obtained, they may not be amenable to further direct numerical computations.

With the new era of computing, the relevance of numerical methods increased many times. The given data may not be exact as it comes usually from some measurements and the methods for numerical solutions may introduce some errors. So the final result may be carefully interpreted considering the magnitude of error permitted.

> Numerical methods provide constructive methods for drawing valid inferences from a set of data points when traditional exact methods may not give satisfactory answers.

Accuracy of numbers

Two types of numbers are discussed in numerical methods. First one is exact numbers like 2,4,7/2, 4.3 etc and the other one is approximate numbers which cannot be expressed by a finite number of digits like 1/3, e, etc. These approximate numbers are written within a degree of accuracy required. The digits that are used to represent a number are called significant digits(figures). Each of 8765,2.135 and 0.7896 have four significant digits, 0.00897,0.000368 and 0.0000432 have three significant digits. The numbers 32000, 5600 have 2 significant figures.

When there are a number of digits in an approximate number, the number of digits to be kept are limited and process of chopping off other digits is called rounding off. Naturally this results in error called rounding off error. This is unavoidable in computations just as Π is represented by 3.1416.Rounding off error is reduced by retaining one more significant figure than required till the last step of computation .For example, 865250 can be rounded off to 865200 and 37.46235 to four significant digits as 37.46.

Roots of an equation

Finding the roots of an equation f(x)=0 is a basic problem. If the equation is simple like a quadratic equation, formulas are available to compute them. But if the degree of a polynomial is large or if the equation contains a transcendental function, finding the root becomes very difficult. So the roots must be found by numerical methods.

Regula falsi (method of false position) method

This is the oldest method for finding the root of a nonlinear equation f(x)=0.

Step 1: choose two points a and b such that f(a) and f(b) are opposite in sign.

Step 2: Choose $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$. If f(a) and f($x_1$) are opposite in sign, root lies between a and x_1, replace b by x_1. Otherwise replace a by x_1. Then continue to generate next approximation.

> When conventional methods fail to find a solution for a problem, its solution is obtained by numerical methods. False position method is an example.

Illustration 2.3.1

Find a root of the equation $f(x)=x^3-2x-5=0$

Solution

f(2)=-1, f(3)=16, so root lies between 2 and 3.

Choose a=2, b=3. $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$$x_1 = \frac{2(16) - 3(-1)}{16 - -1} = \frac{35}{17} = 2.058823529$$

Now f(x_1) <0, so root lies between x_1 and b=3.

Choose $x_2 = \frac{2.058823529(16) - 3(\ 0.390799917)}{16.390799917}$ =2.08126366.

f(x_2) <0, so root lies between x_2 and b=3.

So x_3= 2.08963921, continuing, x_4=2.092739575,

x_5 =2.09388371, 2.0943305452,2.094460846,…

Correct value is 2.0945

Illustration 2.3.2

Find a root of the equation correct to three decimal places if 2x= $\log_{10} x$ +7 has a root between 3 and 4.

Solution

Let f(x)=2x- $\log_{10} x$ -7

Choose a=3, b=4, then f(3)=-1.4771, f(4)=0.3979

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3(0.3979) - 4(-1.4771)}{0.3979 + 1.4771} = 3.7878$$

So the root lies between 3 and 3.7878.So choose a=3,b=3.7878

$$x_2 = \frac{3(-0.002787) - 3.78(-1.4771)}{-0.002787 + 1.4771} = 3.7893$$

$f(x_2)$=0.000041, so x=0.3789 is the root correct to three decimal places.

Newton-Raphson method

This method required fewer steps than the previous one but requires more computing time.

Consider f(x)=0. Let x_0 be an approximate root of the equation. Then the iterative method is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \ldots,$ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. When the derivative has a large value, this is suitable.

Numerical methods for solving a linear or non linear equation starts from an approximate root and proceed by iteration.

Illustration 2.3.3

Find a root of the equation $f(x)=x^3-2x-5=0$

Solution

$f(x)=x^3-2x-5=0$

$$f(x) = x^3 - 2x - 5,\ f'(x) = 3x^2 - 2$$

Choose x_0=2, then $f(x_0) = -1, f'(x_0) = 10$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$, so x_1=2-(-1/10)=2.1

$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.094568$$

Illustration 2.3.4

Find a real root of the equation $f(x)=xe^x-1=0$

Solution

$$f(x) = xe^x - 1,\ f'(x) = e^x(x+1)$$

Choose x_0=1, then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n e^{x_n} - 1}{e^{x_n}(x_n + 1)}$

$$x_1 = \frac{1}{2}\left(1+\frac{1}{e}\right) = 0.6839397, f(x_1) = 0.3553424, f'(x_1) = 3.337012$$

$$x_2 = 0.6839397 - \frac{0.3553424}{0.337012} = 0.5774545 f(x_1) = 0.3553424, f'(x_1) = 3.337012$$

Continuing, $x_3 = 0.5672297, x_3 = 0.5671433$.

2.3.2 Numerical differentiation

Suppose x and y are two dependent variables whose exact functional relation y=f(x) is unknown. Given a set of values (x_i, y_i), computing the derivative of y with respect to x is called numerical differentiation. For this, tables for computing quantities called forward and backward differences are required.

Difference tables

Suppose the function y=f(x) is tabulated for the equally spaced values $x=x_0+h, x_0+2h, \ldots, x_0+nh$ denoted by $x_1, x_2, \ldots x_n$ giving y values as $y_0, y_1, y_2, \ldots y_n$ respectively.

The differences $y_1 - y_{0,}\ y_2 - y_{1,} \ldots, y_n - y_{n-1}$ are called first forward differences denoted by

$\Delta y_0 = y_1 - y_{0,}\ \Delta y_1 = y_2 - y_{1,} \ldots, \Delta y_{n-1} = y_n - y_{n-1}$ where Δ is called forward difference operator.

> It is difficult to identify a direct function in a tabulated data. There derivatives and integrals can be determined by numerical methods.

Similarly, the second forward difference is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \Delta^2 y_1 = \Delta y_2 - \Delta y_1, = y_2 - y_{1,} \ldots, \Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

$$In \text{ general}, \Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

This is usually given in forward difference table:

x	y_0	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Similarly, backward differences are defined using a backward difference operator as:

First backward difference:

$$\nabla y_1 = y_1 - y_{0,}\ \nabla y_2 = y_2 - y_{1,} \ldots,\ \nabla y_n = y_n - y_{n-1}$$

Second and higher differences:

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \ldots,\ \nabla^2 y_{n+1} = \nabla y_{n+1} - \nabla y_n$$

In general, $\nabla^p y_{r+1} = \nabla^{p-1} y_{r+1} - \Delta^{p-1} y_r$

This is also represented by a table called backward difference table as above but no additional computation is required.

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$

Illustration 2.3.5

Consider the data points (1,3010),(2,3424),(3,3802),(4,4105),(5,4472),(6,4771),(7,5051),(8,5315). Prepare a forward difference table.

Solution

x	y	Δ	Δ^2	Δ^3	Δ^4
1	3010				
		414			
2	3424		−36		
		378		−39	
3	3802		−75		+178
		303		+139	
4	4105		+ 64		−271
		367		−132	
5	4472		− 68		+181
		299		+ 49	
6	4771		−19		− 46
		280		+3	
7	5051		−16		
		264			
8	5315				

Derivatives using forward differences

Consider the function y=f(x) tabulated for equally spaced x values $x=x_0+h, x_0+2h, \ldots, x_0+nh$ denoted by $x_1, x_2, \ldots x_n$ giving $y=y_0, y_1, y_2, \ldots y_n$.

$$\left(\frac{dy}{dx}\right)_{(x_0)} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

Then $$\left(\frac{d^2y}{dx^2}\right)_{(x_0)} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right]$$

Now using backward differences:

$$\left(\frac{dy}{dx}\right)_{(x_n)} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_0 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_n)} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right]$$

If the derivative is required for a value at the beginning of given x values, forward difference formula is used and if the derivative is required at the end of the given x values, backward difference formula is used.

Illustration 2.3.6

Given that

x: 1.0 1.1 1.2 1.3 1.4 1.5 1.6

y: 7.989 8.403 8.781 9.129 9.451 9.750 10.031

find first and second derivatives at x=1.1 and at x=1.6

Solution

x y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.989						
	0.414					
8.403		-0.036				
	0.378		0.006			
8.781		-0.030		-0.002		
	0.348		0.004		0.002	
9.129		-0.026		0.000		-0.003
	0.322		0.004		-0.001	
9.451		-0.023		-0.001		
	0.299		0.005			
9.750		-0018				
	0.281					
10.031						

$$h = 0.1,\ x_0 = 1.1,\ \Delta y_0 = 0.378,\ \Delta^2 y_0 = -0.03,\ \Delta^3 y_0 = 0.004...$$

$$\left(\frac{dy}{dx}\right)_{(x_0)} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right]$$

$$\left(\frac{dy}{dx}\right)_{(1.1)} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(0) + \frac{1}{5}(-0.001)\right]$$

$$= 3.941$$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_0)} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + ...\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1.1)} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(0) - \frac{5}{6}(-0.001)...\right]$$

$$=-3.3167$$

Now using the above table for backward difference operator:

$$h = 0.1,\ x_n = 1.6,\ \nabla y_n = 0.281,\ \nabla^2 y_n = -0.018,\ \Delta^3 y_0 = 0.005, \Delta^4 y_n = -0.001$$

$$\left(\frac{dy}{dx}\right)_{(x_n)} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\Delta^4 y_0 + ...\right]$$

$$\left(\frac{dy}{dx}\right)_{(1.6)} = \frac{1}{(0.1)}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001)\right]$$

$$= 2.732$$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_n)} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1.6)} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(-0.001) + \frac{5}{6}(-0.001)...\right]$$

$$= -1.475$$

Illustration 2.3.7

From the following table of values of x and y

x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.04529	4.9530	6.0496	7.3891	9.0250

find first and second derivatives at x=1.2 and x=2.2

Solution

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Here

$h = 0.2,\ x_0 = 1.2,\ y_0 = 3.3201$

$$\left(\frac{dy}{dx}\right)_{(x_0)} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_0)} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right]$$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205.$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318.$$

$h = 0.2,\ x_n = 2.2,\ \nabla y_n = 9.0250$

$$\left(\frac{dy}{dx}\right)_{(x_n)} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_0 + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(x_n)} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right]$$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014)\right]$$

$$= 9.0228.$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014)\right] = 8.992.$$

There are other formulas also for computing the derivatives.

When the given x values are not equally spaced, the above formulas can not be used.

2.3.3 Numerical integration

Given a set of tabulated values for two variables x

(which are equally spaced) and y which have an unknown functional relation y=f(x), the process of evaluating a definite integral is called numerical integration. Two formulas that are frequently used are given below:

Suppose values of x are x_0, $x_0+h, x_0+2h, \ldots, x_0+nh$ denoted by $x_0, x_1, x_2, \ldots x_n$ giving y values $y_0, y_1, y_2, \ldots y_n$ respectively.

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

Illustration 2.3.8

Find from the following table the area bounded by the curve and the x-axis from x=7.47 to x=7.52.

x:	7.47	7.48	7.49	7.50	7.51	7.52
y:	1.93	1.95	1.98	2.01	2.03	2.06

Solution

$$Area = \int_{7.47}^{7.52} f(x)dx,$$

here h=0.1, y_0=1.93, y_1=1.95, y_2=1.98, y_3=2.01, y_4=2.03, y_5=2.06

$$So,\ using,\ \int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$$\int_{7.47}^{7.52} f(x)dx = \frac{0.1}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$=0.0996$$

Simpson's 1/8 rule

Suppose values of x are x_0, $x_0+h, x_0+2h, \ldots, x_0+nh$ denoted by $x_0, x_1, x_2, \ldots x_n$ giving y values $y_0, y_1, y_2, \ldots y_n$ respectively.

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\ldots+y_{n-1})+2(y_2+y_4+y_6+\ldots+y_{n-2})\right]$$

If a definite integral is given for evaluation, interval for integration is divided into an even number of sub-intervals to apply the formula.

Illustration 2.3.9

Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by Simpson's one third rule.

Solution

Divide the interval of integration into even number of equal sub-intervals.

Let y be the integrand.

x:	0	1	2	3	4	5	6
y:	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$h=1$

$$\int_0^6 f(x)dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$=1.3662$$

Illustration 2.3.10

A solid of revolution is formed by rotating about the x-axis the area between the x-axis, the lines x=0, x=1 and the curve through the points with the following coordinates. Estimate the volume of the solid formed correct to three decimal places.(volume of revolution is $V=\pi\int_0^1 y^2 dx$)

x	y
0.00	1.0000
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415

Solution

x	y^2
0.00	1.0000
0.25	0.9793
0.50	0.9195
0.75	0.8261
1.00	0.7081

Here h=0.25

So volume = $V = \pi \int_0^1 y^2 dx$

$$V = \int_0^1 f(x)dx = \frac{\pi(0.25)}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$= \frac{\pi(0.25)}{3}\left[(1 + 0.7081) + 4(0.9793 + 0.8261) + 2(0.9195)\right]$$

=2.8192

2.3.4 Iterative methods

An iterative method is one in which successive approximations to a solution are generated in sequence, with each approximation getting closer to the true solution.

> Most of the numerical methods involve iterations. These methods help to improve an initial approximate solution in successive steps.

One important iterative method is for solving a system of linear equations namely Gauss-Seidal iteration method. This is explained using an example as below.

Consider $a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

Write it as $x=1/\ a_1\ [d_1 - b_1y - c_1z]$

$y=1/\ b_2\ [d_2 - a_2x - c_2z]$

$z=1/\ c_3\ [d_3 - a_3x - b_3y]$

Start with an initial approximation x_0, y_0, z_0.

Substitute y_0, z_0 in first equation to get x_1

Substitute x_1, z_0 in second equation to get y_1

Substitute x_1, y_1 in third equation to get z_1

This iteration continues till desired accuracy is obtained.

Illustration 2.3.11

Solve by iteration method:

$20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$

Solution

$$x=\frac{1}{20}[17-y+2z]$$

$$y=\frac{1}{20}[-18-3x+z]$$

$$z=\frac{1}{20}[25-2x+3y]$$

Put $x_0=y_0=z_0=0$ as the initial solution.

Now substitute $y_0=z_0=0$ in first equation, $x_1=0.8500$,

Substitute x_1, z_0 in second equation to get $y_1=-1.0275$

Substitute x_1, y_1 in third equation to get $z_1=1.0109$

Now start from first equation: substitute y_1, z_1, $x_2=1.0025$

Substitute x_2, z_1 in second equation to get $y_2=-0.9998$

Substitute x_2, y_2 in third equation to get $z_2=0.9998$

For third iteration,

substitute y_2, z_2, in first equation, $x_3=1.0000$

Substitute x_3, z_2 in second equation to get $y_3=-1.0000$

Substitute x_3, y_3 in third equation to get $z_3=1.0000$

Thus solution is $x=1$, $y=-1$, $z=1$

Other iterative methods are also available not only in solving linear equations but for solving differential equations etc. also.

Summarized Overview

Numerical methods for solving problems were available for a long period but the difficulty is in carrying out the lengthy steps of calculations. With the arrival of computers, it became very easy. Several methods exist for solving a particular problem numerically and each of these method has its own limitations and advantages. Iterative methods are the most common method and they approximate solutions whether in differentiation, in integration or in differential equations. Discussions of numerical methods in differentiation and integration provide a different view of practical problems. Most often data points are available but no function defining will be obtained. So it is a challenge to find derivative and integral from the set of tabulated values.Numerical methods do this.

Assignments

1. Apply regula falsi method to solve: $4e^x \sin x - 1 = 0$ given root being between 0 and 0.5.
2. By Newton-Raphson method solve $x \sin x + \cos x = 0$
3. Find the first and second derivative for the function tabulated below:

x:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y:	1.000	1.0247	1.0488	1.0723	1.0954	1.1180	1.1401

4. Evaluate $\int_0^4 e^x\,dx$ by Trapezoidal and Simpson's rule
5. Solve by iterative method:

   $54x+y+z=110$, $2x+15y+6z=72$, $-x+6y+27z=85$

Suggested Reading

1. ***Introductory Methods of Numerical Analysis***, S S Sastri, 5th ed., PHI Learning, 2012
2. C. L. Liu, D. P. Mohapatra – *Elements of Discrete Mathematics*, 3rd ed., McGraw-Hill, 2008.

Reference

1. *Numerical Analysis*, 10th ed., Cengage Learning, 2015, Richard L. Burden, J. Douglas Faires .
2. Discrete Mathematics- Richard Johnsonbaugh, Eighth Edition, Pearson

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

4 UNIT

Matrix Decomposition Methods

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand various matrix factorisations
- understand significance of common methods
- compare different decompositions
- apply factorisations in problems

Background

Matrix decomposition is the process of factorising a given matrix into a product of simpler matrices which have some special structure. In many applications, data set is represented by matrices and their orders are very large. This causes many problems in using them for various computations. So it is desirable to find factorisations of matrices. There are different methods by which a matrix can be factorised. Basically the type of factorisation depends on the application for which it is required. Some of the methods are full rank factorization (for any matrix);diagonal or similar canonical factorization (for diagonalizable matrices); orthogonally similar canonical factorization (for symmetric matrices);*LU* factorization and *LDU* factorization (for nonnegative definite matrices and some others, including non-square matrices);*QR* factorization (for any matrix);singular value decomposition, SVD, (for any matrix); square root factorization (for nonnegative definite matrices); and Cholesky factorization (for nonnegative definite matrices). These decompositions simplify computations and reduces dimension of the given data without significant loss in the information contained.

Keywords

Matrix factorisation, singular value decomposition, QR method, LU method, dimensionality reduction

Discussion

2.4.1 Matrix Decomposition

Writing a matrix A as the product of two or more matrices B,C,…where B,C,… have some specified desirable properties, is called matrix decomposition or factorisation. For solving a system of linear equations, this factorisation is useful. Other than that, it is useful for purposes like eigen analysis.This factorization is performed using a sequence of matrices and some of the matrices derived such as inverse of a matrix.

LU, QR and SVD are the factorization methods discussed here.

Matrix multiplication which combines two or more matrices into a single matrix, can be viewed as a synthesis of data and matrix decomposition, where a matrix is written as product, as analysis

2.4.2 LU Factorisation

Remember, a square matrix in which $a_{ij}=0$ for $i>j$, i.e., all elements below the diagonal are zeros, is called an upper triangular matrix .

A square matrix in which $a_{ij}=0$ for $i<j$, i.e., all elements above the diagonal are zeros, is called a lower triangular matrix .

LU factorization is motivated by solving a sequence of system of linear equations $Ax=b_1, Ax=b_2, \ldots, Ax=b_p$. When inverse of A exists, one could solve them by finding this and $x=A^{-1}b_1$, $x=A^{-1}b_2$, etc. LU factorisation provide more efficient method.

LU factorisation is usually applied for solving systems of linear equations. Whenever coefficient matrix is non singular, this factorisation is unique.

Suppose A is an m x n matrix that can be transformed to row reduced echelon form by a sequence of Gaussian elimination to generate zeros below the diagonal in a given column. Then A can be expressed as A=LU where L is an m x m lower triangular matrix with 1s on the diagonal and U is an m x n echelon form of A. Such a factorisation is called LU factorisation of A. The matrix L is called unit lower triangular matrix, it is invertible. For example,

$$A=\begin{bmatrix}1&0&0&0\\ *&1&0&0\\ *&*&1&0\\ *&*&*&1\end{bmatrix}\begin{bmatrix}\bullet&*&*&*&*\\ 0&\bullet&*&*&*\\ 0&0&0&\bullet&*\\ 0&0&0&0&0\end{bmatrix}\text{ is in LU form.}$$

To find LU factorisation, following steps are applied:

1.Reduce A to an echelon form U by a sequence of row transformations, if possible.

2. Place entries in L such that the same sequence of row transformations reduce L to I.

Illustration 2.4.1

Find an LU factorisation:

$$A=\begin{bmatrix}2&4&-1&5&-2\\ -4&-5&3&-8&1\\ 2&-5&-4&1&8\\ -6&0&7&-3&1\end{bmatrix}$$

Solution

A has 4 rows, so L should be of order 4 x 4.

The first column of L is obtained from the first column A after dividing each element by the pivot 2.

$$L=\begin{bmatrix}1&0&0&0\\ -2&1&0&0\\ 1&-&1&0\\ -3&-&-&1\end{bmatrix}$$

Comparing first columns of A and L, it can be seen that row operations that create zeros in the first column of A will also create zeros in the first column of L. This correspondence of row transformations is retained for rest of L when A is transformed to echelon form step by step as below:

$$A=\begin{bmatrix}\boxed{2}&4&-1&5&-2\\ \boxed{-4}&-5&3&-8&1\\ \boxed{2}&-5&-4&1&8\\ \boxed{-6}&0&7&-3&1\end{bmatrix}, Next\ R_2\to R_2+2R_1, R_3\to R_3-2R_1, R_4\to R_4+3R_1$$

$$A_1=\begin{bmatrix}2&4&-1&5&-2\\ 0&\boxed{3}&1&2&-3\\ 0&\boxed{-9}&-3&-4&10\\ 0&\boxed{12}&4&12&-5\end{bmatrix}, next, R_3\to R_3+3R_2, R_4\to R_4-4R_2$$

$$A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & \boxed{4} & 7 \end{bmatrix}, next, R_4 \to R_4 - 2R_3$$

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{5} \end{bmatrix} = U$$

In each step, elements marked decide the row reduction of A to U.

Divide them by the pivot and place that order in L.

$$\begin{bmatrix} 2 & & & \\ -4 & 3 & & \\ 2 & -9 & 2 & \\ -6 & 12 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

$$\text{Now } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Thus, LU=A

For any given matrix, there are actually many different LU decompositions. If A has non zero determinant (non singular) then L and U are unique.

Illustration 2.4.2

Find an LU factorisation:

$$A = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} \boxed{6} & 18 & 3 \\ \boxed{2} & 12 & 1 \\ \boxed{4} & 15 & 3 \end{bmatrix}, Next\ R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$$

$$A_1 = \begin{bmatrix} 6 & 18 & 3 \\ 0 & \boxed{6} & 0 \\ 0 & \boxed{3} & 1 \end{bmatrix}, Next\ R_3 \to R_3 - \frac{1}{2}R_2$$

$$\begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{bmatrix}$$

Then A=LU

2.4.3 QR Factorisation

A matrix Q is called if $QQ^T=I$. Then columns of Q form orthogonal vectors with norm 1, usually called orthonormal basis. The columns are linearly independent vectors. For example,

$$Q = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ is orthogonal.}$$

> QR decomposition is helpful in determining the rank of A and is said to be "rank-revealing"of data.It is useful for computations for overdetermined systems and for nonsquare matrices.

Suppose A is an m x n matrix whose columns are linearly independent vectors. The A can be factorised as A=Q R where Q is an m x n matrix whose columns are linearly independent and orthogonal vectors and R is an n x n upper triangular invertible with diagonal elements all positive. There are different method for doing this. One such method is Gram-Schmidt process.

Illustration 2.4.3

Find QR factorisation:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution

Consider each column as a vector:

Let $C_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\|C_1\| = \sqrt{1^2 + 1^2} = \sqrt{2},\ \|C_2\| = \sqrt{2},\ \|C_3\| = \sqrt{2}$$

Let $u_1 = C_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$Let\ e_1 = \frac{1}{\|u_1\|} u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\text{Let } u_2 = C_2 - \langle C_2, e_1 \rangle e_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix},\ \|u_2\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + 1^2} = \sqrt{\frac{3}{2}}$$

$$e_2 = \frac{1}{\|u_2\|} u_2 = \frac{1}{\sqrt{\frac{3}{2}}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$\langle C_3, e_1 \rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, e_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \langle C_3, e_1 \rangle = \frac{1}{\sqrt{2}}, \langle C_3, e_1 \rangle e_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Let $u_3 = C_3 - \langle C_3, e_1\rangle e_1 - \langle C_3, e_2\rangle e_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \frac{1}{\sqrt{6}}\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \\ \frac{2}{6} \end{bmatrix} = \begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\|u_3\| = \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{3}}$$

$$e_3 = \frac{1}{\|u_3\|} u_3 = \frac{1}{\sqrt{\frac{4}{3}}}\begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{\sqrt{3}}{2}\begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Now $Q = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$R = \begin{bmatrix} \langle C_1, e_1\rangle & \langle C_2, e_1\rangle & \langle C_3, e_1\rangle \\ 0 & \langle C_2, e_2\rangle & \langle C_3, e_2\rangle \\ 0 & 0 & \langle C_3, e_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

Illustration 2.4.4

Find QR factorisation:

$$A = \begin{bmatrix} 1 & \sqrt{5} \\ 2 & 0 \\ 0 & -\sqrt{5} \end{bmatrix}$$

Solution

$$e_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} \frac{4}{3\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \\ \frac{-5}{3\sqrt{5}} \end{bmatrix}$$

As above, $Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ 0 & \frac{-5}{3\sqrt{5}} \end{bmatrix}$, $R = = \begin{bmatrix} \sqrt{5} & 1 \\ 0 & 3 \end{bmatrix}$

The A=QR.

QR method is commonly used in regression analysis.

2.4.4 Singular Value Decomposition

Diagonalisation of a matrix A was defined as writing $A=PDP^{-1}$. Here D is a diagonal matrix. But because of the conditions for diagonalisable matrices, not all matrices can be diagonalised. But a factorisation of the form $A= QDP^{-1}$ is possible for any m x n matrix. Singular Value Decomposition is one such decomposition and it is one of the most widely applied factorisation. Basic idea is from the definition of eigen values and eigen vectors.

> SVD is the most widely used matrix factorization method. Eigen values are defined only for square matrices and singular values are eigen values of a matrix from a non square matrix which are always non negative.

Singular value

A square matrix A is said to be symmetric if $A^T=A$.

Suppose A is an m x n matrix. Then A^TA is symmetric and it can be diagonalised. Let $\{v_1,v_2,\ldots,v_n\}$ be the set of eigen vectors of A^TA which are linearly independent orthogonal vectors in R^n each having unit norm (called an orthonormal basis).Let λ_1, $\lambda_2,\ldots\lambda_n$ be the corresponding eigen values. These eigen values can be proved to be non negative using the definition of norm of A, so they can be arranged in ascending order and suppose this is $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$.

The singular values of A are the square root of the eigen values of A^TA and they are denoted by $\sigma_1, \sigma_2, ... \sigma_n$, *i.e.*, $\sigma_i = \sqrt{\lambda_i}$. For example,

if $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$, *then* $A^T A$ has eigen values $\lambda_1 = 360, \lambda_2 = 90, \lambda_3 = 0$.

so that $\sigma_1 = 6\sqrt{10},\ \sigma_2 = 3\sqrt{10},\ \sigma_3 = 0.$

Singular Value Decomposition (SVD)

Suppose A is an m x n matrix with rank r. Then there exists an m x n matrix $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$ where D is an r x r diagonal matrix. Diagonal elements in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq ... \sigma_r > 0$. Also, there exists an m x m orthogonal matrix U and n x n orthogonal matrix V such that $A = U \ \Sigma \ V^T$. Any such factorisation where

1. U and V are orthogonal
2. $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$ with positive diagonal values in D is called a singular value decomposition (SVD) of A.

The matrices U and V are not uniquely determined by A but diagonal entries of D are singular values of A.

The Columns of U are called left singular vectors of A and columns of V are called right singular vectors of A.

Illustration 2.4.5

Find SVD:

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

Solution

There are 3 steps:

Step 1:

Find the eigen values of A^TA and the corresponding set of normalised eigen vectors which are orthogonal (orthonormalized). This is called Orthogonal Diagonalisation of A^TA.

$$A=\begin{bmatrix}4 & 11 & 14\\ 8 & 7 & -2\end{bmatrix},\quad A^T=\begin{bmatrix}4 & 8\\ 11 & 7\\ 14 & -2\end{bmatrix},\quad A^TA=\begin{bmatrix}80 & 100 & 40\\ 100 & 170 & 140\\ 40 & 140 & 200\end{bmatrix}$$

Eigen values of A^TA are 36,90 and 0.

So singular values are $\sigma_1=6\sqrt{10},\ \sigma_2=3\sqrt{10},\ \sigma_3=0$. The corresponding eigen vectors with unit norm are the columns of V. $=\begin{bmatrix}1/3 & -2/3 & 2/3\\ 2/3 & -1/3 & 2/3\\ 2/3 & 2/3 & 1/3\end{bmatrix}$

Step 2:

Finding V and Σ

Non zero singular values are the diagonal elements of D = $\begin{bmatrix}6\sqrt{10} & 0\\ 0 & 3\sqrt{10}\end{bmatrix}$ and

$$\Sigma=\begin{bmatrix}D & O\\ O & O\end{bmatrix}=\begin{bmatrix}6\sqrt{10} & 0 & 0\\ 0 & 3\sqrt{10} & 0\end{bmatrix}$$

Step 3:

constructing U

If A has rank r, select normalised vectors from $Av_1, Av_2, \ldots Av_r$. They are the first r columns of U. Here rank of A=2.

$$u_1=\frac{1}{\sigma_1}Av_1=\frac{1}{6\sqrt{10}}\begin{bmatrix}18\\ 6\end{bmatrix}=\begin{bmatrix}\frac{3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{bmatrix},\quad u_2=\frac{1}{\sigma_2}Av_2=\frac{1}{3\sqrt{10}}\begin{bmatrix}3\\ -9\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{10}}\\ \frac{-3}{\sqrt{10}}\end{bmatrix}$$

Since these two columns form a basis for R^2, no additional vectors are required in U.

Thus, $A=U\ \Sigma\ V^T$ where

$$U=\begin{bmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}}\end{bmatrix},\quad \Sigma=\begin{bmatrix}6\sqrt{10} & 0 & 0\\ 0 & 3\sqrt{10} & 0\end{bmatrix},$$

$$V^T=\begin{bmatrix}1/3 & 2/3 & 2/3\\ -2/3 & -1/3 & 2/3\\ 2/3 & -2/3 & 1/3\end{bmatrix}$$

2.4.5 Dimensionality Reduction

Dimension of a data set describes the number of features or attributes in the data set. As we are considering many high resolution images containing millions of pixels for several applications, we frequently obtain large data sets with high dimension. Some of these features are irrelevant and some may not have much values. For practical purposes, it is a better to reduce this dimension without suffering much loss of information. This is the purpose of dimensionality reduction. When data is represented by matrices, orders of the matrices will be large and reducing their order so that these smaller order matrices are in some well-defined way close to the original matrix.

A good example is given by UV decomposition . Given a large matrix M, it is factorized into a product of two matrices U and V. Number of columns of U is smaller and number of rows of V is smaller compared to M. But together they represent the information in M.

Principal Component Analysis (PCA)

Suppose a data set is given containing n-tuples which lie in a high dimensional space. The goal is to find the direction along which these data points line up the best. The set of n-tuples is represented by a matrix M and then apply SVD on this matrix. First the eigen vectors for M^TM are found. The eigen vector corresponding to the largest eigen value is called principal eigen vector .Then it is seen that points are lying along the principal eigen vector with small deviations. This is the direction in which most points are spread. This means the variance of the data is the maximum along this direction. Then the second largest eigen value and the corresponding eigen vector are determined. The variance of distance from the first axis is greatest for this direction. This is continued. In terms of projections, this implies the high dimensional data points are projected onto the axis of principal eigen vector and second eigen vector and so on. Thus the original data points are represented in lower dimensions without much loss.

> Principal component analysis (PCA) is one of the central applications of the SVD, providing a statistical interpretation of the data-driven, hierarchical coordinate system used to represent high-dimensional correlated data.

Mean and Covariance

Consider a px n matrix of observations $[X_1 \; X_2 \; X_3 \ldots X_n]$. The sample mean

$$M = \frac{1}{N}[X_1 + X_2 + X_3 + \ldots + X_n].$$

The mean deviation form is given by $B = \begin{bmatrix} \dot{X}_1 & \dot{X}_2 & \dot{X}_3 & \ldots & \dot{X}_n \end{bmatrix}$ where $\dot{X}_i = X_i - M$

The covariance matrix of the is the p x p matrix S defined by

$S = \dfrac{1}{N-1} BB^T$. The diagonal entry S_{ij} in S is called variance of the component x_j of any vector X in the data set.

The total variance of the data is the sum of the diagonal elements which is trace of S.

Any entry s_{ij} in matrix S is called covariance of x_i and x_j.

If covariance s_{ij} is zero, then they are said to be uncorrelated.

When the covariance matrix is diagonal, then most or all of the x variables are uncorrelated, making the analysis of the multivariate data $X_1, X_2, \ldots Xn$ simple.

So the target is to make this possible.

Assume that the multi variate data is already in mean deviation form $S = \frac{1}{N-1} BB^T$

The goal of PCA is to find an orthogonal p x p matrix $P = \begin{bmatrix} U_1 & U_2 & U_3 \ldots U_p \end{bmatrix}$ that determines a change of variables X=PY given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_p \end{bmatrix} = \begin{bmatrix} U_1 & U_2 & U_3 \ldots U_p \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \ldots \\ y_p \end{bmatrix}$$ such that the new variables in y are uncorrelated and are arranged in decreasing variance.

Since P is orthogonal, it is clear that the covariance matrix of Y points is P^TSP. Thus such a P makes this product diagonal.

Let D be a diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda p$ of S on the diagonal, arranged in the order of decreasing magnitude , and let P be an orthogonal matrix whose columns are the corresponding unit eigenvectors $u_1, u_2, \ldots, up$.

Then $S= PDP^T$ and $P^TSP= D$.

Principal components of the data are the unit eigen vectors $u_1, u_2, \ldots, up$. The first principal component is the eigenvector corresponding to the largest eigenvalue of S, the second principal component is the eigenvector corresponding to the second largest eigenvalue, and so on.

Suppose $c_1, c_2, \ldots, cp$ be the elements of the first principal component u_1. This determines the required new variable y_1 as $y_1 = u_1^T X = c_1x_1 + c_2x_2 + \ldots + c_px_p$.

Thus y is the linear combination of original variables x with elements of the unit eigen vector u as weights.

Similarly,u_2 determines y_2 and so on.

Reducing the dimension

Principal Component Analysis is significant when the actual variation in multivariate dat depends only on p variables even the data contains n variables. An orthogonal change of variables X=PY as in PCA does not change the total variance computed from the data.

Thus, the total variance of x values=total variance of y values-trace of D=sum of eigen values.

So, the variance of $y_j = \lambda_j$ /trace of S=fraction of the total variance captured by y_j. Based on this quantity, those which have lesser values can be neglected, thus making a reduction in dimension.

Illustration 2.4.6

Explain PCA using $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}$ where four rows – one for each point – and two columns, corresponding to the x-axis and y-axis.

Solution

Compute $M^TM = = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix}$

The eigen values are $\lambda = 58$ and $\lambda = 2$.

Eigen vectors are obtained by solving, $\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 58 \begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 30 & 28 \\ 28 & 30 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

Then the unit eigen vectors are $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $u_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. They are the principal components.

After having the principal components, to compute the percentage of variance (information) accounted for by each component, we divide the eigenvalue of each component by the sum of eigenvalues. Then we find the principal eigen vector contains 97% and the second one contains 3% of the variance of the data. So we may discard second component without much information loss. (In this way the dimensionality can be reduced by keeping only p of the n principal components)

The matrix of eigen vectors is $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Now using X=PY or $y_1 = u_1^T X = c_1x_1 + c_2x_2 + ... + c_px_p$ the uncorrelated y variables can be determined.

Illustration 2.4.7

Find the principal components and determine the possible reduction in dimension if the covariance matrix S for a data is given as $S = \begin{bmatrix} 2382.78 & 2611.84 & 2136.20 \\ 2611.84 & 3106.47 & 2553.90 \\ 2136.20 & 2553.90 & 2650.71 \end{bmatrix}$

Solution

The eigen values are λ_1 =7614.23, λ_2=427.63, λ_3=98.1

and the associated unit eigen vectors or principal components are

$$u_1 = \begin{bmatrix} 0.5417 \\ 0.6295 \\ 0.5570 \end{bmatrix},\ u_2 = \begin{bmatrix} -0.4894 \\ -0.3026 \\ 0.8179 \end{bmatrix},\ u_3 = \begin{bmatrix} 0.6834 \\ -0.7157 \\ 0.1441 \end{bmatrix} \text{ respectively.}$$

Then y_1= 0.5417 x_1 +0.6295 x_2+0.5570 x_3

y_2= -0.4894 x_1 -0.3026 x_2+0.8179 x_3

y_3= 0.6834 x_1 -0.7157 x_2+0.1441 x_3

$$D = \begin{bmatrix} 7614.23 & 0 & 0 \\ 0 & 427.63 & 0 \\ 0 & 0 & 98.1 \end{bmatrix}$$

The variances of the variables y_1, y_2, and y_3 appear on the diagonal of D, and obviously the first variance in D is much larger than the other two. As we shall see, this fact will permit us to view the data as essentially one-dimensional rather than three-dimensional

The total variance of the data is

Trace of D= 7614.23 + 427.63 + 98.10 = 8139.96 = trace of S

The percentages of the total variance explained by the principal components are

First component : $\frac{7614.23}{8139.96} = 93{:}5\%$

Second component : $\frac{427.63}{8139.96} = 5.3\%$

Third component : $\frac{98.10}{8139.96} = 1.2\%$

This fact will permit us to view the data as essentially one-dimensional rather than three-dimensional.

Summarized Overview

Matrix factorisation is the process of writing a matrix into a product of matrices. When dimension of the matrix is very large, computations become cumbersome. The factorisations help to get a view of the structure of the given matrix. This is an area that has wide applications in image compression and signal processing.Dimensionality reduction and computational efficiency are their advantages.

LU decomposition is used in solving a system of linear equations. QR decomposition involves orthogonal matrix. Singular Value Decomposition is for any matrix. Principal Component Analysis is one of the most important dimension reduction techniques based on SVD. Its theory and application enables to understand the fundamental concepts discussed in all these topics. This study will help to understand applications of matrices.

An interesting problem where SVD and PCA were applied initially in a very successful way is to detect relation between faces. One of the most striking demonstrations of SVD/PCA is the so-called eigenfaces example. In this problem, PCA (i.e., SVD on mean-subtracted data) is applied to a large library of facial images to extract the most dominant correlations between images. The result of this decomposition is a set of eigen faces that define a new coordinate system. Images may be represented in these coordinates by taking the dot product with each of the principal components.

Assignments

1. Write down the algorithm for finding SVD and use it to find SVD for a 4 x 4 matrix.
2. If $S = \begin{bmatrix} 164.12 & 32.73 & 81.04 \\ 32.73 & 539.44 & 249.13 \\ 81.04 & 249.13 & 189.11 \end{bmatrix}$ is the covariance matrix for a data, find the principal components and compute the percentage of total variance in these components.
3. Find the SVD for $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$

4. Find QR factorisation for $A=\begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$

5. Find LU factorisation: $\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$

Suggested Reading

1. *Introduction to Linear Algebra*, Gilbert Strang .5th ed., Wellesley-Cambridge Press, 2016

Reference

1. *Linear Algebra and Its Applications*, David C. Lay, Steven R. Lay, Judi J. McDonald 5th ed., Pearson, 2015.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

BLOCK 3

Probability and Statistics

1
UNIT

Probability Basics

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand fundamental concepts of Probability
- apply Conditional Probability and Independence
- use Bayes' Theorem to update probabilities based on new evidence
- enhance logical reasoning and decision-making through the application of probability principles

Background

Probability is the branch of mathematics that deals with measuring the likelihood of uncertain events. At its core is the sample space, which includes all possible outcomes of a random experiment like tossing a coin (sample space: heads or tails) or rolling a die (sample space: numbers 1 to 6). An event is any subset of this sample space, such as getting an even number when rolling a die. In everyday life, probability is everywhere, for example, weather forecasts use probabilities to predict rain. Conditional probability helps us to understand how the likelihood of an event changes when we know another event has occurred. For example, the probability that a person has a disease, given that they have evaluated positive. This leads to the concept of independence, where two events do not influence each other, like tossing two coins, the outcome of one does not affect the other. When events are not independent, Bayes' Theorem becomes a powerful tool to update probabilities based on new information. For example, in a medical diagnosis if a patient evaluates positive for a disease, Bayes' Theorem helps to calculate the actual probability they have the disease by considering the test's accuracy and the general prevalence of the disease. Together, these basic probability concepts form a foundation for reasoning under uncertainty in fields like healthcare, finance, machine learning, and everyday decision-making.

Consider the simple case of tossing a coin. Although we do not know whether the result will be a head or a tail, we are certain that one of the two outcomes will occur. Similarly, when a die is rolled, any one of the numbers 1 to 6 may appear, but it is impossible to predict which number will come up. Experiments like these, where the outcome is uncertain, but the set of all possible outcomes is known, are called random experiments. The theory of probability is the mathematical framework that analyzes and interprets the results of such experiments, helping us to measure and understand the likelihood of different outcomes.

The word probability or chance is used commonly in our daily conversations. For example, the chance of winning a game before the start of a game are equal. We often say that it is very probable that it will rain tomorrow. All these terms, chance, probable etc. convey the same meaning. In such of these cases we talk about chance or probability which is taken to be a quantitative measure of certainty.

In this section we will define and explain the various terms which are used in the definition of probability.

Keywords

Sample Space, Events, Conditional Probability, Independence, Bayes' Theorem

Discussion

3.1.1 The Concept of Probability

In statistics, probability is a fundamental concept that provides a numerical measure of the uncertainty associated with events in a random experiment. Represent this uncertainty, we assign a probability value between 0 and 1, where 1 indicates absolute certainty that the event will occur, and 0 means the event will not occur. For example, if the probability of an event is 1/4, it implies there is a 25% chance that the event will happen and a 75% chance that it will not happen. Probability allows us to make informed predictions and decisions in the presence of uncertainty. This numerical assignment enables us to quantify and convey our expectations about the likelihood of different outcomes. Expressing probability numerically and understanding its implications have practical applications across various fields, such as risk assessment and decision-making. From evaluating the chances of success in a business venture to predicting outcomes in games of chance or making informed choices under uncertain conditions, probability and its numerical representation provide a crucial tool for quantifying and managing uncertainty in a wide range of scenarios.

3.1.2 Random Experiment

Random experiments can be defined as experiments that can be performed many times under the same conditions and their outcome cannot be predicted with complete certainty.

For example, tossing a coin and throwing a die are random experiments.

Set of all events of a random experiment is a sample space.

Sample Space

A sample space can be defined as the list of all possible outcomes of a random experiment.

For example, tossing a coin the sample space is $\{H, T\}$.

Trail and event

Consider an experiment of tossing a coin. Here tossing a coin is a trail and getting a head or tail is an event.

Sample Space is $\{H, T\}$. Events are $\{H\}$and $\{T\}$.

Exclusive events

All possible outcomes in any trial is known as exclusive events.

For example, in tossing a coin the possible outcomes are a head and a tail. Hence, we have two exclusive events in throwing a coin.

A specific outcome of an experiment is the event.

Mutually exclusive events

Two events are called Mutually exclusive when the occurrence of one affects the occurrence of the other.

In other words, A and B are Mutually exclusive events and if A happens then B will not happens and vice versa.

For example, in throwing a die all six faces numbered 1 to 6 are mutually exclusive since if anyone of these faces comes, the possibility of others, in the same trial, is ruled out.

In tossing a coin the event of head or tail are mutually exclusive events.

Equally Likely

Two events are said to be equally likely if one of them cannot be expected in preference to the other.

For example, in tossing a coin head or tail are equally likely events.

Exhaustive Events

The total number of possible outcomes in any trial is known as Exhaustive Events.

For example, in tossing a coin, the possible outcomes are getting a head or a tail. Hence there are two exhaustive events.

Independent Events

Two events are said to be independent when the actual happening of one does not influence in any way happening of the other.

For example, in tossing a coin, the event of getting a head in the first toss is independent of getting a head in the second toss and third toss etc.

Probability

Probability of happening an event

$$E = \frac{\textit{Number of Favourable cases}}{\textit{Total Number of Cases}}$$

Probability of happening of an event is p and probability of not happening of an event is q and $p + q = 1$.

Probability values are always assigned on a scale from 0 to 1. A probability near zero indicates an event is unlikely to occur; a probability near one indicates an event is almost certain to occur. other probabilities between 0 and 1 represent degrees of likelihood that an event will occur.

For example, if we consider the event "rain tomorrow," we understand that when the weather report indicates "a near-zero probability of rain," it means almost no chance of rain. however, if a 0.90 probability of rain is reported, we know that rain is likely to occur. A probability of 0.50 indicates that rain is equally likely to occur or not.

Probability is a measure of how likely an event is to occur, ranging from 0 (impossible) to 1 (certain).

3.1.3 Addition law of probability

Let S be the sample space of a given experiment. Let A and B be two events of S. $A \cup B$ denote the event that A *or* B (or both) occur when the experiment is performed. $A \cap B$ denotes the event that both A and B occur together.

Then, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

This rule can be extended to three or more events, for example:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

If two events A and B are mutually exclusive, then.

$P(A \cup B) = P(A) + P(B)$

3.1.4 Multiplication Law of probability

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent events, then.

$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$

i.e., The probability of independent events $\boldsymbol{A}$ and $\boldsymbol{B}$ occurring is the product of the probabilities of the events occurring separately.

3.1.5 Conditional probability

Suppose a bag contains six balls, three red and three white. Two balls are chosen (without replacement) at random, one after the other. Consider the two events, R is the event that the first ball chosen is red,

W is the event that the second ball chosen is white.

We easily find $\boldsymbol{P(R)} = \frac{3}{6} = \frac{1}{2}$. If the first ball chosen is red, then the bag subsequently contains two red balls and three white. In this case $\boldsymbol{P(W)} = \frac{3}{5}$. However, if the first ball chosen is white then the bag subsequently contains three red balls and two white. In this case $\boldsymbol{P(W)} = \frac{2}{5}$.

i.e., the probability that W occurs is clearly dependent upon whether the event R has occurred. The probability of W occurring is conditional on the occurrence or otherwise of R. The conditional probability of an event B occurring given that event A has occurred is written $\boldsymbol{P(B|A)}$. In this example $\boldsymbol{P(W|R)} = \frac{3}{5}$ and $\boldsymbol{P(W|R')} = \frac{2}{5}$

which shows that the probability that W occurs is clearly dependent upon whether the event R has occurred. The probability of W occurring is conditional on the occurrence or otherwise of R. The conditional probability of an event B occurring given that event A has occurred is written $\boldsymbol{P(B|A)}$.

The conditional probability $\boldsymbol{P(B|A)}$ is defined as

$$\boldsymbol{P(B|A)} = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A} = \frac{P(A \cap B)}{P(A)}$$

OR

$$\boldsymbol{P(A \cap B) = P(B|A) \times P(A)}$$

Conditional probability is the likelihood of an event occurring given that another event has already occurred.

3.1.6 Baye's theorem

Bayes' Theorem is a fundamental concept in probability theory that allows us to update the probability of an event based on new evidence or information. It connects prior knowledge (prior probability) with new data (likelihood) to calculate a revised probability (posterior probability).

Let $\boldsymbol{A_1, A_2, \ldots A_n}$ be $\boldsymbol{n}$ mutually exclusive and exhaustive events with $\boldsymbol{P(A_i) \neq 0}$ for $\boldsymbol{i = 1, 2, \ldots n}$. Let B be an event such that $\boldsymbol{P(B) > 0}$, Then.

$$P(A_i|B) = \frac{P(A_i).\ P(B|A_i)}{\sum_{i=1}^{n} P(A_i).\ P(B|A_i)}$$

Bayes' Theorem calculates the probability of an event based on prior knowledge of related conditions or events.

Illustration 3.1.1

The probability that a car being filled with petrol will also need an oil change is 0.30; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and filter need changing is 0.15.

(i) If the oil had to be changed, what is the probability that a new oil filter is needed?

(ii) If a new oil filter is needed, what is the probability that the oil must be changed?

Solution:

$\boldsymbol{P(O) = 0.30}$ = Probability that oil must be changed.

$\boldsymbol{P(F) = 0.40}$ = Probability that a new oil filter is needed

$\boldsymbol{P(O \cap F) = 0.15}$ = Probability that both oil and filter need changing

(i) Here we must find the probability that a new oil filter is needed if the oil had to be changed. i.e., the event $\boldsymbol{F}$ depends on $\boldsymbol{O}$.

$$P(F/O) = P(O \cap F)/P(O)$$

$$= \frac{0.15}{0.3} = \frac{1}{2}$$

(ii) The event O depends on F.

$$P(O/F) = P(O \cap F)/P(F)$$

$$= \frac{0.15}{0.4} = \frac{3}{8}$$

Illustration 3.1.2

A person is known to hit the target in three out of four shots, where as another person is known to hit the target in two out of three shots. Find the probability of the targets being hit at all when they both persons try?

Solution

Probability of the first person hit the target =

$P(A) = \frac{3}{4}$

Probability of the second person hit the target = $P(B) = \frac{2}{3}$

The two events are not mutually exclusive since both persons hit the same target.

∴ Required probability P (A or B)

$= P(A) + P(B) - P(A \cap B)$

$= \left(\frac{3}{4} + \frac{2}{3}\right) - \left(\frac{3}{4} \times \frac{2}{3}\right)$ since A and B are independent

$= \frac{17}{12} - \frac{6}{12}$

$= \frac{11}{12}$

Illustration 3.1.3

A bag contains twenty balls, three are coloured red, six are coloured green, four are coloured blue, two are coloured white and five are coloured yellow. One ball is selected at random. Find the probabilities of the following events. (a) the ball is either red or green (b) the ball is not blue (c) the ball is either red, white, or blue.

Solution

P [getting red ball] = $\frac{3}{20}$

P [getting green ball] = $\frac{6}{20}$

P [getting blue ball] = $\frac{4}{20}$

P [getting white ball] = $\frac{2}{20}$

P [the ball is either red or green] = $\frac{3}{20} + \frac{6}{20} = \frac{9}{20}$

P [the ball is blue] = $\frac{4}{20}$,

P [the ball is not blue] = $1 - \frac{4}{20} = \frac{16}{20} = \frac{4}{5}$

P [the ball is either red, white, or blue]

$$= \frac{3}{20} + \frac{2}{20} + \frac{4}{20} = \frac{9}{20}$$

Illustration.3.1.4

If from a pack of cards, a single card is drawn. What is the probability that it is either spade or king?

Solution

$P[A] = P[\text{Spade card}] = 13/52$

$P[B] = P[\text{King card}] = 4/52$

$P[\text{either spade or king}] = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{13}{52} \times \frac{4}{52} = \frac{4}{13}$$

Illustration.3.1.5

The probability that machine A will be performing a usual function in 5 years' time is 1/4 while the probability that the machine B will still be operating usefully at the end of the same period is 1/3. Find the probability that both machines will be performing a usual function?

Solution

P [machine A operating usually] $= \frac{1}{4}$

P [machine B operating usually] $= \frac{1}{3}$

P [both machines will be performing usual function]$= P(A) \times P[B] = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

Illustration.3.1.6

Let five men out of 100 and 25 women out of 1000 are colour blind. A colour-blind person is chosen at random. What is the probability of his being male. (Assume that males and female are in equal proportions).

Solution

Let M denote a person is Male. Let F denotes a person is Female. Let C denote a person is colour blind.

Given $P(M) = \frac{1}{2}$, $\quad P(F) = \frac{1}{2}$

$$P(C|M) = \frac{5}{100}, \qquad P(C|F) = \frac{25}{100}$$

$$P(C|M) = \frac{P(C|M).P(M)}{P(C|M)P(M) + P(C|F)P(F)}$$

$$= \frac{\frac{5}{100} \cdot \frac{1}{2}}{\frac{5}{100} \cdot \frac{1}{2} + \frac{25}{1000} \cdot \frac{1}{2}}$$

$$= \frac{0.05}{0.05 + 0.025} = \frac{2}{3}$$

Illustration.3.1.7

An urn contains 10 W, 3B balls while another urn contains 3 W, 5 B balls. Two balls are drawn from the first urn and put into the second urn. Then a ball is drawn from the latter. What is the probability that it is a white ball?

Solution

Two balls drawn from the $\mathbf{1^{st}}$ urn may be

i. both white (event A_1)

ii. both black (event A_2)

iii. 1 W, 1 B (event A_3)

$$\therefore P(A_1) = \frac{10C_2}{13C_2} = \frac{15}{26}$$

$$P(A_2) = \frac{3C_2}{13C_2} = \frac{1}{26}$$

$$P(A_3) = \frac{10C_1 \times 3C_1}{13C_2} = \frac{10}{26}$$

After the balls are transformed from 1^{st} urn to 2^{nd} urn, the 2^{nd} urn will contain

i) 5W, 5B ii) 3 W, 7 B iii) 4W, 6B

Let B be the event of drawing a white ball from the 2^{nd} urn.

Now $P(B|A_1) = \frac{5C_1}{10C_1} = \frac{5}{10}$

$$P(B|A_2) = \frac{3C_1}{10C_1} = \frac{3}{10}$$

$$P(B|A_3) = \frac{4C_1}{10C_1} = \frac{4}{10}$$

$$\therefore P(B) = \sum_{i=1}^{3} P(B|A_i)\,P(A_i)$$

$$= \frac{5}{10}\cdot\frac{15}{26} + \frac{3}{10}\cdot\frac{1}{26} + \frac{4}{10}\cdot\frac{10}{26} = \frac{59}{130}$$

Illustration.3.1.8

An insurance company has insured 4000 doctors, 8000 teachers, and 12000 businesspeople. The probabilities of a doctor, teacher, and businessperson dying before the age of 58 are 0.01, 0.03, and 0.05, respectively. If one of the insured individuals dies before 58, find the probability that he is a doctor.

Solution

A_1 = The person is a doctor.

A_2 = The person is a teacher

A_3 = The person is a businessperson

A = The death of an insured person

$$P(A_1) = \frac{4000}{4000 + 8000 + 12000} = \frac{1}{6}$$

$$P(A2) = \frac{8000}{4000 + 8000 + 12000} = \frac{1}{3}$$

$$P(A_3) = \frac{12000}{4000 + 8000 + 12000} = \frac{1}{2}$$

$$P(A|A_1) = 0.01, P(A|A_2) = 0.03, P(A|A_3) = 0.05$$

Therefore,

$$P(A_1|A) = \frac{P(A|A_1)P(A_1)}{P(A|A_1)P(A_1)+P(A|A_2)P(A_2)+P(A|A_3)P(A_3)}$$

$$= \frac{0.01 \times \frac{1}{6}}{0.01 \times \frac{1}{6} + 0.03 \times \frac{1}{3} + 0.05 \times \frac{1}{2}}$$

$$= \frac{0.01 \times \frac{1}{6}}{0.01 \times \frac{1}{6} + 0.01 + 0.05 \times \frac{1}{2}} = \frac{1}{22}$$

Summarized Overview

Probability basics provide the foundation for understanding uncertainty and making informed predictions. The sample space represents all outcomes of an experiment, while events are specific outcomes or sets of outcomes within that space. Conditional probability measures the likelihood of an event occurring given that another event has already occurred. Independence describes a situation where the occurrence of one event does not affect the probability of another. Bayes' Theorem is a powerful tool that allows us to update probabilities based on new evidence, making it essential in applications such as medical diagnosis, spam detection, and decision-making under uncertainty.

Assignments

1. It is observed that 50% of mails are spam. There is a software that filters spam mail before reaching the inbox. Its accuracy for detecting a spam mail is 99% and chances of tagging a non-spam mail as spam mail is 5%. If a certain mail is tagged as spam find the probability that it is not a spam mail.
2. Three factories produce light bulbs to supply the market. Factory A produces 20%, 50% of the tools are produced in factories B and 30% in factory C. 2% of the bulbs produced in factory A, 1% of the bulbs produced in factory B and 3% of the bulbs produced in factory C are defective. A bulb is selected at random in the market and found to be defective. what is the probability that this bulb was produced by factory B?
3. A company has two plants to manufacture scooters. Plant I manufactures 80 per cent of the scooters and plant II manufactures 20 per cent. At plant I, 85 out of 100 scooters are rated standard quality or better. At plant II, only 65 out of 100 scooters are rated standard quality or better. (i) What is the probability that scooter selected at random came from plant, I if it is known that the scooter is of standard quality? (ii) What is the probability that the scooter came from plant II, if it is known that the scooter is of standard quality?
4. A man has five coins, one of which has two heads. He randomly takes out a coin and tosses it three times. (i)What is the probability that it will fall head upward all the time? (ii) If it always falls head upward, what is the probability that it is the coin with two heads?

5. Market studies have shown that 30% of chartered accountants leave their jobs to start their own consultancy. Among those who leave their jobs, 60% have a degree in law while 20% of those who do not leave, have a law degree. If a chartered accountant has a law degree, what is the probability he will leave his current job to set up his own consultancy firm?
6. An Airport claims that 85% of their flights are on time. If the claim is correct, what is the probability that in a sample of 20 flights at the Dallas-Fort Worth Airport that 15 or more of the sample flights are on time?
7. A psychological study involving the troops in the Bosnia peacekeeping force was conducted. If 12 percent of the 21,496 troops are females, what is the probability that in a sample of 50 randomly selected individuals that five or fewer are female?

Suggested Reading

1. Fundamentals of Mathematical Statistics, S.C Gupta & V K Kapoor, Sultan Chand & Sons Educational publishers.
2. Statistical Methods, S.P. Gupta, Sultan Chand and Sons, New Delhi.

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1. Sheldon Ross – *A First Course in Probability*, 10th ed., Pearson, 2018.
2. Ronald E. Walpole, Raymond H. Myers, et al. – *Probability and Statistics for Engineers and Scientists*, 9th ed., Pearson, 2016.
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4. Richard Johnson – *Miller and Freund's Probability and Statistics for Engineers*, 9th ed., Pearson, 2016

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

2 UNIT Random Variables and Distributions

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand the concept of discrete and continuous Probability Distributions
- distinguish between the Probability Mass Function Probability Density Function
- understand and use common probability distributions
- model real-world problems using appropriate distributions

Background

Random variables are quantities whose outcomes depend on the result of a random experiment, and they can be either discrete or continuous. A discrete random variable takes countable values (like 0, 1, 2...), while a continuous random variable takes values from an interval (like height or weight). The Probability Mass Function (PMF) is used for discrete random variables and gives the probability of each possible value. For example, the number of heads in three-coin tosses is a discrete random variable, and its probabilities can be described using a PMF. On the other hand, the Probability Density Function (PDF) is used for continuous random variables and gives the relative likelihood for values within a range rather than at exact points or instance. For example, the distribution of people's heights in a city. Several common distributions help to model real-life scenarios. Bernoulli distribution applies to events with two outcomes, like pass/fail in an exam or yes/no in a survey. The Binomial distribution models the number of successes in a fixed number of independent trials, such as counting how many out of 10 customers entering a store to make a purchase, assuming each customer has the same probability of buying. Poisson distribution is useful for counting events that occur randomly over time or space, like the number of calls received by a call centre per hour. Gaussian (or Normal) distribution is used for many natural measurements, such as test scores, blood pressure, or IQ levels, which tend to cluster around an average and show symmetry. Understanding these distributions helps in analyzing uncertainty and making informed decisions in fields ranging from business and engineering to health and education.

Keywords

Random Variables, Probability Mass Function, Probability Density Function, Bernoulli Distribution, Binomial Distribution, Gaussian Distribution, Poisson Distribution

Discussion

3.2.1 Random Variables

Consider an experiment of tossing a coin twice. The outcomes {HH, TT, HT, TH} constitute the Sample space. Let X be the number of heads in each throw. Then X can take the values 0, 1, and 2.

Event	HH	HT	TH	TT
Value	2	1	1	0

X is a variable defined over the sample space of a random experiment called Random variable.

i.e., each of these outcomes can be associate with a number by specifying a rule of association (example: the number of heads in each throw). Such a rule of association is called random variable.

Random variables are denoted by capital letters $\boldsymbol{X}, \boldsymbol{Y}$ etc and any particular value is denoted by small letters $\boldsymbol{x}, \boldsymbol{y}$ etc.

3.2.2 Discrete and continuous Random Variables

There are two types of random variables, discrete random variable, and continuous random variables.

Discrete random variable is a random variable X whose possible values constitute finite set of values or countably infinite set of values.

For example, tossing two coins and find the number of heads in each throw.

The number of items sold at a store on a certain day is also a discrete random variable.

On the other hand, a continuous random variable takes on an uncountably infinite range of values and typically represents continuous phenomena, such as individuals' heights or temperatures at various locations.

A random variable is a numerical value that represents the outcome of a random experiment

3.2.3. Discrete Probability Distributions (Probability Mass function)

For a discrete random variable X, the set of possible values it can assume is denoted by $x_1, x_2, x_3, \ldots$, arranged in some order. The probabilities associated with each of these values are specified by the probability mass function (PMF), denoted by $f(x_k)$, where x_k represents a particular value that X can take. The PMF describes the likelihood of X assuming each of its possible values.

A probability mass function gives the probability that a discrete random variable takes on each of its possible values.

The probability mass function is formally expressed as:

$$P(X = x_k) = f(x_k) \quad k = 1, 2 \ldots.$$

It is convenient to introduce the *probability function*, also referred to as *probability distribution*, given by

$$P(X = x) = f(x)$$

In summary, the probability mass function $f(x)$ specifies the probabilities associated with individual values of a discrete random variable, and the probability distribution $P(X = x)$ generalizes this to express the probability of X being any specific value x.

In general, $f(x)$ is a probability function if

1. $f(x) \geq 0$
2. $\sum f(x) = 1$

For example, consider the random experiment of tossing 3 coins. The sample space

$$S = \{(HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$$

Let the random variable X be defined as 'Number of heads in each throw.' Then X can assume values 0,1,2,3, with corresponding probabilities

$$P[X = 0] = \frac{1}{8} = f(x_1) \geq 0,$$

$P[X = 1] = \frac{3}{8} = f(x_2) \geq 0,$

$P[X = 2] = \frac{3}{8} = f(x_3) \geq 0,$

$P[X = 3] = \frac{1}{8} = f(x_4) \geq 0$

$\boldsymbol{f(x)}$ probability mass function since each probability$\geq \mathbf{0}$ and sum of the probability

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

3.2.4 Continuous probability function

(Probability Density function)

A random variable $\boldsymbol{X}$ is said to be continuous Probability density function is

$\boldsymbol{f_i(x_i) \geq 0}$

$\boldsymbol{\int_{-\infty}^{\infty} f_i(x_i)\, dx_i = 1}$

A function $\boldsymbol{f(x)}$ that satisfies the above requirements is called a *probability function* or *probability distribution* for a continuous random variable, but it is more often called a *probability density function* or *simply density function*. Any function $\boldsymbol{f(x)}$ satisfying Properties 1 and 2 above will automatically be a density function.

> A probability density function (PDF) describes the relative likelihood of a continuous random variable taking on a specific value, with probabilities found over intervals.

Distribution Function

The cumulative distribution function (CDF), denoted by $\boldsymbol{F(x)}$, is a fundamental concept in probability theory and statistics. It provides a comprehensive representation of the probability distribution of a random variable $\boldsymbol{X}$. The CDF is defined for any real number x and is expressed as

$\mathbf{F(x) = P(X \leq x) = \sum_0^x f(x)}$ (for discrete random variable)

$\mathbf{F(x) = \int_{-\infty}^{x} f(x)\ dx}$ (for continuous random variable)

$\mathbf{X}$ is a continuous random variable, probability that $\mathbf{X}$ lies between two different values, say, $\mathbf{a}$ and $\mathbf{b}$, is given by

$$\mathbf{P(a < X < b) = \int_a^b f(x)dx}$$

Illustration.3.2.1

Given the following probability distribution

x	0	1	2	3	4	5	6	7
$f(x)$	0	C	2c	2c	3c	c^2	$2c^2$	$7c^2+c$

Find 1) c 2) $P[X \geq 5]$ 3) Find the distribution function

Solution

1. $c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$

$10c^2 + 9c - 1 = 0$

$10c^2 + 10c - c - 1 = 0$

$10c(c+1) - (c+1) = 0$

$(10c-1)(c+1) = 0$

$c = -1, \;\; c = \dfrac{1}{10}$

Since $P(x) \geq 0, \;\; c = \frac{1}{10}$

2. $P[X \geq 5] = P[X = 5,6,7]$

$= c^2 + 2c^2 + 7c^2 + c$

$= 10\,c^2 + c$

$= \frac{10}{100} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5}$

3.

x	0	1	2	3	4	5	6	7
$f(x)$	0	C	2c	2c	3c	c^2	$2c^2$	$7c^2+c$
F(x)	0	C	3c	5c	8c	$8c+c^2$	$8c+c^2+2c^2$	$8c+c^2+2c^2+7c^2+c$
	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1

Illustration.3.2.2

The probability distribution of a discrete random variable X is given by

$P[X = x\} = kx^2, x = 3,4,5$

= 0 elsewhere

Determine the value of the constant $\boldsymbol{k}$.

Solution

Since X is a discrete random variable $\sum_{x=3,4,5} kx^2 = 1$

i.e., $k\ [3^2 + 4^2 + 5^2] = 1$

$k[50] = 1$

$$k = \frac{1}{50}$$

Illustration.3.2.3

The probability distribution of a discrete random variable X is given by

$P[X = x\} = kx(5 - x)\ , x = 1,2,3,4$

$= 0$ elsewhere

Solution

Since X is a discrete random variable

$\sum_{x=1,2,3,4} kx(5 - x) = 1$

i.e., $k\ [4 + 6 + 6 + 4] = 1$

$k[20] = 1$

$$k = \frac{1}{20}$$

Illustration.3.2.4

Find the value of k if $f(x) = k(2 - x),\ 0 < x < 2$ is a Probability density function.

Solution

Since $f(x)$ is a continuous probability density function $\int_0^2 f(x)\ dx = 1$

$$\int_0^2 k(2 - x)\ dx = 1$$

$$k\left(2x - \frac{x^2}{2}\right)_0^2 = 1$$

$$k\left(4 - \frac{4}{2}\right) = 1$$

$k(4-2) = 1$

$k = \frac{1}{2}$

Illustration.3.2.5

Show that $\boldsymbol{f(x) = \frac{x+1}{2} \quad |x| < 1}$

$\boldsymbol{= 0}$ elsewhere represents the Probability density function of a random variable $\boldsymbol{X}$.

Solution

If $f(x)$ is a Probability density function, then $\int f(x)\ dx = 1$

Consider $\int_{-1}^{1} \frac{x+1}{2}\ dx = \frac{1}{2}\int_{-1}^{1} (x+1) \quad dx$

$$= \frac{1}{2}\left(\frac{x^2}{2} + x\right)_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{2} + 1 - (\frac{1}{2} - 1)\right)_{-1}^{1}$$

$$= \frac{1}{2} \times 2$$

$$= 1$$

Illustration.3.2.6

The probability density function of X be

$f(x) = 12x^2(1-x) \qquad 0 < x < 1$

$\qquad = 0$ elsewhere

Find $P\left(\frac{1}{3} < X < \frac{1}{2}\right)$

Solution

Probability density function of X lies between 1/3 and ½ is

$P\left(\frac{1}{3} < X < \frac{1}{2}\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 12x^2(1-x)\ dx$

$$= 12\int_{1/3}^{1/2} (x^2 - x^3) \quad dx$$

$$= 12\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_{1/3}^{1/2}$$

$$= 12\left(\left(\frac{1}{24} - \frac{1}{64}\right) - \left(\frac{1}{81} - \frac{1}{324}\right)\right)$$

$$= 0.201$$

3.2.5 Binomial Distribution

Let us assume a salesperson is attempting to close deals with potential clients. Each interaction with a client can be viewed as a trial, and the outcome of each trial is either a successful deal (success) or an unsuccessful attempt (failure). Let us denote the probability of successfully closing a deal in any single trial as p, and the probability of failure as $q = 1 - p$.

> The Binomial distribution gives the probability of obtaining a fixed number of successes in a set number of independent trials, each with the same success probability.

Now, suppose the salesperson conducts a series of n trials, trying to close deals with different clients independently. The objective is to understand the probability distribution of the number of successful deals (x) among these n trials. This is where the probability function for a binomial distribution comes into play.

The probability mass function (PMF) for a binomial distribution is given by:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Here, $\binom{n}{x}$ represents the number of ways to choose x successes from n trials, p^x is the probability of having x successes, and q^{n-x} is the probability of having $n - x$ failures.

For example, let us say the salesperson has a 20% success rate $(p = 0.2)$ in closing deals. If the salesperson conducts 10 independent trials, the probability of closing exactly 2 deals $(x = 2)$ can be calculated using the binomial distribution PMF. This probability calculation provides insights into the likelihood of achieving a specific number of successful deals in a given number of trials, which is valuable information for sales forecasting and performance evaluation.

The discrete probability function $P(X = x)$ for the number of successes in n trials, where $x = 0, 1, \dots, n$, is commonly referred to as the binomial distribution. This distribution is so named because, for each value of x, it corresponds to the coefficients of the binomial expansion of $(q + p)^n$, where q and p are the probabilities of failure and success, respectively. The binomial expansion is a mathematical expression obtained by raising the binomial $(q + p)$ to the power of n.

The binomial distribution is further illustrated by the binomial expansion formula, which expands $(q + p)^n$ into a sum of terms, each representing the probability of a specific number of successes. The coefficients $\binom{n}{x}$ in the expansion correspond to the number of ways to choose x successes from n trials.

Overall, the binomial distribution is a powerful tool in probability theory and statistics, widely used in various fields, including economics, to model and analyze phenomena involving repeated trials with binary outcomes.

The discrete probability function is often called the *binomial distribution* since for $x = 0, 1, 2, \ldots, n$, it corresponds to successive terms in the *binomial expansion*

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots p^n$$

$$= \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

Some Properties of The Binomial Distribution

1. Mean (μ)

The mean of a binomial distribution is given by $\mu = np$, where n is the number of trials and p is the probability of success in a single trial. This provides the average number of successes expected in n trials.

2. Variance (σ^2)

The variance of a binomial distribution is calculated using $\sigma^2 = npq$, where $q = 1 - p$ is the probability of failure.

3. Standard Deviation (σ)

The standard deviation is the square root of the variance and is given by $\sigma = \sqrt{npq}$.

Illustration.3.2.7

Seventy-five percent of employed women say their income is essential to support their family. Let **X** be the number in a sample of 200 employed women who will say their income is essential to support their family. What is the mean and standard deviation of X ?

Solution

X is a binomial random variable with $n = 200$ and $p = .75$.

The mean is $\mu = np = 200 \times .75 = 150$, and the standard deviation is $\sigma = \sqrt{npq} = \sqrt{37.5} = 6.12$.

Illustration 3.2.8

A binomial distribution has a mean equal to 8 and a standard deviation equal to 2. Find the values for **n** and **p**.

Solution

The following equations must hold: $8 = np$ and $4 = npq$.

Substituting 8 for np in the second equation gives

$4 = 8q$, which gives $q = 0.5$.

Since $p + q = 1, p = 1 - 0.5 = 0.5$. Substituting 0.5 for p in the first equation gives $\mathrm{n} \times 0.5 = 8$, and it follows that $\mathrm{n} = 16$.

Illustration.3.2.9

If on the average rainfall on 10 days in every 30 days, obtain the probability that rain will fall on at least 3 days of a given week.

Solution

The probability density function foe a binomial distribution is $\boldsymbol{P(X = x) = \binom{n}{x} p^x q^{n-x}}$

Given $p = \frac{10}{30} = \frac{1}{3}, \quad n = 7, \quad q = 1 - \frac{1}{3} = \frac{2}{3}$

$P[X \geq 3] = 1 - P[X < 3]$

$$= 1 - P[X = 0,1,2]$$

$$= 1 - P[X = 0] + P[X = 1] + P[X = 2]$$

$$P(X = 0) = \binom{7}{0}\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{7-0} = \left(\frac{2}{3}\right)^7 = 0.0585$$

$$P(X = 1) = \binom{7}{1}\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{7-1} = \binom{7}{1}\left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right) = 0.2048$$

$$P(X = 1) = \binom{7}{2}\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{7-2} = \binom{7}{2}\left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^2 = 0.3073$$

$P[X \geq 3] = 1 - 0.5706 = 0.4293$

Illustration.3.2.10

Ten coins are thrown simultaneously. Find the probability of getting at least seven heads?

Solution

$\boldsymbol{p}$ = Probability of getting a head $= \frac{1}{2}$

$\boldsymbol{q}$ = Probability of not getting a head $= \frac{1}{2}$

The probability of getting $\boldsymbol{x}$ heads in a random throw of 10 coins is

$$P(x) = \binom{10}{x} p^x q^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10}, \quad x = 0,1,2 \ldots .10$$

probability of getting at least seven heads = $P[X \geq 7]$

$= P[X = 7] + P[X = 8] + P[X = 9] + P[X = 10]$

$$= \binom{10}{7}\left(\frac{1}{2}\right)^{10} + \binom{10}{8}\left(\frac{1}{2}\right)^{10} + \binom{10}{9}\left(\frac{1}{2}\right)^{10} + \binom{10}{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} \left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right]$$

$$= \left(\frac{1}{2}\right)^{10} [120 + 45 + 10 + 1] = \frac{176}{1024} = 0.172$$

Illustration.3.2.11

A die is tossed 3 times. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of success.

Solution

Given $\boldsymbol{n = 3}, \quad \boldsymbol{p = p(getting\ 1\ or\ 6) = \frac{2}{6} = \frac{1}{3}}$

Mean $= \boldsymbol{np = 3 \times \frac{1}{3} = 1}$

Variance $= \boldsymbol{npq = 3 \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{3}}$

Illustration.3.2.12

Suppose that a Central University must form a committee of 5 members from a list of 20 candidates out of whom 12 are teachers and 8 are students. If the members of the committee are selected at random, what is the probability that the majority of the committee members are students?

Solution

$\boldsymbol{p}$ = Probability of selecting a student member $= \boldsymbol{\frac{8}{20} = \frac{2}{5}}$

$\boldsymbol{q}$ = Probability of selecting a teacher member $= \boldsymbol{\frac{12}{20} = \frac{3}{5}}$

Let X denote the number of students selected in the committee. Hence, by binomial probability distribution,

$$P(x) = \binom{5}{x} p^x q^{5-x} = \binom{5}{x}\left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{5-x}, \quad x = 0,1,2,3,4,5.$$

The required probability is given by :

probability of getting at least seven heads = $P[X \geq 3]$

$= P[X = 3] + P[X = 4] + P[X = 5]$

$$= \binom{5}{3}\left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^{5-3} + \binom{5}{4}\left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^{5-4} + \binom{5}{5}\left(\frac{2}{5}\right)^5 \left(\frac{3}{5}\right)^{5-5}$$

$$= \left(\frac{2}{5}\right)^3 \left[10 \times \left(\frac{3}{5}\right)^2 + 5 \times \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) + \left(\frac{2}{5}\right)^2\right]$$

$$= \left(\frac{2}{5}\right)^3 \left[10 \times \frac{9}{25} + \left(\frac{6}{5}\right) + \left(\frac{4}{25}\right)\right]$$

$$= \left(\frac{2}{5}\right)^3 \left[\frac{90 + 30 + 4}{25}\right]$$

$$= \left(\frac{2}{5}\right)^3 \left[\frac{124}{25}\right]$$

$$= \frac{8}{125} \times \frac{124}{25}$$

$$= \frac{992}{3125} = 0.3174$$

3.2.6 Bernoulli distribution

Suppose that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. Each toss or selection is called a *trial*. In any single trial there will be a probability associated with a particular event such as head on the coin, 4 on the die, or selection of a red marble. In some cases, this probability will not change from one trial to the next (as in tossing a coin or die). Such trials are then said to be *independent* and are often called *Bernoulli trials* after James Bernoulli who investigated them at the end of the seventeenth century.

In the special case where $\boldsymbol{n = 1}$, the binomial distribution reduces to the Bernoulli distribution, which represents a single Bernoulli trial. The Bernoulli distribution is characterized by the probability of success $(\boldsymbol{p})$ and the probability of failure $(\boldsymbol{q = 1 - p})$ in a single trial, making it a fundamental building block for the broader binomial distribution.

The mean of a Bernoulli distribution is $\boldsymbol{p}$, and the variance is $\boldsymbol{p(1 - p)}$.

Bernoulli distribution is used when we want to model the outcome of a single trial of an event, whereas Binomial distribution is used If we want to model the outcome of multiple trials of an event. The parameter for a Bernoulli distribution is probability of success p and the parameter for a Binomial Distribution is $\boldsymbol{n}$ and $\boldsymbol{p}$.

If the probability of passing an exam is 80% and failing is 20%, we can use the Bernoulli distribution to model the outcome of a single exam either pass (success) or fail (failure). However, if we want to find the probability that a student passes exactly 4 out of 5 exams, we use the Binomial distribution, which extends the Bernoulli distribution to multiple independent trials.

The Bernoulli distribution models a random variable that has only two possible outcomes, typically labelled as 1 (success) and 0 (failure).

3.2.7 Poisson distribution

Poisson distribution was derived in 1837 by a French mathematician Simeon D. Poisson.

Poisson distribution may be obtained as a limiting case of Binomial probability distribution under the following conditions:

i. n, the number of trials is indefinitely large i.e., $n \to \infty$.

ii. p, the constant probability of success for each trial is indefinitely small

i.e., $p \to 0$.

iii. $np = \lambda$, (say), is finite.

Under the above three conditions the Binomial probability function tends to the probability function of the Poisson distribution given by

$$p(x) = P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , x = 0, 1, 2, 3, \ldots$$

where X is the number of successes (occurrences of the event), $\lambda = np$.

Poisson distribution is a distribution of rare events. Accident occurring in a particular place and death due to malaria are rare events which follow Poisson distribution.

> The Poisson distribution models the probability of a given number of events occurring in a fixed interval of time or space, when events happen independently at a constant average rate.

Some Properties of The Poisson Distribution

1. Mean (λ)

The mean of a Poisson distribution is given by $\lambda = np$, where n is the number of trials and p is the probability of success in a single trial. This provides the average number of successes expected in n trials.

2. Variance (λ)

The variance of a binomial distribution is calculated using $\sigma^2 = \lambda$,

3. Standard Deviation (σ)

The standard deviation is the square root of the variance and is given by $\sigma = \sqrt{\lambda}$.

Illustration.3.2.13

If 5% of the electric bulbs manufactured by a company are defective, use Poisson distribution to find the probability that in a sample of 100 bulbs

i. none is defective,

ii. 5 bulbs will be defective. (Given: $e^{-5} = 0 \cdot 007$)

Solution

Given $n = 100$,

$p =$ probability of defective bulbs $= 5\% = \frac{5}{100}$

$\lambda = np = 100 \times \frac{5}{100} = 5$

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, 2, 3, \ldots$$

i. $P(X = 0) = \frac{e^{-5}5^{0}}{0!} = e^{-5} = 0.007$

ii. $P(X = 5) = \frac{e^{-5}5^{5}}{5!} = e^{-5} = 0.175$

Illustration.3.2.14

A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?

Solution

We are given- $n = 100$.

p - Probability of a defective pin

$$= 5\% = \frac{5}{100} = 0.05$$

λ = Mean' number of defective pins in a box of 100

$$= np = 100 \; x \; 0 \cdot 05$$

$$= 5$$

Since 'p' is small, we may use Poisson distribution.

Probability of x detective pins in a box of 100 is

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-5}\,5^{x}}{x!}, x = 0, 1, 2, 3, \ldots$$

$$P[X > 10] = 1 - P[X \leq 10\,)$$

$$= 1 - \{P[X = 0] + P[X = 1] + \cdots + P[X = 10]$$

$$= 1 - \{\frac{e^{-5}\,5^{0}}{0!} + \frac{e^{-5}\,5^{1}}{1!} + \cdots + \frac{e^{-5}\,5^{10}}{10!}\}$$

$$= 1 - e^{-5}\{1 + 5 + \cdots + \frac{5^{10}}{10!}\} = 0.0137$$

Illustration.3.2.15

The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3. Out of 2000 taxi drivers, find the number of drivers with more than 3 accidents in a year?

Solution

We are given - $n = 2000, \lambda = 3$

$P[X > 3] = 1 - P[X \leq 3)$

$$= 1 - \{P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3]$$

$$= 1 - \{\frac{e^{-3}\, 3^0}{0!} + \frac{e^{-3}\, 3^1}{1!} + \frac{e^{-3}\, 3^2}{2!} + \frac{e^{-3}\, 3^3}{3!}\}$$

$$= 1 - e^{-3}\left\{1 + 3 + \frac{9}{2} + \frac{27}{6}\right\} = 1 - 0.6472 = 0.3528$$

Illustration.3.2.16

Suppose that on an average one house in 1000 in a certain district has a fire during a year. If there are 1000 houses in that district what is the probability that exactly 5 houses will have a fire during the year?

Solution

Given - $n = 1000,\ p = \frac{1}{1000}$

$\lambda = \text{np} = 1000 \times \frac{1}{1000} = 1$

$P[X = 5] = \frac{e^{-1} 1^5}{5!} = 0.003$

Illustration.3.2.17

A car hire firm has 2 cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson distribution with mean 1.5. Calculate the proportion of days on which 1) there is no demand 2) Some demand is refused.

Solution

Given $\boldsymbol{\lambda} = \mathbf{1.5}$

proportion of days with no demand

$$= P(0) = \frac{e^{-1.5} \times 1.5^0}{0!} = 0.2231$$

proportion of days on which some demand is refused

$$= P[X > 2] = 1 - \{P[X = 0] + P[X = 1] + P[X = 2]\}$$

$$= 1 - \{\frac{e^{-1.5} \times 1.5^0}{0!} + \frac{e^{-1.5} \times 1.5^1}{1!} + \frac{e^{-1.5} \times 1.5^2}{2!}\}$$

$$= 1 - e^{-1.5}\{1 + \frac{1.5^1}{1!} + \frac{1.5^2}{2!}\}$$

$$= 0.191$$

3.2.8 Gaussian distribution (Normal Distribution)

One of the most important examples of a continuous probability distribution is the *normal distribution*, sometimes called the *Gaussian distribution*. The density function for this distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

where μ and σ are the mean and standard deviation, respectively. The corresponding distribution function is given by

$$F(x) = P(X \le x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(v-\mu)^2/2\sigma^2} dv$$

If X has the distribution function, we say that the random variable X is *normally distributed* with mean μ and variance σ^2.

If we let Z be the standardized variable corresponding to X, i.e., if we let

$$z = \frac{x-\mu}{\sigma} \approx N(0,1)$$

then the mean or expected value of Z is 0 and the variance is 1. In such cases the density function for Z can be formally placing $\mu = 0$ and $\sigma = 1$, yielding

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

This is often referred to as the *standard normal density function*.

A normal distribution is a symmetric, bell-shaped probability distribution

A graph of the density function, sometimes called the *standard normal curve*. It is a bell-shaped curve. In this graph we have indicated the areas within 1, 2, and 3 standard deviations of the mean (i.e., between $z = -1$ and $+1$, $z = -2$ and $+2$, $z = -3$ and +3) as equal, respectively, to 68.27%, 95.45% and 99.73% of the total area, which is one. This means that

$$P(-1 \le Z \le 1) = 0.6827$$
$$P(-2 \le Z \le 2) = 0.9545$$
$$P(-3 \le Z \le 3) = 0.9973$$

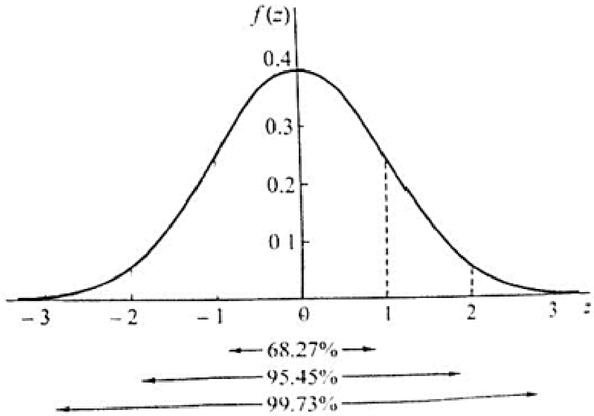
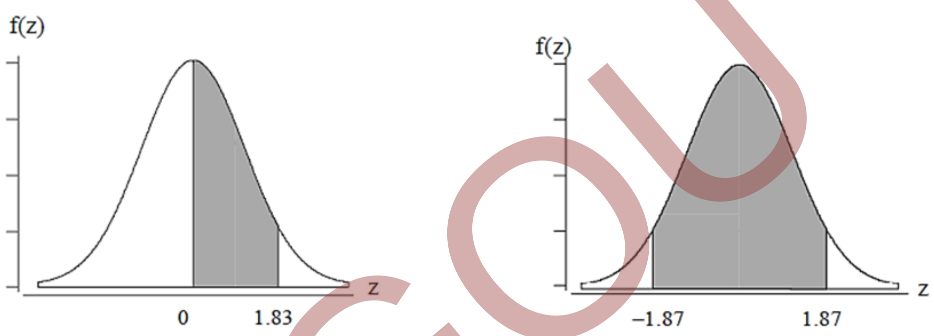


Illustration.3.2.18

Express the areas shown in the following two standard normal curves as a probability statement and find the area of each one.



Solution

The area under the curve on the left is represented as $\mathbf{P(0 < Z < 1.83)}$ and from the standard normal distribution table is equal to .4664. The area under the curve on the right is represented as $\mathbf{P(-1.87 < Z < 1.87)}$ and from the standard normal distribution table is $\mathbf{2 \times .4693 = .9386}$.

Illustration.3.2.19

The distribution of complaints per week per 100,000 passengers for all airlines in a country is normally distributed with $\boldsymbol{\mu = 4.5}$ and $\boldsymbol{\sigma = 0.8}$. Find the standardized values for the following observed values of the number of complaints per week per 100,000 passengers: (a) 6.3; (b) 2.5; (c) 4.5 ; (d) 8.0 .

Solution

(a) The standardized value for 6.3 is found by

$$z = \frac{x-\mu}{\sigma} = \frac{6.3-4.5}{0.8} = 2.25$$

(b) The standardized value for 2.5 is found by

$$z = \frac{x-\mu}{\sigma} = \frac{2.5-4.5}{0.8} = -2.50$$

(c) The standardized value for 4.5 is found by

$$z = \frac{x-\mu}{\sigma} = \frac{4.5-4.5}{0.8} = 0.00.$$

(d) The standardized value for 8.0 is found by

$$z = \frac{x-\mu}{\sigma} = \frac{8.0-4.5}{0.8} = 4.38$$

Illustration.3.2.20

The net worth of senior citizens is normally distributed with mean equal to $\$225{,}000$ and standard deviation equal to $\$35{,}000$. What percent of senior citizens have a net worth less than $\$300{,}000$?

Solution

Let $\mathbf{X}$ represent the net worth of senior citizens in thousands of dollars. The percent of senior citizens with a net worth less than $\$300{,}000$ is found by multiplying $\boldsymbol{P(X < 300)}$ times 100. The probability $\boldsymbol{P(X < 300)}$ is shown in figure below.

The event $\boldsymbol{X < 300}$ is equivalent to the event $\boldsymbol{Z < \frac{300-225}{35} = 2.14}$. The probability that $\boldsymbol{Z < 2.14}$ is represented as the shaded area in figure. The probability that $\boldsymbol{Z}$ is less than 2.14 is found by adding $\boldsymbol{P(0 < Z < 2.14)}$ to 0.5, which equals $\boldsymbol{.5 + .4838 = .9838}$. We can conclude that $\mathbf{98.38\%}$ of the senior citizens have net worths less than $\mathbf{\$300{,}000}$.

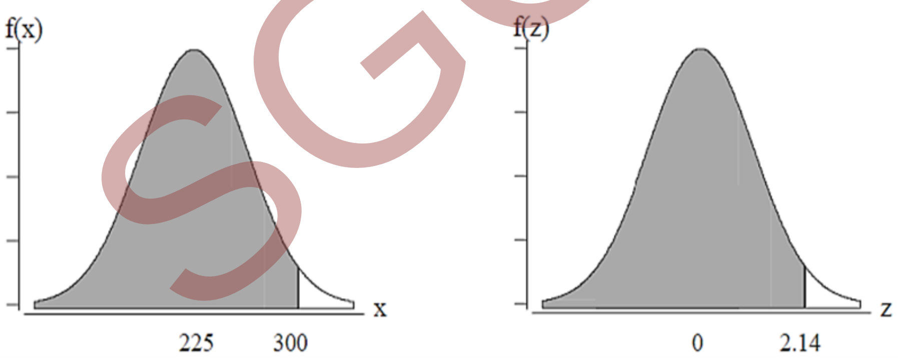


Illustration.3.2.21

The average test marks in a particular class are 79 and standard deviation is 5. If the marks are normally distributed how many students in a class of 200 did not receive marks between 75 and 82?

Solution

Given $\mu = 79,\ \sigma = 5,\ n = 200$

$$z = \frac{x-\mu}{\sigma} = \frac{75-79}{5} = -0.8$$

$$z = \frac{x - \mu}{\sigma} = \frac{82 - 79}{5} = 0.6$$

$$P[75 < X < 82] = P[-0.8 < Z < 0.6]$$

$$= 0.2881 + 0.2257 = 0.5138$$

$$1 - P[75 < X < 82] = 1 - 0.5138 = 0.4862$$

Number of students = $0.4862 \times 200 = 97$

Summarized Overview

Random variables are quantities that can take different values based on outcomes of a random process and are classified as discrete (countable outcomes, e.g., number of heads in coin tosses) or continuous (uncountable outcomes, e.g., height, time). Discrete random variables are described by a Probability Mass Function (PMF), which gives the probability of each possible value, while continuous random variables use a Probability Density Function (PDF), representing probabilities over intervals. Common distributions include the Bernoulli distribution (binary outcomes), Binomial distribution (number of successes in a fixed number of independent trials), Poisson distribution (number of events in a fixed interval), and Gaussian or Normal distribution, which models many natural phenomena with a symmetric bell-shaped curve.

Assignments

1. Four bad apples are mixed accidentally with 20 good apples. Obtain the probability distribution of the number of bad apples in a draw of 2 apples at random.
2. Obtain the probability distribution of the number of sixes in two tosses of a die
3. Suppose that a Central University has to form a committee of 5 members from a list of 20 candidates out of whom 12 are teachers and 8 are students. If the members of the committee are selected at random, what is the probability that the majority of the committee members are students?

4. . Assume that half the population is vegetarian so that the chance of an individual being a vegetarian is 1 2. Assuming that 100 investigators each take sample of 10 individuals to see whether they are vegetarians, how many investigators would you expect to report that three people or less were vegetarian?
5. It is known from past experience that in a certain plant there are on the average 4 industrial accidents per month. Find the probability that in a given year there will be less than 4 accidents. Assume Poisson distribution. (e– 4 = 0·0183)
6. In a certain factory turning out optical lenses, there is a small chance 1/500 for any one lens to be defective. The lenses are supplied, in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective, two defective, three defective lenses respectively in a consignment of 20,000 packets. You are given that e –0·02 = 0·9802
7. The average test marks in a particular class are 79. The standard deviation is 5. If the marks are distributed normally, how many students in a class of 200 did not receive marks between 75 and 82?
8. A set of examination marks is approximately normally distributed with a mean of 75 and standard deviation of 5. If the top 5% of students get grade A and the bottom 25% get grade F, what mark is the lowest A and what mark is the highest F?
9. (i) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution. (ii) What % of the items differ from the mean by a number not more than 5?

Suggested Reading

1. Fundamentals of Mathematical Statistics, S.C Gupta & V K Kapoor, Sultan Chand & Sons Educational publishers.
2. Statistical Methods, S.P. Gupta, Sultan Chand and Sons, New Delhi.

Reference

1. Sheldon Ross – *A First Course in Probability*, 10th ed., Pearson, 2018.
2. Ronald E. Walpole, Raymond H. Myers, et al. – *Probability and Statistics for Engineers and Scientists*, 9th ed., Pearson, 2016.
3. John Freund, Irwin Miller – *Probability and Statistics for Engineers*, 8th ed., Pearson, 2010.
4. Richard Johnson – *Miller and Freund's Probability and Statistics for Engineers*, 9th ed., Pearson, 2016

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

3

UNIT

Descriptive Statistics

Learning Outcomes

After completion of this unit, the learner will be able to:

- demonstrate the ability to calculate the mean, median, and mode for a given dataset
- recognizing how averages are used in real life scenarios
- understand the concept of correlation
- understand the concept of skewness and kurtosis as measures of asymmetry and peaked-ness in distributions
- develop problem-solving skills by applying skewness and kurtosis calculations to real-world datasets

Background

One of the most important objectives of statistical analysis is to get one single value that describe the characterises of the entire mass of the data. Such a value is called central value or average. The word average is very commonly used in day-to-day life. For example, we often talk of average height of a boy, average income etc. But in statistics the term average has a different meaning. It may be defined as the value of the distribution which is considered as the most representative for the group. Since the average represents the entire data, its value lies between two extremes, the largest and the smallest items. Here we consider three types of average, Mean, Median and Mode.

In practice we come across a large number of problems involving the use of two or more than two variables. If two quantities vary in such a way that movement of one are accompanied by movement of other, these quantities are corelated. For example, there exist some relationship between price of a commodity and amount demanded, increase in rain fall up to a point and production of rice etc. The degree of relationship between the variables under consideration is measured through the corelation analysis. The measure of corelation is called correlation coefficient. It helps us in determining the degree of relationship between two or more variables.

If the variables do not have a relationship with each other, then there is no correlation.

Skewness and kurtosis are statistical measures used to analyse the shape and peaked-ness of the distribution of data. Skewness measures the asymmetry of a distribution, indicating whether the data is symmetrically distributed around the mean. A skewness value of zero suggests a perfectly symmetrical distribution, while positive or negative values indicate skew to the right or left, respectively.

Kurtosis, on the other hand, assesses the peaked-ness or flatness of a distribution. It indicates the tail behaviour and concentration of data around the mean. A kurtosis value of zero implies a normal distribution, while positive values denote peaked-ness (leptokurtic), and negative values indicate flatness (platykurtic). Understanding skewness and kurtosis helps analysts assess the normality and characteristics of datasets, which is fundamental in various statistical analyses and modelling techniques.

Keywords

Mean, Median, Mode, Standard Deviation, Skewness and kurtosis, Correlation

Discussion

3.3.1 Mean

In everyday language, what many refer to as an 'average' is formally known to statisticians as the 'arithmetic mean.' This widely employed measure is derived by summing up all the items in a dataset and then dividing this total by the number of items. In simpler terms, it provides a single, representative value that encapsulates the central tendency of the entire set, making it a commonly used and easily understandable measure of the 'average' in statistical analysis. So, the average is frequently measured as the measure of central tendency.

> The mean is the average value of a set of numbers, calculated by dividing their sum by the number of values.

Computation of arithmetic mean

1. Individual Series

If x is the variable which takes the values $x_1, x_2, \ldots\ldots x_n$ over n items then mean of x, denoted by $\bar{x}$ is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum x}{n}$$

where $\sum$ denote the summation over all values of x.

2. Discrete series

There are two ways for calculating the mean for individual data series.

i. Direct method

Mean $\bar{x} = \frac{\Sigma f \times x}{\Sigma f}$ where f - frequency

ii. Short cut method

$\bar{x} = A + \frac{\Sigma f \times d}{\Sigma f}$, A – Assumed mean

3. Continuous series

i. Direct method

Mean $\bar{x} = \frac{\Sigma f \times x'}{\Sigma f}$ where, x'- mid-point of various class

f-frequency

Short cut method

$\bar{x} = A + \frac{\Sigma f \times d'}{\Sigma f}$ where A-Assumed mean

d' - Deviations of mid-points from the assumed mean $ie,\ (x' - A)$

f - frequency

iii. Step deviation method

This method is used when class intervals are equal

Arithmetic mean $\bar{x} = A + \frac{\Sigma f \times d'}{\Sigma f} \times c$

where, A-Assumed mean

d'- Deviations of mid-points from the assumed mean $ie, d' = \frac{x-A}{c}$, f-frequency, c- class interval.

Properties of arithmetic mean

The arithmetic mean of a distribution has the following mathematical properties.

1. The sum of the deviations of the observations from the arithmetic mean in a data set is always zero. i.e., $\sum(x-\bar{x}) = 0$
2. The sum of squares of the deviations of the observations in a data is least when the deviations are taken from the arithmetic mean. i.e., $\sum(x-a)^2$ is least when $a = \bar{x}$
3. If the mean of n observations, $x_1, x_2, \ldots . x_n$ is $\bar{x}$ then the mean of the observation,

   $(x_1 \pm a), (x_2 \pm a), \ldots \ldots, (x_n \pm a)$ *is* $(\bar{x} \pm a)$.

4. If the mean of $\boldsymbol{n}$ observation, $\boldsymbol{x_1, x_2, \ldots . x_n}$ is $\boldsymbol{\bar{x}}$ and if each observation is multiplied by $\boldsymbol{p}$, $\boldsymbol{p \neq 0}$, then the mean of the new observation is $\boldsymbol{p\bar{x}}$.

Illustration 3.3.1

The monthly income (Rs) of 10 families in a town are 280, 180, 96, 98, 104, 75, 80, 94, 100,75. Find the mean income of the family.

Solution

The mean income is $\bar{x} = \frac{\sum x}{n}$

$$= \frac{280+180+96+98+104+75+80+94+100+75}{10}$$

$$= \frac{1182}{10}$$

$= 118.2$

Illustration 3.3.2

The weight of 20 diabetic patients was gathered from a primary health centre by a researcher. The weights of 20 diabetic individuals are shown in the table below.

Weight (in kg)	49	53	54	55	66	68	70	80
No of patient	1	2	4	5	3	2	2	1

Calculate the mean weight of the diabetic patients

Solution

Direct method

Weight (in kg)	No of diabetic patients (f)	$f \times x$
49	1	49
53	2	106
54	4	216

55	5	275
66	3	198
68	2	136
70	2	140
80	1	80
Total	N= 20	1200

$$\bar{x} = \frac{\Sigma(f \times x)}{N}$$

$$= \frac{1200}{20}$$

$= 60$

Short cut method

Assumed Average = 65

Weight (in kg) x	No. of diabetic patients f	$dx = x - 65$	$f \times dx$
49	1	-16	-16
53	2	-12	-24
54	4	-11	-44
55	5	-10	-50
66	3	1	3
68	2	3	6
70	2	5	10
80	1	15	15
	20		**−100**

$$\bar{x} = A + \frac{\Sigma(f \times dx)}{N} = 65 - \frac{100}{20} = 65 - 5 = 60$$

Illustration 3.3.3

From the following data of profits in a shop, determine the mean profit by short cut method and step deviation method.

Profit per shop	0-20	20-40	40-60	60-80	80-100
No. of shops	6	12	22	7	3

Solution

Short cut method

Assumed Mean = 50

Profit per shop	No. of shops f	Mid value x'	$d' = x' - 50$	$f \times d'$
0-20	6	10	-40	-240
20-40	12	30	-20	-240
40-60	22	50	0	0
60-80	7	70	20	140
80-100	3	90	40	120
Total	50			-220

$$\bar{x} = A + \frac{\Sigma(f \times d')}{\Sigma f} = 50 - \frac{220}{50} = 45.6$$

Step deviation method

Assumed Mean = 50

Profit per shop	No. of shops f	Mid value x'	$d' = \frac{x - 50}{20}$	$f \times d'$
0-20	6	10	-2	-12
20-40	12	30	-1	-12
40-60	22	50	0	0
60-80	7	70	1	7
80-100	3	90	2	6
Total	50			-11

$$\bar{x} = A + \frac{\Sigma(f \times d')}{\Sigma f} \times c = 50 - \frac{11}{50} \times 20 = 45.6$$

Illustration 3.3.4

200 people were interviewed by a public opinion polling agency. The frequency distribution gives the age of the people interviewed. Calculate the mean age.

Age group (yrs)	80-89	70-79	60-69	50-59	40-49	30-39	20-29	10-19
Frequency	2	2	6	20	56	40	42	32

Solution

Here the classes are of the inclusive type. Before computing the mean, the inclusive class should be converted into an exclusive class (no difference between the upper limit of one class and the lower limit of the other class interval) to get the actual class limit.

Assumed Mean = 44.5

Age group	f	x	Mid value x'	$d' = \frac{x' - 44.5}{10}$	$f \times d'$
80-89	2	79.5-89.5	84.5	4	8
70-79	2	69.5-79.5	74.5	3	6
60-69	6	59.5-69.5	64.5	2	12
50-59	20	49.5-59.5	54.5	1	20
40-49	56	39.5-49.5	44.5	0	0
30-39	40	29.5-39.5	34.5	-1	-40
20-29	42	19.5-29.5	24.5	-2	-84
10-19	32	9.5-19.5	14.5	-3	-96
Total	200				-174

$$\bar{x} = A + \frac{\Sigma(f \times d')}{\Sigma f} \times c$$

$$= 44.5 - \frac{174}{200} \times 10$$

$$= 44.5 - 8.7$$

$= 35.8$

3.3.2 Median

The median is the middle value of a distribution when data are arranged in either ascending or descending order. Unlike the mean, which is calculated by summing all values and dividing by the total number of items, the median represents a positional average. 'Position' refers to the place of a value within a series. The median's position in a series is such that an equal number of items lie on either side of it. For example, if the incomes of 5 persons are 100, 120, 150, 160, and 180, then the median income would be 150. Even if the series is changed to 10, 25, 150, 200, and 300, the median remains 150. In the case of the mean, a change in the value of a single term alters the mean. The median may be defined as the value of the variable that divides the group into two equal parts, with one part comprising all values greater than the median and the other containing all values less than the median. The middle value serves as the median for an odd number of data points, while the average of the two middle values represents the median for an even number of data points. For instance, if there are 10 values, the median is the average of the 5th and 6th values.

> The median is the middle value in an ordered dataset, dividing it into two equal halves.

Computation of Median

1. For individual series

Sort the data into ascending or descending order.

ii. Use the formula.

Median = $\left(\frac{n+1}{2}\right)^{th}$ item

Illustration 3.3.5

Wages (Rs)	100	150	80	90	160	200	140

From the following data of the weekly wages of 7 workers compute the median wage.

Solution

Arrange the data in ascending order

Wages (Rs): 80 90 100 140 150 160 200

Apply the formula, Median = $\left(\frac{n+1}{2}\right)^{th}$ item

= $\left(\frac{7+1}{2}\right)^{th}$ item = 4^{th} item

The 4^{th} item in the series is 140.

∴ Median wage is 140

2. For discrete series

Arrange the data in ascending or descending order

Calculate cumulative frequency (c f). (The cumulative frequency is the total of frequencies, in which the frequency of the first-class interval is added to the frequency of the second class and then the sum is added to the frequency of the third-class interval and so on)

Determine $\frac{N+1}{2}$ where N is the total frequency

Median is the value for the $\left(\frac{N+1}{2}\right)^{th}$ item of the data

Illustration 3.3.6

From the following data find the median income

Income:	100	150	80	200	250	180
No of employees:	24	26	16	20	6	30

Solution

Income arranged in ascending order	f	c f
80	16	16
100	24	40
150	26	66
180	30	96
200	20	116
250	6	122
Total	122	

Median = $\left(\frac{N+1}{2}\right)^{th}$ item

= $\left(\frac{122+1}{2}\right)^{th}$ item

= $\left(\frac{123}{2}\right)^{th}$ item = 61.5^{th} item

∴ Median is the value in the data which comes in the 61.5^{th} position, which is the value of the item having cumulative frequency 66. Since cumulative frequency of 66 comes under the income 150, median is the value in the data that comes in the 66^{th} position,

∴ Median = 150

For continuous series

i. Convert inclusive classes to the exclusive class (if any). i.e., no difference between the upper limit of one class and the lower limit of the other class interval

ii. Calculate the cumulative frequencies (c f)

iii. Calculate $\frac{N}{2}$, where N is the total frequency

iv. Identify the class having cumulative frequency $\frac{N}{2}$, which is the median class.

v. Find median by using this formula.

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c$$

where,

l – Lower limit of the median class.

m – Cumulative frequency of the class preceding the median class.

f – Frequency of the median class.

c – Class interval of the median class.

Illustration 3.3.7

Income (Rs in 1000's):	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No of household:	18	15	12	22	13	12	8

The following table shows the household income of 80 families.

Find the median income.

Solution

The cumulative distribution table is

Class	f	c f
0-10	18	18
10-20	15	33
20-30	12	45
30-40	22	67
40-50	13	80
50-60	12	92
60-70	8	100
	N = 100	

$\frac{N}{2} = \frac{100}{2} = 50$

The class having cumulative frequency 50 is 30-40

∴ Median class is 30-40

Median = $l + \frac{\frac{N}{2} - m}{f} \times c$

l – Lower limit of the median class - 30

m – Cumulative frequency of the class preceding the median class - 45

f – Frequency of the median class - 22

c – Class interval of the median class - 10

$= 30 + \frac{50 - 45}{22} \text{ x } 10 = 30 + \frac{5 \times 10}{22} = 30 + \frac{50}{22} = 30 + 2.273$

= 32.273

Illustration 3.3.8

Find the median mark from the following frequency table giving the distribution of marks of 100 students.

Mark:	0-9	10-19	20-29	30-39	40-49
No of students:	5	15	32	41	7

Solution

Here the classes are of the inclusive type. Before computing the median, the inclusive class should be converted into an exclusive class to get the actual class limit.

Marks	Actual class	F	c f
0-9	-0.5 - 9.5	5	5
10-19	9.5 – 19.5	15	20
20-29	19.5 – 29.5	32	52
30-39	29.5 - 39.5	41	93
40-49	39.5 – 49.5	7	100
		N = 100	

$\frac{N}{2} = \frac{100}{2} = 50$

The class having cumulative frequency 40 is 19.5 – 29.5

∴ Median class is 19.5 - 29.5

$$\text{Median} = l + \frac{\frac{N}{2} - m}{f} \times c$$

$$= 19.5 + \frac{(50-20)}{32} \text{x } 10 = 19.5 + \frac{300}{32} = 19.5 + 9.375$$

$$= 28.875$$

Illustration 3.3.9

The following table gives the distribution of monthly wages of 600 middle-class families in a certain city. Find the median income of the families.

Monthly income (Rs)	Below 75	75-150	150-225	225-300	300-375	375-450	450 above
No of families	69	167	207	65	58	24	10

Solution

Age	F	c f
0-75	69	69
75-150	167	236
150-225	207	443

225-300	65	508
300-375	58	566
375-450	24	590
450 above	10	600
	600	

$\frac{N}{2} = \frac{600}{2} = 300$

the class having cumulative frequency 300 is 150-225

∴ median class is 150-225

$$\text{median} = l + \frac{\frac{N}{2} - m}{f} \times c$$

$$= 150 + \frac{300 - 236}{207} \times 75$$

$$= 150 + \frac{64}{207} \times 75$$

$$= 150 + \frac{4800}{207}$$

$$= 150 + 23.188$$

$$= 173.188$$

3.3.3 Mode

Mode or Model value is that value in the series which occurs or repeat itself with greatest number of times. The mode of a distribution is the value at the point around which the items tend to be most heavily concentrated. For example, if seven men are receiving daily wages of Rs. 5, 6, 7, 7, 7, 8 and 10, it is clear that the model value is Rs. 7 per day. If we have a series such that 2, 3, 5, 6, 7, 10, it is apparent that there is no mode.

The mode is the value that appears most frequently in a data set

Relation between Mean, Median and Mode

There exists a relationship between Mean, Median and Mode for moderately asymmetric distribution. For a symmetric distribution Mean, Median and Mode will have identical value.

The relation is

Mean – Mode = 3 (Mean – Median)

OR

Mode = 3 Median-2 Mean

Computation of mode

i. For individual series

The mode in individual observations is the most occurring value in a series.

Illustrations 3.3.10

Find the mode of the following set of numbers:

10,13,18,9,10,16,11,8,7,10 and 19

Solution

Here 10 occurs maximum number of times, i.e., 3 times, mode = 10

i. For Discrete series

Observation with highest frequency is considered as the mode in the discrete series.

ii. For continuous series

Steps

$$Mode = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c$$

where,

$\boldsymbol{l}$ – Lower limit of the modal class.

$\boldsymbol{f_1}$ - Frequency of the modal class.

$\boldsymbol{f_0}$ – frequency of the preceding class to the modal class.

$\boldsymbol{f_2}$ – frequency of the succeeding class to the modal class.

$\boldsymbol{c}$ – Class interval of the modal class

Grouping Table and Analysis Table

The item with the highest frequency is referred to as a mode. However, if the maximum frequency is repeating or if the maximum frequency occurs at the beginning or end of the distribution or if there are irregularity in the distribution it may be impossible to find the mode simply by looking at the distribution. In rare circumstances, the frequency concentration may be more concentrated around a frequency that is lower than the highest frequency. A grouping table and an analysis table should be developed to determine the correct modal value in such circumstances.

Steps for calculation

i. Construct a six-column grouping table.

ii. In column (1), record the frequency in relation to the item.

iii. The frequencies in column (2) are arranged in twos, starting at the top. Their totals are calculated, and the highest total is highlighted.

iv. The frequencies are grouped in twos again in column (3), leaving the first frequencies. The highest total is once again noted.

v. The frequencies in column (4) are arranged in threes, starting at the top. Their totals are calculated, and the highest total is highlighted.

vi. The frequencies are grouped in threes again in column (5), leaving the initial frequency. Their totals are calculated, and the highest total is highlighted.

vii. The frequencies are grouped in threes again in column (6), leaving the first and second frequencies. After totalling the frequencies, the highest total is identified and highlighted again.

viii. Create an analysis table to find the modal value or modal class that the largest frequencies cluster around for the longest periods of time. Place the column number on the left-hand side of the table and the item sizes on the right-hand side. Mark 'X' in the relevant box corresponding to the values they represent to input the values against which the highest frequencies are found. The mode is the set of values with the most 'X' marks against them.

Illustration 3.3.11

The following table shows the frequency distribution of the marks of 100 students. Find the mode.

Marks:	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No.of students:	4	12	18	22	21	19	10	3	1

Solution

There are no irregularities in the distribution, hence the model class is 30-40.

c = 10 f_1=22, $f_2 = 21$, $f_0 = 18$

$$Mode = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c \; = 30 + \frac{22-18}{2\times22-18-21} \times 10 \; = 30 + \frac{4}{5} \times 10$$

$$= 30 + \frac{40}{5} = 30 + 8 = 38$$

Illustration 3.3.12

The following table shows the monthly income of 130 families. Calculate the mode value.

Income (in'000):	10-25	25-40	40-55	55-70	70-85	85-100
No of Families:	12	9	17	16	20	16

Solutions

Since there are irregularity in the distribution, we must construct the grouping table and analysis table because determining the modal value is tough by examination.

(a)Grouping table

Income (In'000) x	F (1)	Grouping in twos (2)	(3)	Grouping in threes (4)	(5)	(6)
10-25	12	21		38		
25-40	9		26		42	
40-55	17	33				53
55-70	16		36	52		
70-85	20	36				
85-100	16					

In column 1 the highest frequency is 20 corresponds to the 70-85. So, we put X mark in 70-85. In column 2 the highest frequency is 36 corresponds to 70-85 and 85-100. So, we put X mark in70-85 and 85-100. In column 3 the highest frequency is 36 corresponds to 55-70 and 70-85. So, we put X mark in 55-70 and 70-85. In column 4 the highest frequency is 52 corresponds to 55-70, 70-85 and 85-100. So, we put X mark in 55-70, 70-85 and 85-100. In column 5 the highest frequency is 42 corresponds to 25-40, 40-55 and 55-70. So, we put X mark in 25-40, 40-55 and 55-70. In column 6 the highest frequency is 53 corresponds to 40-55, 55-70 and 70-85. So, we put X mark in 40-55, 55-70 and 70-85.

(b) Analysis table

Variable / F column	10-25	25-40	40-55	55-70	55-70	70-85	85-100
1						X	
2						X	X
3					X	X	
4					X	X	X
5		X	X	X			
6			X	X	X		
Total	-	1	2	2	3	4	2

The greatest total (4) is noted to be against 70-85.

$$Mode = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times c \;= 70 + \frac{20-16}{2\times20-16-16} \times 15 \;= 70 + \frac{4}{8} \times 15$$

$$= 70 + \frac{60}{8}$$

$$= 70 + 7.5$$

$$= 77.5$$

Illustration 3.3.13

The mean of 10 observations is 20 and median is 15. Find the mode of the observations.

Solution

Given mean = 20, median = 15.

The relation between mean, median and mode is

Mode = 3 Median-2 Mean

$$Mode = 3 \times 15 - 2 \times 20 = 45 - 40 = 5$$

3.3.4 Standard deviation

The most important and widely used measure of dispersion is Standard deviation. It is the positive square root of the mean of the squares of deviation from the arithmetic mean. It is denoted by the Greek letter σ (sigma). It cannot be negative. The standard deviation concept was introduced by Karl Pearson in 1893. It is the most used methods of dispersion since it is free from some defects of other measures of dispersion.

The square of the Standard deviation σ^2 is termed as variance It has the same properties as Standard deviation.

Standard deviation measures how much the values in a dataset deviate, on average, from the mean.

Coefficient of variation

The coefficient of variation is calculated by dividing the standard deviation by the arithmetic mean, which is given as a percentage. It is the most popular way of comparing the consistency or stability of two or more sets of data. The series for which the CV is greater is said to be more variable or less consistent or less stable. On the other hand, the series for which CV is less is said to be less variable or more consistent or more stable.

$$CV = \frac{\textbf{Standard Deviation}}{\textbf{Mean}} \times \mathbf{100}$$

$$= \frac{\sigma}{\bar{x}} \text{ x } 100$$

Computation of Standard deviation and Coefficient of variation

i. For individual series

Standard deviation = $\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$

Variance = $\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n}$

where,

$\bar{x}$ -Actual mean of the observation

n - Total number of items

Illustration.3.3.14

Calculate the standard deviation for the following data

1,3,5,7,4

Solution

$\bar{x} = \frac{\Sigma x}{n} = \frac{20}{5} = 4$

X	(x -4)	$(x-4)^2$
1	-3	9
3	-1	1
5	1	1
7	3	9
4	0	0
		20

Standard deviation = $\sqrt{\frac{\Sigma(x-\bar{x})^2}{N}} = \sqrt{\frac{20}{5}} = \sqrt{4} = 2$

CV = $\frac{\text{Standard Deviation}}{\text{Mean}} \times 100 = \frac{2}{4} \times 100 = 50\%$

For discrete series

Standard deviation = $\sqrt{\frac{\Sigma f \times x^2}{\Sigma f} - \bar{x}^2}$

Where, f – Frequency, $\bar{x} = \frac{\Sigma f \times x}{\Sigma f}$

Illustration 3.3.15

An arithmetic test was given to 100 students. The following is the time in minutes required to finish the test:

Time (in minute):	18	19	20	21	22	23	24	25	26	27
No of students:	3	7	11	14	18	17	13	8	5	4

Calculate the standard deviation of their test completion time as well as the coefficient of variation.

Solution

x	f	fx	x^2	fx^2
18	3	54	324	972
19	7	133	361	2527
20	11	220	400	4400
21	14	294	441	6174
22	18	396	484	8712
23	17	391	529	8993
24	13	312	576	7488
25	8	200	625	5000
26	5	130	676	3380
27	4	108	729	2916
	N = 100	2238		50562

$$\bar{x} = \frac{\Sigma f \times x}{\Sigma f} = \frac{2238}{100} = 22.38$$

$$\text{Standard deviation} = \sqrt{\frac{\Sigma f \times x^2}{\Sigma f} - \bar{x}^2}$$

$$= \sqrt{\frac{50562}{100} - 22.38^2}$$

$$= \sqrt{505.62 - 500.8644}$$

$$= \sqrt{4.7556} = 2.181$$

$$CV = \frac{\text{Standard Deviation}}{\text{Mean}} \text{ x } 100$$

$$= \frac{2.181}{22.38} \text{x} 100 = 9.75\%$$

A	32	28	47	63	71	39	10	60	96	14
B	19	31	48	53	67	90	10	62	40	80

Illustration. 3.3.16

The score of 2 batsman A and B in 10 innings during a certain match are as under. Who is the better batsman? Who is more consistent?

Solution

In order to decide as to which of the two batsman, A or B, is better player, we should find their average score. The one whose average is higher will be considered as a better batsman.

T determine the consistency we should determine the coefficient of variation. The less this coefficient of variation is more consistent.

A			B		
X	(x -46)	$(x - 46)^2$	X	(x -50)	$(x - 50)^2$
32	-14	196	19	-31	961
28	-18	324	31	-19	361
47	1	1	48	-2	4
63	17	289	53	3	9
71	25	625	67	17	289
39	-7	49	90	40	1600
10	-36	1296	10	-40	1600
60	14	196	62	12	144
96	50	2500	40	-10	100
14	-32	1024	80	30	900
		6500			5968

A's $\bar{x} = \frac{\sum x}{n} = \frac{460}{10} = 46$

B's $\bar{x} = \frac{\sum x}{n} = \frac{500}{10} = 50$

A

Standard deviation = $\sqrt{\frac{\sum (x-\bar{x})^2}{N}}$

$= \sqrt{\frac{6500}{10}} = \sqrt{650} = 25.5$

$$CV = \frac{\text{Standard Deviation}}{\text{Mean}} \text{ x } 100 = \frac{25.5}{46} \text{x}100 = 55.4\%$$

B

Standard deviation = $\sqrt{\frac{\Sigma(x-\bar{x})^2}{N}}$

$$= \sqrt{\frac{5968}{10}} = \sqrt{596.8} = 24.43$$

$$CV = \frac{\text{Standard Deviation}}{\text{Mean}} \text{ x } 100 = \frac{24.43}{50} \text{x}100 = 48.8\ \%$$

B is better batsman since his average is 50 as compared to 46 of A. B is more consistent since the coefficient of variation of B is less than the coefficient of variation of A

For continuous series

Standard deviation = $\sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}}$

OR

Standard deviation = $\sqrt{\frac{\Sigma(fx^2)}{\Sigma f} - (\bar{x})^2}$

Where,

f – Frequency

x – Mid value

Illustration. 3.3.17

The marks of 75 students in a class is given below.

Mark:	1-3	3-5	5-7	7-9	9-11	11-13	13-15
No of students:	1	9	25	35	17	10	3

Find the standard deviation and the coefficient of variation of the data.

Solution

Mark	Mid value (x)	F	f x	$(x-8)^2$	$f(x-8)^2$
1-3	2	1	2	36	36
3-5	4	9	36	16	144
5-7	6	25	150	4	100
7-9	8	35	280	0	0
9-11	10	17	170	4	68
11-13	12	10	120	16	160
13-15	14	3	42	36	108
Total		100	800		616

$\overline{x} = \frac{\Sigma fx}{\Sigma f} = \frac{800}{100} = 8$

Standard deviation = $\sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{616}{100}} = 2.48$

3.3.5 Skewness

Let us consider the following 3 distributions.

Distribution A

X	1	2	3	4	5	6	7	8	9
F	2	3	5	7	8	7	5	3	2

Distribution B

X	1	2	3	4	5	6	7	8	9
F	2	3	5	7	8	7	5	3	2

Distribution C

X	1	2	3	4	5	6	7	8	9
F	2	3	5	7	8	7	5	3	2

For distribution A, Mean = 5, Median = 5, Mode = 5

For distribution B, Mean = 4.06, Median = 4, Mode = 3

For distribution C, Mean = 5.94, Median = 6, Mode = 7

In distribution A the value of Mean, Median and Mode are identical. Hence it is known as symmetric distribution.

In distribution B, Mean is maximum, and Mode is least. The excess value on the right-hand side. Hence it is known as positively skewed distribution.

In distribution C, Mean is least, and Mode is maximum. The excess value on the left-hand side. Hence it is known as negatively skewed distribution.

Skewness measures the asymmetry of a probability distribution around its mean.

Measure of Skewness

Measure of Skewness tells us the direction and extend of asymmetry in a series and permit us to compare two or more series.

Karl Pearson's coefficient of Skewness,

$$SK = \frac{Mean - Mode}{Standard\ Deviation}$$

OR

$$SK = \frac{3(Mean - Median)}{Standard\ Deviation}$$

Illustration.3.3.18

Compute Karl Pearson's coefficient of Skewness from the following table

Value:	6	12	18	24	30	36	42
Frequency:	4	7	9	18	15	10	5

Solution

Value x	F	xf	x^2	fx^2
6	4	24	36	144
12	7	84	144	1008
18	9	162	324	2916
24	18	432	576	10368
30	15	450	900	13500
36	10	360	1296	1290
42	5	210	1764	8820
Total	68	1722	5040	49716

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{1722}{68} = 25.32$$

$$SD = \sqrt{\frac{49716}{68} - (25.32)^2}$$

$$= \sqrt{731.12 - 641.10} \quad = 9.48$$

Mode = 24

Coefficient of Skewness SK = $\frac{25.32 - 24}{9.48}$

$= \frac{1.32}{9.48}$

$= 0.139$

Illustration. 3.3 19

Compute Karl Pearson's coefficient of Skewness from the following table.

Value:	5-7	8-10	11-13	14-16	17-19
Frequency:	14	24	38	20	4

Solution

Value	9X	F	Mid x	xf	x^2	fx^2
5-7	4.5-7.5	14	6	84	36	404
8-10	7.5-10.5	24	9	216	81	1944
11-13	10.5-13.5	38	12	456	144	5472
14-16	13.5-16.5	20	15	300	225	4500
17-19	16.5-19.5	4	18	72	344	1296
Total		100		1128		13716

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{1128}{100} = 11.28$$

$$SD = \sqrt{\frac{13716}{100} - (11.28)^2} = \sqrt{137.16 - 127.24}$$

$$= 3.15$$

Model class 10.5-13.5

$$Mode = 10.5 + \frac{(38-24)\times 2}{2\times 38 - 24 - 20}$$

$$= 10.5 + \frac{28}{32} = 10.5 + 0.875$$

$$= 11.375$$

$$SK = \frac{11.28 - 11.375}{3.15} = -\frac{0.095}{3.15} = -0.03$$

Illustration.3.3.20

Consider the following distributions:

	Distribution A	Distribution B
Mean	100	90
Median	90	80
Standard Deviation	10	10
Coefficient of variation	10	11.11

Solution

Distribution A

$$SK = \frac{3\,(Mean - Median)}{Standard\ Deviation}$$

$$SK = \frac{3(100-90)}{10}$$

$$= \frac{3 \times 10}{10}$$

$= 3$

Distribution B

$$SK = \frac{3\,(Mean - Median)}{Standard\ Deviation}$$

$$SK = \frac{3(90-80)}{10}$$

$$= \frac{3 \times 10}{10}$$

$= 3$

Illustration.3.3.21

Pearson's measure of skewness of a distribution is 0·5. Its median and mode are respectively 42 and 36. Find the Coefficient of Variation.

Solution

Given Median = 42, Mode = 36, Pearson's coefficient of skewness $= 0 \cdot 5$

$$SK = \frac{Mean - Mode}{Standard\ Deviation}$$

$$0.5 = \frac{Mean - 36}{Standard\ Deviation}$$

By using the empirical relationship between mean, median and mode.

Mode= 3 Median – 2 Mean

$\Rightarrow$ 2$\times$ Mean = 3 Median – Mode $= 3 \times 42 - 36$

$= 126 - 36 = 90$

$\Rightarrow$ Mean $= \frac{90}{2} = 45$

$$0.5 = \frac{45 - 36}{Standard\ Deviation}$$

$$S\,D = \frac{45 - 36}{0.5} = \frac{9}{0.5} = 18$$

Coefficient of variation of $A = \frac{\sigma}{\bar{x}} \times 100 = \frac{18}{45} \times 100 = 40$

Moments

The term moment originates from mechanics, where it refers to the capacity of a force to rotate a pivoted lever. In statistics, moments are statistical measures that provide information about certain characteristics of a distribution.

In a frequency distribution, there are four moments: the first, second, third, and fourth moments. These moments describe properties such as the mean, variance, skewness, and kurtosis of the distribution.

Moments can be classified as raw or central moments. Raw moments are measured about any arbitrary point, often denoted as A. When A is taken to be zero, these moments are called moments of the origin. Central moments are calculated with respect to a specific point, often the arithmetic mean.

The first raw moment of the origin is the mean, while the first central moment is zero. The second raw and central moments correspond to the mean square deviation and variance, respectively. The third and fourth moments are particularly useful in measuring skewness and kurtosis.

Three types of moments are:

1. Moments about arbitrary point,

2. Moments about mean

3. Moments of origin

Moment about Arbitrary point A

For an individual series

If $x_1, x_2, \dots, x_n$ are the n observations of a variable X, then

The r^{th} moment of x about the arbitrary point A is denoted by μ_r' and is defined as

$$\mu_r' = \frac{\sum (x - A)^r}{n}, \qquad r = 0,1,2 \dots . r$$

For Discrete and Continuous series

$$\mu_r' = \frac{\sum f \times (x - A)^r}{n}, \qquad r = 0,1,2 \dots . r$$

Moment about Origin

When we take an arbitrary point $A = 0$ then, we get the moments about origin.

For an individual series

The r^{th} moment of x about zero is denoted by μ_r' and is defined as

$$\mu_r' = \frac{\sum (x - 0)^r}{n} = \frac{\sum x^r}{n}, \qquad r = 0,1,2 \dots . r$$

For Discrete and continuous Series

$$\mu_r' = \frac{\sum f \times (x)^r}{n}, \qquad r = 0,1,2 \dots . r$$

Moment about Mean (Central Moment)

For an individual series

If $x_1, x_2, \ldots, x_n$ are the n observations of a variable X, then

The r^{th} moment of x about mean is denoted by μ_r' and is defined as

$$\mu_r = \frac{\sum(x - \bar{x})^r}{n}, \qquad r = 0,1,2 \ldots . r$$

For a grouped Series

$$\mu_r = \frac{\sum f \times (x - \bar{x})^r}{\sum f}, \qquad r = 0,1,2 \ldots . r$$

Relation between Moments about Mean and Moments about Arbitrary Point 'A'

$$\mu_1 = 0,$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$\mu_3 = \mu_3' - 3 \mu_2' \mu_1' + 2 \mu_1'^3$$

$$\mu_4 = \mu_4' - 4 \mu_3' \mu_1' + 6 \mu_2' \mu_1'^2 - 3 \mu_1'^4$$

3.3.6 Kurtosis

Besides average, variance and skewness the fourth characteristics used for description and comparison of frequency distribution is the peaked-ness of the distribution. Measure of peaked-ness is called the measure of kurtosis.

Kurtosis refers to the degree of flatness or peaked ness in the region above the mode of the frequency curve. The degree of kurtosis of a distribution is measured relative to the peaked-ness of the normal curve. For a normal curve the distribution is symmetric about the mean half the values fall below the mean and half above the mean

Kurtosis is a statistical measure that describes the heaviness of a distribution's tails relative to a normal distribution.

The measure of Kurtosis tells us the extent to which a distribution is more peaked or flat topped than the normal curve. If the curve is more peaked than the normal curve, it is leptokurtic. On the other hand, if the curve is more flat topped than the normal curve, it is platykurtic. The normal curve is mesokurtic (Fig 3.3.1).

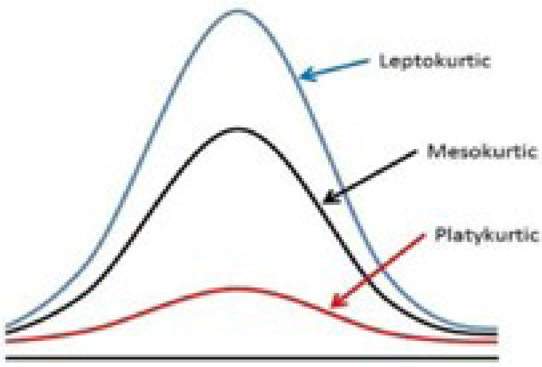


Fig 3.3.1 Kurtosis

Measure of Kurtosis

The most important Measure of Kurtosis is the value of the coefficient β_2, it is defined as

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2}$$

Where μ_4 is the fourth moment about mean and μ_2 is the second moment about mean.

For normal curve $\beta_2 = 3$.

When $\beta_2 > 3$ the curve is more peaked than normal curve. i.e, leptokurtic.

When $\beta_2 < 3$ the curve is less peaked than normal curve. i.e, platykurtic.

When $\beta_2 = 3$ the curve is a normal curve. i.e, mesokurtic.

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ is the skewness.

Illustration.3.3.22

Find the Kurtosis from the following table

Value:	0-10	10-20	20-30	30-40
Frequency:	1	3	4	2

Solution

	f	*mid*	*xf*	d = x - 22	*fd*	*fd*²	fd^3	fd^4
0-10	1	5	5	-17	-17	289	-4913	83521
10-20	3	15	45	-7	-21	147	-1029	7203
20-30	4	25	100	3	12	36	108	324
30-40	2	35	72	13	26	338	4394	57122
Total	10	60	220		0	810	-1440	148170

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{220}{10} = 22$$

$$\mu_1 = \frac{\sum fd}{\sum f} = 0, \quad \mu_2 = \frac{\sum fd^2}{\sum f} = \frac{810}{10} = 81,$$

$$\mu_3 = \frac{\sum fd^3}{\sum f} = -\frac{1440}{10} = -144,$$

$$\mu_4 = \frac{\sum f d^4}{\sum f} = \frac{148170}{10} = 14817$$

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{14817}{81^2} = 2.258$$

Illustration.3.3.23

For a distribution, it has been found that the first four moments about 27 are 0, 256, –2871 and 188462 respectively. Obtain the first four moments about zero. Also calculate the value of $\boldsymbol{\beta_1}$ and $\boldsymbol{\beta_2}$, and comment.

Solution

$$A = 27;\ \mu_1' = 0,\ \ \mu_2' = 256,\ \ \mu_3' = -2871,\ \mu_4{}' = 188462$$

Mean $= A + \mu_1{}' = 27 + 0 = 27$

Hence, the moments about 'A = 27' are same as moments about mean.

i.e,

$$\mu_2 = \mu_2' = 256,\quad \mu_3 = \mu_3' = -2871,\ \ \mu_4 = \mu_4{}' = 188462$$

Moments about Origin

$$\mu_1' = Mean = 27$$

We know that $\mu_2 = \mu_2' - {\mu_1'}^2$, then $\mu_2' = \mu_2 + {\mu_1'}^2$

$$\mu_2' = \mu_2 + {\mu_1'}^2 = 256 + 272 = 256 + 729 = 985$$

$$\begin{aligned}\mu_3' &= \mu_3 + 3\,\mu_2\,\mu_1' + {\mu_1'}^3 \\ &= -2871 + 3 \times 256 \times 27 + 273 \\ &= -2871 + 20736 + 19683 = 37548\end{aligned}$$

$$\mu_4{}' = \mu_4 + 4\,\mu_3\,\mu_1' + 6\,\mu_2\,{\mu_1'}^2 + {\mu_1'}^4.$$

$$= 188462 + 4 \times (-2871) \times 27 + 6 \times 256 \times 272 + 274$$

$$= 188462 - 310068 + 1119744 + 531441$$

$$= 1529579$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-2871)^2}{(256)^3} = 0.4913$$

Since $\beta_1 \neq 0$, the distribution is not symmetrical.

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{188462}{(256)^3} = 2.876$$

Since $\beta_2 < 3 \Rightarrow$ Distribution is moderately platy-kurtic.

3.3.7 Correlation

In practice we come across a large number of problems involving the use of two or more than two variables. If two quantities vary in such a way that movement of one are accompanied by movement of other, these quantities are corelated. For example, there exist some relationship between price of a commodity and amount demanded, increase in rain fall up to a point and production of rice etc. The degree of relationship between the variables under consideration is measured through the corelation analysis. The measure of corelation is called correlation coefficient. It helps us in determining the degree of relationship between two or more variables.

> Correlation is a statistical measure that describes the strength and direction of a relationship between two variables.

If the variables do not have a relationship with each other, then there is no correlation.

Methods of Studying Correlation

To determine the linearity and non- linearity among the variables, and the extent to which they are correlated, various methods are used to define and measure the correlation among the variables. The various methods of studying correlation coefficient

1. Karl Pearson's Coefficient of Correlation
2. Spearman's Rank Correlation

Correlation coefficient

Degree of relationship between two variables is called coefficient of correlation. It is an algebraic method of measuring correlation. Coefficient of correlation is denoted by the symbol $\boldsymbol{r}$ and r lies between -1 and $+1$.

i.e., $-1 \leq r \leq 1$

Properties of Correlation coefficient

i. Coefficient of correlation lies between -1 and $+1$.

ii. When $\boldsymbol{r}$ lies between 0 and 1, the correlation is positive, when $\boldsymbol{r}$ lies between -1and 0, the correlation is negative. If $\boldsymbol{r} = \mathbf{0}$ there is no correlation.

iii. If $r = +1$ it is perfect positive correlation.

If $r = -1$ it is called perfect negative correlation.

iv. It is a pure number lies between -1 and +1 and has no units.

v. Correlation coefficient does not change with reference to change of origin and scale. By origin, we mean that there will be no effect on the correlation coefficients if any constant is subtracted from the value of X and Y. By scale, we mean that if the value of X and Y is either multiplied or divided by some constant, then the correlation coefficients will not change.

Karl Pearson's Coefficient of Correlation

Of several mathematical methods of measuring correlation Karl Pearson's coefficient of correlation is the most widely used method for measuring correlation. It is popularly known as Pearson coefficient of correlation. It is denoted by the symbol "r". It is also known as Product Moment Method.

Computation of correlation coefficient

$$r(x,y) = \frac{Cov\ (x,y)}{\sigma(x)\sigma(y)}$$

where $Cov\ (x,y)$= covariance of (x,y). Covariance is a statical measure that quantifies the degree to which two variables change together. It is the sum of the product of the average of the observations from arithmetic mean.

$$Cov\ (x,y) = \frac{\sum(x-\bar{x})(y-\bar{y})}{n}$$

$\sigma(x) = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$ is the standard deviation of x.

$\sigma(y) = \sqrt{\frac{\sum(y-\bar{y})^2}{n}}$ is the standard deviation of y.

$\bar{x} = \frac{\sum x}{n}$, $\bar{y} = \frac{\sum y}{n}$ are the arithmetic means.

So, $r(x,y) = \dfrac{\frac{\sum(x-\bar{x})(y-\bar{y})}{n}}{\sqrt{\frac{\sum(x-\bar{x})^2}{n}}\sqrt{\frac{\sum(y-\bar{y})^2}{n}}}$

$$= \frac{\sum(x-\bar{x})\ (y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2}\ \sqrt{\sum(y-\bar{y})^2}} = \frac{\sum dx\ dy}{\sqrt{\sum dx^2 \sum dy^2}}$$

where $dx = x - \bar{x}$, $\quad dy = y - \bar{y}$

or

$$r(x,y) = \frac{n\sum xy - \sum x \sum y}{\sqrt{n(\sum x^2) - (\sum x)^2}\sqrt{n(\sum y^2) - (\sum y)^2}}$$

Illustration 3.3.24

Find Karl Pearson's correlation coefficient between x and y for the following data,

$n = 15, Cov\ (x,y) = 8.13, \sigma x = 3.01, \sigma y = 3.03$

Solution

$$r = \frac{Cov\ (x,y)}{\sigma(x)\ \times\ \sigma(y)}$$

$$= \frac{8.13}{3.01 \times 3.03}$$

$$= \frac{8.13}{9.12} = 0.89$$

Illustration 3.3.25

Given: $\sum x = 125, \quad \sum y = 100, \quad \sum x^2 = 650, \quad \sum y^2 = 436,$ $\sum xy = 520 \text{ and } n = 25,$ obtain the value of Karl Pearson's correlation coefficient $r(X,Y)$.

Solution

$$r(x,y) = \frac{n\sum xy - \sum x \sum y}{\sqrt{n(\sum x^2) - (\sum x)^2}\sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$= \frac{25 \times 520 - 125 \times 100}{\sqrt{25 \times 650 - 125^2}\ \sqrt{25 \times 436 - 100^2}}$$

$$= \frac{13000 - 12500}{\sqrt{16250 - 15625}\sqrt{10900 - 10000}}$$

$$= \frac{500}{\sqrt{625}\sqrt{900}}$$

$$= \frac{500}{25 \times 30}$$

$$= \frac{500}{750}$$

$$= 0.67$$

Summarized Overview

Descriptive statistics summarize and describe the essential features of a dataset. Measures of central tendency mean (arithmetic average), median (middle value), and mode (most frequent value), indicate the dataset's center. Measures of dispersion, variance (average squared deviation from the mean) and standard deviation (square root of variance), show how spread out the data is. Skewness measures the asymmetry of the distribution, while kurtosis measures the "peakedness" or heaviness of tails compared to a normal distribution. Covariance indicates the direction of the linear relationship between two variables, whereas correlation standardizes this measure to range between –1 and +1, reflecting both strength and direction of the relationship.

Assignments

1. The marks obtained by 10 students in a class test is given below.

Roll No:	1	2	3	4	5	6	7	8	9	10
Marks:	60	30	70	50	40	50	60	20	70	30

2. The heights in inches of 70 employees in an office are given below. Find the mean height of an employee

Height (in inches)	60	62	63	65	67	68
No of employees	5	10	12	18	15	10

3. Find the median wage of the following distribution

Wages (Rs):	20-30	30-40	40-50	50-60	60-70
No of labourers:	3	5	20	10	5

4. Calculate mode from the following data

Weight in kg:	25	30	35	40	45	50	55	60
No of person:	50	70	80	180	70	30	20	10

5. Below are the profits earned by 100 sole proprietorship businesses.

Profit in '000:	0-10	10-20	20-30	30-40	40-50	50-60
No of companies:	8	12	20	30	20	10

Calculate the standard deviation and the coefficient of variation of the data.

6. Calculate the coefficient of skewness from the following data:

Variable	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency:	12	16	26	38	22	15	7	4

7. Find the variance, skewness and kurtosis of the following distribution by the method of moments:

Variable	0-10	10-20	20-30	30-40
Frequency:	1	4	3	2

Suggested Reading

1. Fundamentals of Mathematical Statistics, S.C Gupta & V K Kapoor, Sultan Chand & Sons Educational publishers.
2. Statistical Methods, S.P. Gupta, Sultan Chand and Sons, New Delhi.

Reference

1. Sheldon Ross – *A First Course in Probability*, 10th ed., Pearson, 2018.
2. Ronald E. Walpole, Raymond H. Myers, et al. – *Probability and Statistics for Engineers and Scientists*, 9th ed., Pearson, 2016.
3. John Freund, Irwin Miller – *Probability and Statistics for Engineers*, 8th ed., Pearson, 2010.
4. Richard Johnson – *Miller and Freund's Probability and Statistics for Engineers*, 9th ed., Pearson, 2016

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

4 UNIT

Statistical Inference

Learning Outcomes

After completion of this unit, the learner will be able to:

- explain the concept of estimators and bias
- interpret and construct confidence intervals
- perform and interpret hypothesis tests
- explain the principle of least squares
- apply and compare regression techniques

Background

Statistical inference is the process of drawing conclusions about a population based on sample data. It enables us to make educated guesses, predictions, or decisions using limited information. At its core are estimators, which are formulas or rules used to estimate unknown population parameters, such as the mean or proportion. For instance, the sample mean is an estimator of the population mean. An estimator is considered unbiased if, over many samples, its average value equals the true population parameter. If it consistently overestimates or underestimates, it is biased, which could lead to incorrect conclusions.

To express the uncertainty in our estimates, we use confidence intervals. A confidence interval gives a range of reasonable values for a population parameter along with a confidence level (commonly 95%), indicating how likely it is that the interval contains the true value. For example, if a 95% confidence interval for average daily sales is [450, 550], we are 95% confident that the actual average lies within this range.

Another key part of inference is hypothesis testing, which helps us to decide whether the observed data provides enough evidence to reject a stated assumption (null hypothesis) about a population.

A Z-test is used when the sample size is large $(n > 30)$, and the population variance is known. A t-test is used for smaller samples, $(n < 30)$, or when the population variance is unknown. The Chi-square (χ^2) test is a statistical test used to examine whether there is a significant difference between observed frequencies and expected frequencies in one or more categories.

Keywords

Estimators, Confidence Intervals, Hypothesis Testing, Regression, Principle of Least Squares, Bayesian Linear Regression

Discussion

3.4.1 Population and Sample

Assume that a government has decided to implement a new social welfare program aimed at providing financial assistance to low-income families in a city. To determine the eligibility criteria and the level of assistance, the government needs to understand the income distribution of the population. The authorities will collect income data from a representative sample of households. This could provide initial insights to make calculations and predictions about the total population eligible for financial assistance.

As illustrated above, in practical scenarios, the need often arises to draw meaningful and reliable conclusions about a substantial collection of individuals or objects. However, attempting to examine the entire population, can present challenges due to its huge size or inherent complexities. To overcome these challenges, investigating only a smaller subset of this population emerges, and this subset is referred to as a sample. The primary objective behind working with samples is to make informed assessments and inferences about the larger population based on the findings derived from the sample.

The term population within the context of statistical analysis takes on a specialized meaning. In statistics, the idea of a population extends beyond the mere count of individuals or objects. It pertains to the complete set of observations or measurements that hold significance for a particular inquiry.

Sampling entails selecting and studying a representative portion of the population which becomes crucial in statistical analysis. It serves as a practical approach to grasp insights and generalizations about the population without the need to analyze the entirety of it. By judiciously selecting and studying samples, economists can infer meaningful insights, such as trends, patterns, and correlations, which can subsequently be applied to the entire population.

A statistic refers to a numerical summary derived from a sample, such as the sample mean or sample standard deviation. On the other hand, a parameter is a fixed numerical value that characterizes aspects of the population, such as the population mean or population variance. It is often unknown and needs to be estimated using sample statistics.

A population is the entire group of individuals or items of interest, while a sample is a subset of the population used for analysis.

3.4.2 Estimation

An estimator is a statistical formula or rule used to estimate an unknown population parameter based on sample data. since we usually cannot measure the entire population, we take a sample and use that sample to estimate parameters like the mean, proportion, variance, etc.

Estimation is the process of inferring the value of a population parameter using sample data.

Imagine a company wants to estimate the average salary of its employees. The company has 10,000 employees, but it surveys only 200 of them and calculates the average salary from that sample.

Scenario:1

The 200 employees are randomly chosen from different departments, job levels, and regions. The sample mean salary turns out to be ₹20,000. If this sampling process is repeated many times, the average of all those sample means would get closer and closer to the true population mean salary. This shows that the sample mean is an unbiased estimator of the population mean.

Scenario:2

Now suppose the 200 employees are selected only from the company's headquarters, where salaries are generally higher than in other branches. The sample mean salary comes out to be ₹70,000, while the actual average salary across all employees is ₹20,000. Here, the method of sample selection has systematically pushed the estimate away from the true value, making the sample mean a biased estimator.

The bias occurred not because of the formula used, but because the sampling method did not fairly represent the population.

A good estimator should possess the following properties:

1. Unbiased
2. Efficient
3. Sufficient

4. Consistent

3.4.3 Standard Error

The term "standard error" holds particular significance in the context of sampling distributions. While the standard deviation quantifies variability within a set of data points, the standard error specifically relates to the variability of sample statistics. It provides an estimation of how much the sample statistic is expected to deviate from the true population parameter, considering the inherent variability introduced by sampling.

i.e. The standard deviation of sampling distribution of a statistic is known as its standard error, and it is denoted by (S.E)

> Standard error measures the variability of a sample statistic (like the mean) from the true population value across repeated samples.

3.4.4 Confidence Interval

The probability of an estimator to lie between the interval is called the confidence level. 95% confidence level of a parameter θ to lie in the interval (c_1, c_2), we mean $P[c_1 < \theta < c_2) = 0.95$. We usually denote the confidence level by $1-\alpha$ and α is called significance level.

i.e. confidence level = 1 - significance level. 95% confidence level means 5% significance level.

The interval (c_1, c_2), is called confidence interval i.e. the interval within which the population parameter is expected to lie.

> A confidence interval is a range of values, derived from sample data, that is likely to contain the population parameter with a specified level of confidence.

3.4.5 Testing of Hypothesis

A hypothesis is like an educated guess or assumption that you make about something, which you then test to see if it is true. In statistics, a hypothesis gives direction to an investigation, it tells us what we are trying to prove or disprove.

For example, if a farmer thinks a new fertilizer will help plants grow faster, that becomes a hypothesis. The farmer can test it by applying the fertilizer to some plants and comparing their growth with plants that did not get it.

In statistical terms, a hypothesis is defined as a statement that can be tested using scientific methods. For example, a hypothesis might claim that "students who study for more than two hours daily score higher than those who don't." By gathering and analyzing relevant data, this statement can be tested to see if it holds true or not. Testing ensures that conclusions are based on objective evidence rather than subjective opinion. That is, it is a statement about a parameter that we test using the value of a statistic. For this purpose, the hypothesis should possess the following characteristics.

i) It should be clear and precise

ii) It should be capable of being tested

iii) It should state the relationship between variables.

> Testing of hypothesis is a statistical method used to decide whether there is enough evidence in a sample to support or reject a claim about a population.

3.4.6 Null and Alternate Hypothesis

We set up a hypothesis that assumes that there is no significant difference between the sample statistic and the corresponding population parameter or between two sample statistics. Such a hypothesis of no difference is called null hypothesis and is represented as $\boldsymbol{H_0}$. A hypothesis that is complement to the null hypothesis is the alternative hypothesis and is denoted by $\boldsymbol{H_1}$. A procedure for deciding whether to accept or to reject a null hypothesis or hence to reject or accept the alternative hypothesis, is called test of hypothesis.

There are usually two types of hypotheses involved: the null hypothesis (H_0), which assumes there is no effect or no difference (e.g., "the fertilizer has no impact"), and the alternative hypothesis (H_1), which assumes there is an effect (e.g., "the fertilizer helps plants grow faster").

> The null hypothesis states that there is no effect or difference, while the alternate hypothesis states that there is an effect or difference.

For example, suppose one wishes to test whether the mean of a population $\boldsymbol{\mu}$ is 100. We can take $H_0: \mu = 100$. If this null hypothesis is rejected, then it would mean any of the three possible alternatives.

i.e. $\mu \neq 100$, $\mu < 100$, $\mu > 100$

Thus we can test $H_0: \mu = 100$ against

i) $H_0: \mu = 100$, $H_1: \mu \neq 100$

ii) $H_0: \mu = 100$, $H_1: \mu < 100$

iii) $H_0: \mu = 100$, $H_1: \mu > 100$

The hypothesis $\mu = 100$ is called simple hypothesis and the hypothesis, $\mu < 100$, $\mu > 100$ are called composite hypothesis.

3.4.7 Critical region

If we are prepared to conclude that the difference between a sample statistic and the corresponding population parameter is significant when the sample statistic falls within a certain range, that range is called the critical region or rejection region.

This region complementary to the rejection region is called acceptance region.

The critical region is the set of values of a test statistic that leads to the rejection of the null hypothesis in a hypothesis test.

3.4.8 Errors in hypothesis

In hypothesis testing, we make decisions based on sample data. However, these decisions are not always perfect, errors can occur. The two common types of errors are Type I and Type II errors.

Type I error

A Type I error is rejecting the null hypothesis H_0 when it is true. The probability of type I error is denoted by α and is called the level of significance.

Errors in hypothesis testing occur when we make incorrect conclusions about a population based on sample data.

For example, imagine a new COVID-19 test that is designed to detect whether a person is infected. The null hypothesis (H_0) states that the person does not have COVID-19, while the alternative hypothesis (H_1) states that the person does have COVID-19.

i.e. H_0: Person does not have COVID-19

H_1: Person has COVID-19

The test fails to detect the virus, so we do not reject the null hypothesis (we accept the person as healthy), even though the person is truly infected.

Type II error

A Type II error is accepting the null hypothesis H_0 when it is false. The probability of type II error is denoted by β and is called the power of the test.

For example,

A dangerous item (like a weapon) is missed during screening.

H_0: No weapon is present.

H_1: A weapon is present.

Type II error: The weapon is there, but the system fails to detect it.

One tail and two-tail tests

A test of any statistical hypothesis where the alternative hypothesis is one tailed (right tailed or left tailed) is called one tailed test (Fig 3.4.1).

For example, in a test for testing the mean of a population in a single tailed test we assume that the null hypothesis $H_0: \mu = \mu_0$ against the alternate hypothesis

$H_1: \mu > \mu_0$ (Right tailed)

$H_1 : \mu < \mu_0$ (Left tailed)

is called one tailed test.

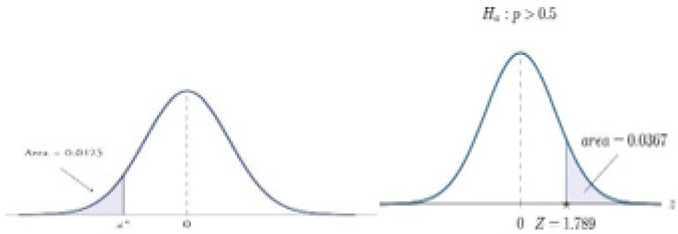


Fig 3.4.1 Left tailed curve and Right tailed curve

In a test of any statistical hypothesis where the alternative hypothesis is two tailed, Fig 3.4.2 we assume that the null hypothesis

$$H_0 : \mu = \mu_0$$

against the alternate hypothesis

$$H_1 : \mu \neq \mu_0$$

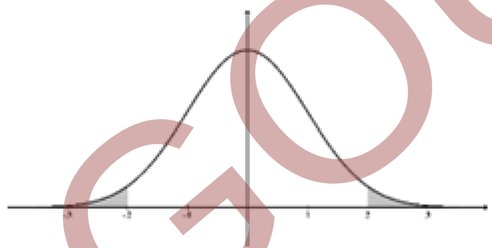


Fig 3.4.2 Two tailed curve

3.4.9 Procedure for testing of hypothesis

Applying a statistical test to any null hypothesis follows eight general steps:

1. Provide a statement of the null hypothesis.
2. Set the level of risk associated with the null hypothesis (significance level).
3. Select the appropriate test statistic $z = \frac{t-\mu}{SE}$
4. Compute the test statistic value (also known as the obtained value).
5. Determine the value (the critical value) needed for rejection of the null hypothesis using the appropriate table of critical values for that particular statistic.

If $|z| < 1.96$, H_0 may be accepted at 5% level of significance

If $|z| > 1.96$, H_0 may be rejected at 5% level of significance

If $|z| < 2.58$, H_0 may be accepted at 1% level of significance

If $|z| > 2.58$, H_0 may be rejected at 1% level of significance

For a single tail test (Right tailed or Left tailed) we compare the computed value of $|z|$ with 1.645 (at 5% LOS) and 2.33 (at 1% LOS) and accept or reject H_0 accordingly.

6. Compare the obtained value with the critical value.
7. If the obtained value is more extreme than the critical value, the null hypothesis must be rejected.
8. If the obtained value does not exceed the critical value, the null hypothesis cannot be rejected.

3.4.10 Test of significance with respect to Mean

We usually conduct the test of significance with respect to mean when the sample consists of 'n' $(n > 30)$ independent observations drawn from a normal distribution. The n independent random variables, follow a normal distribution with mean (μ), and variance (σ^2).

Symbolically, $X \sim N(\mu, \sigma)$

The following are the key steps to conduct the test of significance with respect to mean.

Null Hypothesis $H_0: \mu = \mu_0$ (a value)

$Z = \frac{\bar{X}-\mu}{SE}$ which follows a standard normal distribution.

If the level of significance $\alpha = 0.05$

$$|Z| = \frac{|\bar{X} - \mu_0|}{\frac{\sigma}{\sqrt{n}}}$$

If $|Z| > 1.96 \Rightarrow$ Reject the Null Hypothesis.

If $|Z| < 1.96 \Rightarrow$ Do does not reject the Null Hypothesis.

If the level of significance $\alpha = 0.01$

$|Z| > 2.58 \Rightarrow$ Reject Null Hypothesis

$|Z| < 2.58 \Rightarrow$ Do does not reject Null Hypothesis

Illustration.3.4.1

A motor car company claims that their car average is 35 miles per gallon of petrol. A random sample of 50 cars was tested and found to given an average of 32 miles per gallon. With a standard deviation of 1.2 gallons, test the null hypothesis μ = 35 against $H_1: \mu \neq 35$. 5% LOS.

Solution

Given μ = 35, $\sigma = 1.2,\ \bar{x} = 32,\ n = 50$

Let $H_0: \mu = 35$, $H_1: \mu \neq 35$

As the sample size n is large the test statistic is

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$Z = \frac{32 - 35}{\frac{1.2}{\sqrt{50}}}$$

$$= 17.25 > 1.96$$

H_1 is two sided and hence 5% LOS, from the normal table, $Z_{\frac{\alpha}{2}} = 1.96$. Since $|Z| > 1.96$

Reject the Null Hypothesis.

Illustration.3.4.2

Suppose that a census of city dwellers reveals an average family size of 4.2 with a standard deviation of 0.5. A random sample of 100 city side families reveals a family size of 4.29. We wish to test whether the family size in the city side is the same as in the city. 5% LOS.

Solution

Given μ = 4.2, $\sigma = 0.5,\ \bar{x} = 4.29\ ,\ n = 100$

$H_0: \mu = 4.2$

$H_1: \mu \neq 4.2$

$$|Z| = \frac{|\overline{X} - 4.2|}{\frac{0.5}{10}} = \frac{|4.29 - 4.2|}{0.05} = \frac{0.09}{0.05} = \frac{9}{5} = 1.8$$

1.8 < 1.96 ⟹ Do does not reject H_0 at 5% LOS.

Illustration.3.4.3

A random sample of boots owned by 40 soldiers in a desert region showed an average life of 1.08 years with a standard deviation of 0.5 years. Under standard conditions, the boots are known to have an average life of 1.28 years. Is there a reason to assert at a level of significance of 0.01 that use in deserts causes the mean life of such boots to decrease. 1% LOS.

Solution

Given μ = 1.28, $\sigma = 0.5,\ \bar{x} = 1.08,\ n = 40$

$H_0: \mu = 1.28$

$H_1 : \mu < 1.28$

$$|Z| = \frac{|\bar{x} - 1.28|}{\frac{0.5}{\sqrt{40}}} = \frac{|1.08 - 1.28|}{\frac{0.5}{6.32}} = \frac{0.2}{0.08} = 2.5 > 1.65 \text{ at 1\% LOS}$$

Reject H_0

Illustration.3.4.4

It is claimed that a random sample of 100 tyres with mean life of 15269 kms is drawn from a population of tyres which has a mean life of 15200 kms and standard deviation is 124.8 kms. Test the validity of the claim at 5% LOS.

Solution

Given μ = 15200, $\sigma = 124.8,\ \bar{x} = 15269,\ n = 100$

$H_0 : \mu = 15200$

$H_1 : \mu \neq 15200$

$$|Z| = \frac{|15269 - 15200|}{\frac{1248}{\sqrt{100}}} = \frac{69}{124.8} = 0.055 < 1.96$$

Do not reject H_0.

Illustration.3.4.5

An educator claims that the average I.Q of American college students is at most 110 and that in a study made to test his claim, 150 American college students had an average I Q of 111.2 with a standard deviation 7.2. At 1% LOS test the claim of the educator.

Solution

$H_0 : \mu = 110$

$H_1 : \mu > 110$

$$|Z| = \frac{|111.2 - 110|}{\frac{7.2}{\sqrt{150}}} = 2.0412 < 2.33$$

Do not reject H_0. Claim of the educator is valid.

3.4.11 t-test

A t-test is a statistical test used to compare the means of two groups to determine if there is a significant difference between them. It helps assess whether the differences observed in sample data are due to actual effects or just random chance. The t-test assumes that the data is normally distributed and that the variances of the two groups are equal (in the case of an independent t-test).

When the size of the sample is less than 30, then that sample is small sample. For small sample we use Student's t test or t test.

3.4.12 Hypothesis concerning Mean

To test whether the difference between the sample mean $\bar{x}$ and the population mean μ is significant, we use the statistic,

$$t = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \text{ follows } N(0,1)$$

Where $\bar{x}$- sample mean, μ – population mean, σ – population standard deviation, n- number of observations.

If the standard deviation of the sample is given directly, then the test statistic,

$$t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n-1}}}$$

If the calculated value of *t* exceeds the tabulated value of t (using t table) at given level of significance and degree of freedom, then then the null hypothesis H_0 is rejected.

If the calculated value of *t* is less than the tabulated value of t at given level of significance, then then the null hypothesis H_0 is accepted.

3.4.13 Student t test for difference of Mean

There are situations where the distribution of difference in means. Then we need to adopt the hypothesis tests for difference.

If the population standard deviation is not known, we use the test statistic $t = \frac{\bar{x}_1 - \bar{x}_2}{SE}$

Which follows t – distribution with $n_1 + n_2$ - 2 degree of freedom, where

$$SE = \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

If σ is not known, we may assume that $\sigma = \frac{\sqrt{n_1 s_1^2 + n_2 s_2^2}}{n_1 + n_2 - 2}$

Illustration.3.4.6

A machinist is expected to make engine parts with axil diameter of 1.75 cm. A random sample of 10 parts shows a mean diameter 1.85 cm with a SD of 0.1 cm. On the basis of the sample, would you say that the work of the machinist is interior? Is the claim acceptable at 5% LOS.

Solution

Here, $n = 10$, $\bar{x} = 1.85$, $s = 0.1$, $\mu = 1.75$

$H_0: \bar{x} = \mu$

$H_1: \bar{x} \neq \mu$

Two tailed test is used.

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n-1}}}$$

$$t = \frac{1.85 - 1.75}{\frac{0.1}{\sqrt{10-1}}}$$

$$= \frac{0.1}{\frac{0.1}{3}} = 3$$

Degree of freedom = $n - 1 = 10 - 1 = 9$

From t table with degree of freedom $= 9$ $t_{0.05} = 2.26$

$|t| = 3 > 2.26$

Therefore, H_0 is rejected and H_1 is accepted.

Illustration.3.4.7

The mean life of a sample of 25 bulbs is found as 1550 hours and SD of 120 hrs. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 hrs. Is the claim acceptable at 5% LOS.

Solution

Here, $n = 25$, $\bar{x} = 1550$, $s = 120$, $\mu = 1600$

$H_0: \bar{x} = \mu$

$H_1: \bar{x} < \mu$

One tailed test is used.

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n-1}}}$$

$$t = \frac{1550 - 1600}{\frac{120}{\sqrt{25-1}}}$$

$$= \frac{-50 \times \sqrt{24}}{120} = -2.04$$

Degree of freedom = $n-1=25-1=24$

From t table with $\vartheta=24$ $t_{0.05}=1.71$

$|t|=2.04>1.71$

Therefore H_0 is rejected and H_1 is accepted.

Illustration.3.4.8

The average number of articles produced by two machines per day are 200 and 250 with standard deviation 20 and 25 respectively on the basis of records of 25 days production. Can you regard both the machines equally effective at 1% LOS.

Solution

$H_0: \mu_1=\mu_2$

$H_1: \mu_1 \neq \mu_2$

$n_1=25,\ n_2=25, \overline{x_1}=200, \overline{x_2}=250, s_1=20,\ s_2=25$

So that $\sigma=\sqrt{\frac{n_1 s_1^2+n_2 s_2^2}{n_1+n_2-2}}=\sqrt{\frac{25\times 20^2+25\times 25^2}{25+25-2}}$

$=\sqrt{533.85}=23.1$

The SE $=\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}=23.1\sqrt{\frac{1}{25}+\frac{1}{25}}=6.53$

The test statistic $t=\frac{\overline{x_1}-\overline{x_2}}{SE}$

$=\frac{200-250}{6.53}=-7.65$

From t table with $\vartheta=25+25-2=48$ $t_{0.01}=2.58$

$|t|=7.65>2.58$

Therefore H_0 is rejected and H_1 is accepted.

Illustration.3.4.9

The mean height and SD height of 8 randomly chosen soldiers are 166.9 and 8.29 cm respectively. The corresponding vales of 6 randomly chosen sailors are 170.3 and 8.5 cm respectively . Based on this data, can we conclude that soldiers are, in general, shorter than sailors?

Solution

$H_0: \overline{x_1}=\overline{x_2}$

$H_1: \overline{x_1} \neq \overline{x_2}$

Here $n_1 = 8,\ n_2 = 6, \overline{x_1} = 166.9, \overline{x_2} = 170.3, s_1 = 8.29,\ s_2 = 8.5$

So that $\sigma = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}} = \sqrt{\frac{8\times8.29^2+6\times8.5^2}{8+6-2}} = \sqrt{\frac{983.3}{12}} = 9.05$

The SE $= \sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = \sqrt{\frac{983}{12}}\sqrt{\frac{1}{8}+\frac{1}{6}} = 9.05 \times 0.54 = 4.887$

The test statistic $t = \frac{\overline{x_1} - \overline{x_2}}{SE}$

$$= \frac{166.9 - 170.3}{4.887} = -0.695$$

From t table with $\vartheta = 8 + 6 - 2 = 12\ t_{0.01} = 1.78$

$|t| = 0.695 < 1.78$

Therefore H_0 is accepted and H_1 is rejected.

3.4.14 Chi-square test of goodness of fit

Suppose we are given a set of observed frequencies from an experiment and wish to determine whether these results support a specific hypothesis or theory. To address this, Karl Pearson developed the chi-square test, a statistical method used to assess the significance of the discrepancy between the observed frequencies and the theoretical frequencies predicted by the hypothesis.

This test is known as Chi-square test, denoted by χ^2, of goodness of fit.

The test statistic $\chi^2 = \sum \frac{(O-E)^2}{E}$

Where O – observed frequency

E – expected frequency

χ^2 is used to test whether the difference between observed and expected frequencies are significant.

Illustration.3.4.10

The number of automobile accidents per week in a certain community are as follows: 12,8,20,2,14,10,15,6,9,4. Are these frequencies in agreement with a belief that accident conditions were the same during this 10week period

Solution

H_0: The accident conditions were the same

Observed frequency (O)	Expected frequency (E)	$O - E$	$\frac{(O-E)^2}{E}$
12	10	2	0.4
8	10	-2	0.4
20	10	10	10.0
2	10	-8	6.4
14	10	4	1.6
10	10	0	0
15	10	5	2.5
6	10	-4	1.6
9	10	-1	0.1
4	10	-6	3.6
	100		26.6

Calculated $\chi^2 = \sum \frac{(O-E)^2}{E} = 26.6$

No. of observations $n = 10$

∴ degree of freedom $n - 1 = 9$

χ^2 table value degree of freedom $9 = 16.9 < 26.6$

The null hypothesis is rejected.

3.4.15 Regression

Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of the data.

In regression analysis there are two types of variables. The variable whose value is influenced or is to be predicted is called dependent variable and the variable which influences the values or is used for prediction is called independent variable. In regression analysis, independent variable is also known as regressor or predictor or explanatory variable while the dependent variable is also known as regressed or explained variable.

> Regression is a statistical method used to model and analyze the relationship between a dependent variable and one or more independent variables.

3.4.16 Method of Least Squares

The least-squares regression method is a technique commonly used in Regression Analysis. It is a mathematical method used to find the best fit line that represents the relationship between an independent and dependent variable. It aims to minimize the sum of the squared differences between the observed data and the corresponding values predicted by the model. To understand the least-squares regression method let's get familiar with the concepts involved in formulating the line of best fit.

The mathematical expression capturing the connection between an independent variable and a dependent variable takes the form of a linear regression line, often represented as a straight-line equation y = ax + b, where 'a' and 'b' are constants. When we are presented with an observed pair (x, y), they signify the values of the related independent and dependent variables.

For each data point, calculate the difference between the observed value and the predicted value derived from the model. This difference is referred to as residuals, essentially capturing the disparity between actual and estimated outcomes.

By applying mathematical optimization techniques to minimize the sum of squares of these residuals, we reach a simplified yet powerful outcome. This process allows us to uncover the most suitable linear regression line that best fits the data, making the relationship between variables clearer and more predictive.

The method of least squares is a statistical technique that estimates parameters by minimizing the sum of the squared differences between observed and predicted values.

Apply mathematical optimization to minimize the sum of squares of the residual, we get the normal equations,

$$\sum x = Na + b\sum y$$

$$\sum xy = a\sum y + b\sum y^2$$

Solving these equations and substituting the values of a and b in the equation y = ax+b we get the equation of the regression lines.

If the variables in a bivariate distribution are related, we will find that the points in the scatter diagram will cluster round some curve called the "curve of regression". In a bivariate distribution where variables are interconnected, the scatter diagram's points tend to cluster around a curve known as the "curve of regression". Should this curve adopt a linear shape, it's termed the "line of regression", signifying a linear regression between the variables. Conversely, if the curve deviates from a straight line, the regression is termed "curvilinear."

For the scenario of two variables, x and y, we end up with two regression lines: one for x on y and the other for y on x. The regression line of y on x yields the most probable y values for given x values, while the regression line of x on y provides the most probable x values for given y values. Consequently, we have two regression equations that aid in understanding the predictive relationships between these variables.

Illustration.3.4.11

From the following data, obtain the two regression equations by the method of least square

X	10	6	10	6	8
Y	6	2	10	4	8

Solution

X	Y	$x \times y$	x^2	y^2
10	6	60	100	36
6	2	12	36	04
10	10	100	100	100
6	4	24	36	16
8	8	64	64	64
Total-40	30	260	336	220

Regression equation y on x is given by y = a + bx

To determine the value of constants "a" and "b", the following two normal equations are to be solved;

$\sum y = Na + b\sum x$

$\sum xy = a\sum x + b\sum x^2$

Substituting the values in the equation, we get;

$30 = 5a + 40b$

$260 = 40a + 336b$

Multiplying equation 1 by 8 we get;

$240 = 40a + 320b$

$260 = 40a + 336b$

Subtracting equation (4) from (3), we get;

$b = \frac{20}{16} = 1.25$

This value of "b" can be substituted in equation (1), we get the value of "a". That is;

$30 = 5a + 40x1.25$

$30 = 5a + 50$

$a = {}^{-}4$

Substituting the values of "a" and "b" in the regression equation, we get the regression line of y on x, y = 12x - 4

Now we can calculate the regression equation of x on y, that is given by the equation; x = a + by, and the two normal equations are

$$\sum x = N a + b \sum y$$

$$\sum xy = a \sum y + b \sum y^2$$

Substituting the values in the equation, we get;

$$40 = 5a + 30b$$

$$260 = 30a + 220b$$

Multiplying equation (1) by 6, we get

$$240 = 30a + 180b$$

$$260 = 30a + 220b$$

$b = \frac{20}{40} = 0.5$

Substituting the value of "b" in equation (1), we get;

a = 25 = 5

the regression line of x on y is $x = 5 + 0.5y$

Illustration.3.4.12

From the following data, obtain the two regression equations by the method of least square

x	6	2	10	4	8
y	9	11	5	8	7

Solution

x	Y	$x \times y$	x^2	y^2
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32

8	7	64	49	56
Total=30	40	220	340	214

Regression equation y on x is given by $y = a + b\,x$

two normal equations are

$$\sum y = Na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Substituting the values in the equation, we get;

$$40 = 5a + 30b$$

$$214 = 30a + 220b$$

Solving the equations we get,

$$a = 11.9, \quad b = -0.65$$

Regression equation y on x is given by $y = 11.9 - 0.65\,x$

Regression equation x on y is given by $x = a + b\,y$

two normal equations are

$$\sum x = Na + b\sum y$$

$$\sum xy = a\sum y + b\sum y^2$$

Substituting the values in the equation, we get;

$$30 = 50a + 40b$$

$$214 = 40a + 340b$$

Solving the equations we get,

$$a = \frac{41}{385}, \quad b = \frac{95}{154}$$

Regression equation y on x is given by $y = \frac{41}{385} + \frac{95}{154}\,x$

3.4.17 Bayesian Linear Regression

Bayesian Linear Regression is a probabilistic approach to linear regression that incorporates prior beliefs about the model parameters and updates these beliefs using observed data through Bayes' Theorem. Unlike classical linear regression, which provides point estimates for the model coefficients, Bayesian Linear Regression treats the coefficients as random variables with a prior distribution, typically chosen as a

Gaussian for mathematical convenience. When new data is introduced, the prior is updated to form a posterior distribution, which reflects both the prior beliefs and the evidence from the data. This approach not only provides a more flexible framework for inference but also quantifies uncertainty in the model predictions through the posterior distribution. It is particularly useful when dealing with small datasets or when prior information is available, as it avoids overfitting and provides regularization inherently. Additionally, predictions in Bayesian Linear Regression are made by integrating over the posterior distribution of the parameters, resulting in predictive distributions rather than single-point estimates, thus offering a complete probabilistic view of uncertainty in both model parameters and predictions.

> Bayesian Linear Regression is a statistical method that applies Bayes' Theorem to estimate the distribution of model parameters, incorporating prior beliefs and observed data.

Summarized Overview

Statistical inference involves drawing conclusions about a population based on sample data, using tools like *estimators*, which provide approximate values of population parameters. An *unbiased estimator* yields estimates that, on average, equal the true parameter value. *Confidence intervals* give a range of reasonable values for a parameter, expressing estimation uncertainty. *Hypothesis testing* evaluates assumptions about a population through tests like the *Z-test* (used when population variance is known), *t-test* (for small samples or unknown variance), and *Chi-square test* (for categorical data or variance comparisons). In regression analysis, *linear regression* models the relationship between a dependent and one or more independent variables. The *principle of least squares* estimates regression coefficients by minimizing the sum of squared errors between observed and predicted values. *Bayesian linear regression* incorporates prior beliefs about model parameters and updates them with data, producing a probabilistic interpretation of regression outcomes.

Assignments

1. A light bulb company claims that the 100-watt light bulb it sells has an average life of 1200 hours with a standard deviation of 100 hours. For testing the claim 50 new bulbs were selected randomly and allowed to burn out. The average lifetime of these bulbs was found to be 1180 hours. Is the company's claim is true at 5% level of significance?
2. In two samples of women from Punjab and Tamilnadu, the mean height of 1000 and 2000 women are 67.6 and 68.0 inches respectively. If population standard deviation of Punjab and Tamilnadu are same and equal to 5.5 inches then, can the mean heights of Punjab and Tamilnadu women be regarded as same at 1% level of significance?
3. A machine produces a large number of items out of which 25% are found to be defective. To check this, company manager takes a random sample of 100 items and found 35 items defective. Is there an evidence of more deterioration of quality at 5% level of significance?
4. In a large population 30% of a random sample of 1200 persons had blue eyes and 20% of a random sample of 900 persons had the same blue eyes in another population. Test the proportion of blue-eyes persons is same in two populations at 5% level of significance.
5. In a sample of 100 MSc. Economics first year students of a University, it was seen that 54 students came from Science background and the rest are Large Sample Tests from other background. Can we assume that 50% of the students are from Science background in MSc. Economics first year students in the University at 1% level of significance?
6. A manufacturer claims that a special type of projector bulb has an average life 160 hours. To check this claim an investigator takes a sample of 20 such bulbs, puts on the test, and obtains an average life 167 hours with standard deviation 16 hours. Assuming that the life time of such bulbs follows normal distribution, does the investigator accept the manufacturer's claim at 5% level of significance?
7. The expected lifetime of electric light bulbs produced by a given process was 1500 hours. To test a new batch a sample of 10 was taken which showed a mean lifetime of 1410 hours. The standard deviation is 90 hours. Test the hypothesis that the mean lifetime of the electric light bulbs has not changed, using a level of significance of $\alpha = 0.05$.
8. The means of two random samples of sizes 10 and 8 drawn from two normal populations are 210.40 and 208.92 respectively. The sum of squares of the deviations from their means is 26.94 and 24.50 respectively. Assuming that the populations are normal with equal variances, can samples be considered to have been drawn from normal populations having equal mean.

9. The following data relate to the age of husbands and wives. Obtain the two regression equations and determine the most likely age of husband when the age of wife is 25 years.

X	25	28	30	32	35	36	38	39	42	55
Y	20	26	29	30	25	18	26	35	35	46

The following table shows the exports of raw cotton and the imports of manufactured goods into India for seven years

Exports (in Crores of Rs	42	44	58	55	89	98	60
Imports (in Crores of Rs)	56	49	53	58	67	76	58

Obtain the two regression equations and estimate the imports when exports in particular year were to the value of Rs 70 crores?

Suggested Reading

1. Fundamentals of Mathematical Statistics, S.C Gupta & V K Kapoor, Sultan Chand & Sons Educational publishers.
2. Statistical Methods, S.P. Gupta, Sultan Chand and Sons, New Delhi

Reference

1. Sheldon Ross – *A First Course in Probability*, 10th ed., Pearson, 2018.
2. Ronald E. Walpole, Raymond H. Myers, et al. – *Probability and Statistics for Engineers and Scientists*, 9th ed., Pearson, 2016.
3. John Freund, Irwin Miller – *Probability and Statistics for Engineers*, 8th ed., Pearson, 2010.
4. Richard Johnson – *Miller and Freund's Probability and Statistics for Engineers*, 9th ed., Pearson, 2016

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

BLOCK 4

Calculus and Optimization

1
UNIT

Functions and Limits

Learning Outcomes

After completion of this unit, the learner will be able to:

- identify and describe functions of one and multiple variables
- find the domain and range of given functions
- calculate limits of functions of one and multiple variables
- apply continuity concepts in solving real-world and mathematical problems

Background

In our daily life, we come across many patterns that characterise relations such as father and son, teacher, and student etc. In mathematics, an example of a relation is 'less than' so that x is related to y if $x < y$. Relations are usually defined in terms of an ordered pair. Functions are a special type of relations.

Functions and limits are basic ideas in mathematics that describe how one quantity depends on another and how values behave as we get close to a certain point. A function is like a machine that takes an input (from its domain, the set of all possible inputs) and gives an output (from its range, the set of all possible outputs). For example, the price of a taxi ride might be a function of the distance travelled, distance, the higher the price. Functions can have one variable (like temperature changing with time) or multiple variables (like the speed of a car depending on both fuel level and engine power). A limit tells us what value a function approaches when the input gets very close to a certain point, for instance, as you walk closer to a wall, your distance to it approaches zero. Continuity means the function has no sudden jumps or breaks; you can draw its graph without lifting your pen. These concepts are important in fields like physics, engineering, and economics because they help model and predict real-world situations smoothly and accurately.

Keywords

Functions of Variables, Domain, Range, Limits, Continuity

Discussion

4.1.1 Function

Many scientific laws and engineering principles describe how one quantity changes in relation to another. This concept was formally introduced in 1673 by Gottfried Wilhelm Leibniz, who invented the term *function* to represent the dependence of one quantity on another.

For example, in physics, the distance an object travels depends on its speed and the time it moves; in engineering, the stress on a beam depends on the load applied and its material properties. By expressing these relationships as functions, we can analyze, predict, and control various real-world systems with mathematical precision.

4.1.2 Function of One Variable

A function of one variable is a rule where the output depends on a single input. For example, the height of water in a tank might depend only on the time it has been filling written as $h(t)$. Here, t is the only variable that changes, and each value of t gives exactly one height h.

If a variable y depends on a variable x in such a way that each value of x corresponds to exactly one value of y, then y is called a one variable function of x. For example, if you input a number into a calculator and it always gives you one definite result, like entering 5 into a square function gives 25, that is a function. However, if a single value of x could give two or more different y values, then it would not be considered a proper function.The formula $C = 2\pi r$ shows the circumference (C) of a circle as a function of its radius (r). Here, for every specific value of r, there is exactly one corresponding value of C. For example, if the radius is 3 cm, the circumference will always be $\boldsymbol{C = 2 \times \pi \times 3 = 6\pi\, cm}$. This one-to-one relationship between the radius and the circumference.

i.e., a function f is a rule that assigns one unique output to each input. If the input is represented by x, the output is written as $\boldsymbol{f(x)}$. For example, if $\boldsymbol{f(x) = x^2}$, then for an input of x = 4, the output is $f(4) = 4^2 = 16$.

Function f is a rule that assigns one unique output to each input.

4.1.3 Function of Multiple Variables

A function of multiple variables is a rule where the output depends on two or more inputs. Instead of changing along just one direction (like in single-variable functions), the output here changes based on variations in multiple directions.

For example, the profit of a shop can depends on both the price of an item (p) and the number of items sold (q). We can write this as $\boldsymbol{P(p,q)}$. If either the price changes or the number sold changes, the profit changes too.

The temperature in a city, which might depend on time of day (t) and altitude (h). This can be written as $\boldsymbol{T(t,h)}$.

The fuel efficiency of a car depends on speed, engine size, and road conditions all acting as inputs to the function. In real life, most systems are functions of multiple variables because many factors work together to determine an outcome.

> A function of multiple variables is a rule where the output depends on two or more inputs.

4.1.4 Domain and Range

If x and y are related by the equation $\boldsymbol{y = f(x)}$, then the domain of f is the set of all allowable input values (x-values) for which the function is defined.

The range of f is the set of all possible output values (y-values) that result when x takes on values from the domain.

For example, if $\boldsymbol{f(x) = \sqrt{x}}$, the domain is $\boldsymbol{x \geq 0}$ (since square roots of negative numbers are not real), and the range is $\boldsymbol{y \geq 0}$ (outputs are never negative).

When a function is given by a mathematical formula, the formula itself may place restrictions on the allowable inputs.

For example: consider the function $\boldsymbol{y = \frac{1}{x}, x = 0}$ is not allowed because division by zero is undefined.

If $\boldsymbol{y = \sqrt{x}}$, then negative values of x are not allowed because they produce imaginary numbers, and we are considering only real-valued functions of a real variable.

If a real-valued function of a real variable is defined by a formula without an explicitly stated domain, we assume the domain to be the set of all real numbers for which the formula yields a real value. This set is known as the natural domain of the function.

> set of all input values are domain and the set of all output values are range.

Illustration 4.1.1

Let $A = \{1,2,3,4,5\}$, $B = \{a,b,c,d\}$. $f: A \to B$ such that

$f = \{(1,a)\ (2,b)\ (3,b)\ (4,c)\ (5,c)\}$. Find a) Domain of f b) Range of f

Solution:

Domain of $f = A$ b) Range of $f = \{a, b, c\}$

Illustration 4.1.2

Find the natural domain of $(a) f(x) = x^3$

$$(b) f(x) = \frac{1}{[(x-1)(x-3)]}$$

$$(c)\ f(x) = \sqrt{x^2 - 5x + 6}$$

Solution

a. The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

b. The function f has real values for all real x, except x = 1 and x = 3. Thus, the natural domain is

$$\{x : x = 1 \ and \ x = 3\}$$

c. The function f has real values only when the expression inside the radical (square root) is non-negative. If the expression inside the radical becomes negative, the output would be imaginary, which is not allowed when we are dealing with real-valued functions. Thus, the natural domine is $\{x : x \neq 2 \ and \ x \neq 3\}$.

Illustration 4.1.3

Find the domain and range of each of the following functions.

1. $f(x) = 5x - 3$
2. $g(t) = \sqrt{4 - 7t}$
3. $f(z) = |z - 6| - 3$
4. $g(x) = 8$

Solution

1. The function $f(x) = 5x - 3$ is a linear function, and there is no restriction.

Therefore, the domain is all real numbers.
Domain is $(-\infty, \infty)$

A linear function can produce any real value depending on x. Therefore, the range is also all real numbers. Range is $(-\infty, \infty)$

2. Since we have a square root, the expression inside must be non-negative:

$\mathbf{4 - 7t \geq 0}$

$$t \geq \frac{4}{7}$$

Domain is $\left(-\infty, \frac{4}{7}\right)$, Range is $[0, \infty)$

3. This function contains an absolute value, and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if z=6 and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting three from the absolute value portion and so range will be $[-3, \infty)$ Domain is $(-\infty, \infty)$

4. Domain is $(-\infty, \infty)$, Range-8.

4.1.5 Operations On Functions

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f + g$, $f - g$, fg, and f/g.

Given functions f and g, we define

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)(f/g)(x) = f(x)/g(x)$$

Illustration 4.1.4

Let $f(x) = 1 + \sqrt{x-2}$ and $g(x) = x - 3$

Find the domains and formulas for the functions

$f + g,\ f - g,\ fg,\ f/g, and\ 7f.$

Solution

$$(f + g)(x) = f(x) + g(x) = \left(1 + \sqrt{x-2}\right) + x - 3$$
$$= x - 2 + \sqrt{x-2}$$

$$(f - g)(x) = f(x) - g(x) = \left(1 + \sqrt{x-2}\right) - (x - 3)$$
$$= 4 - x + \sqrt{x-2}$$

$$fg(x) = (1 + \sqrt{x-2})(x - 3)$$

$$\frac{f}{g} = \frac{1 + \sqrt{x-2}}{x-3}$$

The domains of f is $[2, +\infty)$ and g is $(-\infty, +\infty)$, respectively.

The domain of $f + g,\ f - g,\ 7f$ and fg is $[2, +\infty)$

For $f(x)$, we need $x-2 \geq 0$, $x \geq 2$ (because of the square root).

For $g(x)$, denominator $x-3 \neq 0 \rightarrow x \neq 3$.

So, the domine of $\frac{f}{g}$ is $[2,+\infty)-3$

Illustration 4.1.5

Find formulas for $f+g, f-g, fg, and\ f/g$, and state the domains of the functions.

$$f(x) = 2\sqrt{x-1} \text{ and } g(x) = \sqrt{x-1}$$

Solution

$$(f+g)(x) = f(x) + g(x) = \left(2\sqrt{x-1}\right) + \sqrt{x-1}$$
$$= 3\sqrt{x-1}$$

$$(f-g)(x) = f(x) - g(x) = \left(2\sqrt{x-1}\right) - \sqrt{x-1} = \sqrt{x-1}$$

$$fg(x) = \left(2\sqrt{x-1}\right)\sqrt{x-1} = 2(x-1)$$

$$\frac{f}{g} = 2$$

Domine of $f+g, f-g, fg$ is $[1,\infty)$. Domine of f/g is $(1,\infty)$.

Illustration 4.1.6

Show that if $f(x) = \sqrt{x}, g(x) = \sqrt{x}$, and $h(x) = x$, then the domain of fg is not the same as the natural domain of h.

Solution

Domain of $f(x) = \sqrt{x}$ is $[0,\infty)$. domain of $g(x) = \sqrt{x}$ is $[0,\infty)$ since

$$(fg)(x) = \sqrt{x}.\sqrt{x} = x = h(x)$$

domain of $h(x) = x$ is $(-\infty,+\infty)$.

domain of $fg = [0,\infty) \neq (-\infty,+\infty) = \text{domain of } h(x)$

This shows that two functions with the same formula can have different domains depending on how they are defined.

4.1.6 Composite function

If $\boldsymbol{f}$ and g be two functions, then the composite function of $\boldsymbol{f}$ and $\boldsymbol{g}$ is a new function denoted by $\boldsymbol{g} \circ \boldsymbol{f}$ and is defined as $(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$

Illustration 4.1.7

Let $f\ and\ g$ are two functions defined by

$f(x) = x^2, \quad g(x) = x + 5$. Find (gof) and (fog).

Solution

$(gof)(x) = g(f(x)) = g(x^2) = x^2 + 5$,

$(fog)(x) = f(g(x)) = f(x + 5) = (x + 5)^2$

Illustration 4.1.8

Let f *and* g are two functions defined by

$f(x) = 2x + 5, \quad g(x) = \frac{x-5}{2}$. Find (gof) and fog).

Solution

$(gof)(x) = g(f(x)) = g(2x + 5) = \frac{2x+5-5}{2} = x,$

$(fog)(x) = f(g(x)) = f\left(\frac{x-5}{2}\right) = x.$

Illustration 4.1.9

If $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for all $x \in R$ where R is set of real numbers, find a) fog b) gof c) $(fog)oh$ d) $hogof$ e) $fo(goh)$.

Solution

a) $(fog)(x) = f(g(x)) = f(x - 2) = x - 2 + 2 = x,$

b) $(gof)(x) = g(f(x)) = g(x + 2) = x,$

c) $(fogoh)(x) = fog(h(x)) = fog(3x) = f(g(3x)) = f(3x - 2) = 3x - 2 + 2 = 3x$

d) $(hogof)(x) = hog(f(x)) = hog(x + 2) = h(g(x + 2)) = h(x + 2 - 2) = h(x) = 3x$

e) $fo(goh)(x) = f(goh(x)) = f(g(3x) = f(3x - 2) = 3x - 2 + 2 = 3x$.

4.1.7 Limit

The concept of a limit is a fundamental idea in calculus that describes the behavior of a function as its input approaches a certain value. In simple terms, the limit tells us what value a function is getting close to, even if it never actually reaches that value.

For example, imagine walking towards a wall. As you take smaller and smaller steps, your distance from the wall gets closer and closer to zero that "approaching value" is the limit.

Mathematically, we write $\lim_{x \to a} f(x) = l$

which means as x approaches a, the value of f(x) gets closer and closer to L.

For example, the speed of a car at an instant can be found by looking at the average speed over smaller and smaller time intervals, the value it approaches is the instantaneous speed, which is defined using limits.

Consider the function $f\ (x)\ =\ x^2\ -2\ x\ +\ 2$. If we take x-values closer and closer to 2, whether from the left side (values slightly less than 2) or the right side (values slightly greater than 2), the values of $\boldsymbol{f(x)}$get closer and closer to 2.

i.e., The limit of $x^2\ -2\ x\ +\ 2$ is 2 as $\boldsymbol{x}$ approaches 2 from either side.

Mathematically, we write: $\mathbf{\lim_{x\to 2}(x^2\ -2\ x\ +\ 2\ ) = 2}$

This shows that limits capture the value a function approaches, not just the value it takes when we substitute directly (though here direct substitution also gives 2).

4.1.8 Two-sided Limit

$\lim_{x\to a} f(x) = l$ is called a two-sided limit because it requires the values of $\boldsymbol{f(x)}$ to get closer and closer to $\boldsymbol{l}$ as $\boldsymbol{x}$ approaches $\boldsymbol{a}$ from both sides, the left and the right.

However, some functions behave differently on the two sides of a particular $\boldsymbol{x}$ -value In such cases, it becomes necessary to distinguish whether the values of $\boldsymbol{x}$ near $\boldsymbol{a}$ are coming from the left side ($\boldsymbol{x < a}$) or from the right side ($\boldsymbol{x > a}$) when investigating the limiting behavior.

This leads to the concepts of:

- Left-hand limit $\lim_{x\to a^-} f(x)$ approaching a from the left.
- Right-hand limit $\lim_{x\to a^+} f(x)$ approaching a from the right.

If these two limits are equal, then the two-sided limit exists and equals that common value. If they are different, the two-sided limit does not exist.

Consider the function that $f(x) = x^2,\quad if\ x < 3$

$$= 6,\quad if\ \ x \geq 3$$

Left hand limit $\lim_{x\to 3^-} f(x) = 3^2 = 9$

Right hand limit $\lim_{x\to 3^+} f(x) = 6$

$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

$\lim_{x\to 3} f(x)$ does not exist.

Illustration 4.1.10

Does $\lim_{x\to 0} f(x)$ exist where $f(x) = \frac{|x|}{x} = \quad 1,\ if\ x > 0$

$$= -1\quad if\ x < 0$$

Solution

Left hand limit $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = -1$

Right hand limit $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = 1$

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ does not exist.

Illustration 4.1.11

Does $\lim_{x\to 0} f(x)$ exist where $f(x) \;\; = x-1 \quad if\ x \leq 3$

$= \ 3x-7 \quad if\ x > 3$

Solution

Left hand limit $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-1) = 2$

Right hand limit $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3x-7) = 2$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$\therefore \lim_{3\to 0} f(x) = 2$

Illustration 4.1.12

Find $\lim_{x\to 5} f(x)$ where $f(x) \;\; = x^2 - 4x + 3$

Solution

$\lim_{x\to 5} f(x) \; = \lim_{x\to 5} x^2 - 4x + 3 \; = 5^2 - 20 + 3 = 8$

Illustration 4.1.13

Find $\lim_{x\to 2} f(x)$ where $f(x) \;\; = \dfrac{5x^3+4}{x-3}$

Solution

$\lim_{x\to 2} f(x) \; = \lim_{x\to 5} \dfrac{5x^3+4}{x-3} = \dfrac{5.2^3+4}{2-3} = -44$

Illustration 4.1.14

Find $\lim_{x\to 2} f(x)$ where $f(x) = \dfrac{x^2-6x+9}{x-3}$

Solution

$$\lim_{x\to 2} f(x) = \lim_{x\to 5} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x\to 2} \frac{(x-3)^2}{x-3} = \lim_{x\to 2} (x-3) = -1$$

Illustration 4.1.15

Find $\lim_{x\to\infty} \frac{4x^2 - x}{2x^2 - 3}$

Solution

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^2 - x}{2x^2 - 3} = \lim_{x\to\infty} \frac{x^2\left(4 - \frac{1}{x}\right)}{x^2\left(2 - \frac{3}{x}\right)} = \lim_{x\to\infty} \frac{\left(4 - \frac{1}{x}\right)}{\left(2 - \frac{3}{x}\right)} = \frac{4}{2} = 2$$

4.1.9 Continuity

Continuity of a function is a crucial concept in calculus that describes the smooth and unbroken behaviour of the function without any abrupt jumps, gaps, or disruptions. A function is considered continuous at a specific point if its value at that point matches the limit of the function as it approaches that point. In simpler terms, a function is continuous when there are no interruptions or breaks in its graph.

> If a function is continuous then the left limit and right limits exist and are equal

Formally, a function $\boldsymbol{f(x)}$is said to be continuous at $\boldsymbol{x = a}$ if the following conditions are met:

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

For example, consider the function $f(x) = 2x + 3$. This function is continuous everywhere because it is a linear function, and there are no breaks or gaps in its graph. At any point x = a, the value of $f(x)$ matches the limit of the function as x approaches $\boldsymbol{a}$, satisfying the conditions of continuity.

On the other hand, the function $g(x) = \frac{1}{x}$ is not continuous at $x = 0$. The reason is that $g(x)$ becomes unbounded as x approaches 0 from both sides (positive and negative). The function is not defined at $x = 0$, and the limit of $g(x)$ as x approaches 0 is infinite. Thus, the conditions for continuity are not satisfied at $x = 0$, making $g(x)$ discontinuous at that point.

Illustration 4.1.16

Prove that the function $f(x) = x^2 + 1$ is continuous at $x = 2$.

Solution

$f(2) = 5.\quad \lim_{x\to 2} f(x) = 2^2 + 1 = 5$

$\therefore \lim_{x\to 2} f(x) = f(2)$. So, the function is continuous.

Illustration 4.1.17

Prove that the function $f(x) = \frac{1}{x-2}$ is discontinuous at $x = 2$.

Solution

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{1}{x-2} = \frac{1}{0} = \infty$$

Limit does not exist $x = 2$. $\therefore$ the function is discontinuous at $x = 2$.

Illustration 4.1.18

$f(x)$ is defined as $f(x) \; = 2 \;\; when\; x < 0$

$= x \;\; when\; x \geq 0$.

Prove that the function is discontinuous at $x = 0$.

Solution

$RHL = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} 0 + h = 0$

$LHL = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = 2$

$RHL \neq LHL$ Limit does not exist $x = 0 \therefore$ the function is discontinuous at $x = 0$.

Illustration 4.1.19

Determine whether the function $f(x) \; = \frac{x^2-4}{x-2}$ is continuous at x = 2.

Solution

$$f(x) \; = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

$f(2) = 2 + 2 = 4 \; exist$

$\lim_{x\to 2} f(x) \; = \lim_{x\to 2} x + 2 = 4$

$\lim_{x\to 2} f(x) = 4 = f(2)$

$\therefore$ the function is continuous at x = 2.

Illustration 4.1.20

Discuss the continuity of the function f given by

$f(x) = |x|$ at $x = 0$.

Solution

$f(0) = 0$

$f(x) \quad = -x \qquad if\ x < 0$

$\qquad = x \qquad if\ x > 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} -x = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$

$\therefore \lim\limits_{x \to 2} f(x) = 0 = f(0)$

Illustration 4.1.21

Discuss the continuity of the function f defined by $f(x) = \frac{1}{x} \quad x \neq 0$

Solution

Fix any non-zero real number c, we have

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \frac{1}{x} = \frac{1}{c}$

Since $c \neq 0$, $f(c) = \frac{1}{c}$. $\lim\limits_{x \to c} f(x) = f(c)$

hence, f is continuous at every point in the domain of f. Thus, f is a continuous function.

Illustration 4.1.22

Discuss the continuity of the function f at $x = 1$ defined by

$f(x) \quad = x + 2 \qquad if\ x \leq 1$

$\qquad = x - 2 \qquad if\ x > 1$

Solution

Left hand limit $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x - 2) = -1$

Right hand limit $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x + 2) = 3$

$\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

Since the left- and right-hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$.

Summarized Overview

Functions and Limits form the foundation of calculus and mathematical analysis. A function maps each input from its domain (the set of allowed input values) to a unique output in its range (the set of possible output values). Functions can involve one variable (e.g., $f(x) = x^2$) or multiple variables (e.g., $f(x, y) = x^2 + y^2$). The limit of a function describes its behavior as the input approaches a particular value and is crucial for defining continuity and derivatives. A function is said to be continuous at a point if its value matches the limit at that point, indicating no sudden jumps or breaks in the graph. Understanding limits and continuity is essential for analyzing the smoothness and behavior of functions in both single-variable and multivariable contexts.

Assignments

1. Find the value of $\lim\limits_{x\to 0}\left(\frac{x^2-2ax+a^2}{x-a}\right)$
2. Examine the continuity of the function $f(x) = \frac{1}{x} \quad x \neq 0 \quad = 3 \quad x = 0$
3. Find the value of k so that the function, $f(x) = \frac{x^2-16}{x-4} \quad when\ x \neq 4$

   $= k \quad when\ x = 4$ is continuous at $x = 4$.
4. $f(x)$ is defined as $f(x) = x^2 \quad if\ x \neq 1 = 2 \quad if\ x = 1$.

   Prove that the function is discontinuous at $x = 1$.
5. $f(x)$ is defined as $f(x) = x \qquad when\ 0 \leq x < \frac{1}{2}$

   $= 0 \qquad when\ x = \frac{1}{2}$

   $= 1 - x \quad when \frac{1}{2} < x \leq 1$.

   Find $\lim\limits_{x\to \frac{1}{2}} f(x)$
6. Find the value of $\lim\limits_{x\to\infty} \dfrac{9x^2+3x+7}{5x^2+2x+1}$

Suggested Reading

1. Anton, Bivens,Davis: Calculus, John Wiley and Sons: 10th Edition
2. Thomas Jr. G.B Weir MD and Hass J R. Thomas' Calculus Pearson

Reference

1. James Stewart – *Calculus: Early Transcendentals*, 8th ed., Cengage Learning, 2015.
2. E. Kreyszig – ***Advanced Engineering Mathematics***, 10th ed., Wiley, 2011.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

2
UNIT

Differentiation

Learning Outcomes

After completion of this unit, the learner will be able to:

- understand and compute derivatives of basic functions
- understand and apply derivative rules
- ability to find partial derivatives of functions with multiple variables
- apply the chain rule to differentiate composite functions

Background

Differentiation is a core concept in calculus used to measure how a function changes as its input changes. The derivative of a function represents the rate of change or the slope of the function at a point. Basic differentiation rules apply to elementary functions such as polynomials, exponentials, and trigonometric functions. For functions of multiple variables, partial derivatives measure the rate of change with respect to one variable while keeping others constant, crucial in multivariable calculus and optimization problems. The chain rule is a powerful tool used to differentiate composite functions, allowing us to compute derivatives when a variable depends on another variable that in turn depends on a third. Together, these tools provide the foundation for analyzing change in various scientific, economic, and engineering contexts.

Keywords

Derivatives, Partial Derivatives, Chain Rule

Discussion

4.2.1 Differentiability

The derivative of a function is a fundamental concept in calculus that represents the rate at which the function's output changes with respect to its input. In simpler terms, it measures how the function value responds to a slight change in the input variable. The derivative provides information about the slope of the function's graph at a particular point, indicating whether the function is increasing or decreasing and the rate at which it changes.

Differentiation is the process of finding the derivative of a function. Let y is a function of x where x is an independent variable. This is denoted as $y = 4x$. Then the derivative of this function can be defined as follows.

> A differentiable function is a function whose derivative exists at every point in its domain, meaning it has a well-defined and smooth rate of change everywhere.

Let Δ y and Δ x be small increment or change in y and x respectively, then $\frac{\Delta y}{\Delta x}$ is the incremental ratio. The value of incremental ratio when Δ x is very small is called the differential coefficient or derivative of y with respect to x, and is denoted by $\frac{dy}{dx}$.

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

It is denoted as y' or $f'(x)$ or $\frac{dy}{dx}$.

Mathematically, the derivative is defined as the limit of the average rate of change as the interval between two points on the function's graph approaches zero. It is formally expressed as:

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists and h is a small increment in x.

Alternatively, if the function $f(x)$ is expressed as a formula, the derivative can be found using differentiation rules. These rules provide shortcuts for finding the derivative of different types of functions, including constants, powers, sums, products, quotients, and more.

Derivatives have various applications in mathematics, science, and engineering. They are used to analyse the rate of change, determine maximum and minimum points of functions, understand growth and decay processes, and much more. The concept of the derivative is foundational to calculus and plays a pivotal role in understanding the behaviour of functions and their relationships.

4.2.2 Rules of Differentiation

Rule 1. Differentiation of a power of a function.

Derivative of the function $y = x^n$ is $\frac{dy}{dx} = nx^{n-1}$

Rule 2. Multiplication by a constant.

If $y = A\,x^n$, where A is a constant, then $\frac{dy}{dx} = A\,nx^{n-1}$

Rule 3. Derivative of a sum

If $y = u + v$ where $u = f(x)$ and $v = g(x)$ are functions of x, then.

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

For example, if $y = 2x^2 + 4x$, then

$$\frac{dy}{dx} = \frac{d\ (2x^2)}{dx} + \frac{d(4x)}{dx} = 2\frac{d(x^2)}{dx} + 4\frac{dx}{dx}$$

$$= 2 \times 2x + 4 = 4x + 4$$

Rule 4. Derivative of a constant.

If $y = f(x) = c$, where c is a constant, then

$$\frac{dy}{dx} = \frac{dc}{dx} = 0$$

derivative of a constant function is zero.

Rule 5. Derivative of a product.

The derivative of the product $y = uv$, where $u = f(x)$ and $v = g(x)$ is

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = uv' + vu'$$

6. Derivative of a quotient.

The derivative of the quotient $y = \frac{u}{v}$, where $u = f(x)$ and $v = g(x)$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,d(u) - u\,d(v)}{v^2}$$

Some important results of basic differentiation

1. $\frac{d}{dx}(c) = 0$ where c is a constant

2. $\frac{d}{dx}(cf(x)) = cf'(x)$

3. $\frac{d}{dx}(x) = 1$

4. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

5. $\frac{d}{dx}(x^n) = n\,x^{n-1}$

6. $\frac{d}{dx}(e^x) = e^x$

7. $\frac{d}{dx}(sinx) = cosx$

8. $\frac{d}{dx}(cosx) = -sinx$

9. $\frac{d}{dx}(tanx) = sec^2x$

10. $\frac{d}{dx}(secx) = secx\ tanx$

11. $\frac{d}{dx}(cosec\ x) = -cosec\ x\ cot\ x$

12. $\frac{d}{dx}(cotx) = -cosec^2x$

Illustration 4.2.1

Find the derivative of $a)\ y = 3x^7 - 2x^2 + 5\ \ b)\ y = \frac{1}{2}(x^4 + 7)\ \ c)\ y = \frac{x^{\frac{5}{2}} + 2x^2}{x}$

$d)\ y = 2x^6 + x^{-7}\quad e)\ y = \sqrt{x} + \frac{1}{x}$

Solution

a) $y = 3x^7 - 2x^2 + 5$

$\frac{dy}{dx} = 3 \times 7 \times x^{7-1} - 2 \times 2 \times x^{2-1} + 0$

$= 21x^6 - 4x$

b) $y = \frac{1}{2}(x^4 + 7)$

$\frac{dy}{dx} = \frac{1}{2}(4x^3)$

$c)\ y = \frac{x^{\frac{5}{2}} + 2x^2}{x} = x^{\frac{3}{2}} + 2x$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2$

$d)\ y = 2x^6 + x^{-7}$

$\frac{dy}{dx} = 12x^5 - 7x^{-8}$

$e)\ y = \sqrt{x} + \frac{1}{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$$

Illustration 4.2.2

Find the derivative of the following by apply product rule

$a)\ y = (4x^2 - 1)(7x^3 + x)$ $\quad b)\ y = (1 - x)\sqrt{x}$ $\quad c)\ y = (3x^2 + 4)(2x - \frac{1}{3})$

$d)\ y = x\ sinx$ $\qquad e)\ y = x^2\ cosx$

Solution

a) $y = (4x^2 - 1)(7x^3 + x)$

$\frac{dy}{dx} = (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x)$

$= 140x^4 - 9x^2 - 1$

b) $y = (1 - x)\sqrt{x}$

$\frac{dy}{dx} = (1 - x)\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}(-1) = \frac{1-x}{2\sqrt{x}} - \sqrt{x}$

c) $y = (3x^2 + 4)(2x - \frac{1}{3})$

$$\frac{dy}{dx} = (3x^2 + 4)(2) + \left(2x - \frac{1}{3}\right)(6x) = 18x^2 - 2x + 8$$

$d)\ y = x\ sinx$

$$\frac{dy}{dx} = x\ cosx + sinx$$

$e)\ y = x^2\ cosx$

$$\frac{dy}{dx} = x^2(-sinx) + cosx.2x = -x^2 sinx + 2x\ cosx$$

Illustration 4.2.3

Find the derivative of the following by apply Quotient rule

$a)\ y = \frac{x^3+2x^2-1}{x+5}$ $b)\ y = \frac{2x^2+5}{3x-4}$ $c)\ y = \frac{(3x+4)}{(x^2-1)}$ $d)\ y = \frac{sinx}{1+cosx}$ $e)\ y = \frac{x^2-1}{x^4+1}$

Solution

$$a)\ y = \frac{x^3+2x^2-1}{x+5}$$

$$\frac{dy}{dx} = \frac{(x+5)\frac{d}{dx}(x^3+2x^2-1) - (x^3+2x^2-1)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)1}{(x+5)^2}$$

$$= \frac{2x^3+17x^2+20x+1}{(x+5)^2}$$

$$b)\ y = \frac{2x^2+5}{3x-4}$$

$$\frac{dy}{dx} = \frac{(3x-4)\frac{d}{dx}(2x^2+5) - (2x^2+5)\frac{d}{dx}(3x-4)}{(3x-4)^2}$$

$$= \frac{(3x-4)(4x) - (2x^2+5)3}{(3x-4)^2}$$

$$= \frac{6x^2-16x-15}{(3x-4)^2}$$

$$c)\ y = \frac{(3x+4)}{(x^2-1)}$$

$$\frac{dy}{dx} = \frac{(x^2-1)3 - (3x+4)(2x)}{(x^2-1)^2}$$

$$= \frac{-3x^2-8x-3}{(x^2-1)^2}$$

$$d)\ y = \frac{sinx}{1+cosx}$$

$$\frac{dy}{dx} = \frac{(1+cosx)(-cosx) - sinx.(-sinx)}{(1+cosx)^2}$$

$$= \frac{\sin^2 x - \cos^2 x - cosx}{(1 + cosx)^2}$$

$$e)\ y = \frac{x^2 - 1}{x^4 + 1}$$

$$\frac{dy}{dx} = \frac{(x^4 + 1)(2x) - (x^2 - 1)4x^3}{(x^4 + 1)^2}$$

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4 + 1)^2}$$

Illustration 4.2.4

Find $\frac{dy}{dx}$ for the following functions.

1. $y = (3x^2 + 1)(x^3 + 2x)$
2. $y = x^2 e^{2x}$
3. $y = (ax^2 + bx)(cx^4 + dx)$

Solution

1. $\frac{dy}{dx} = (3x^2 + 1)(3x^2 + 2) + (x^3 + 2x)(6x)$

$= 14x^4 + 21x^2 + 2$

2. $\frac{dy}{dx} = x^2 e^{2x} \times 2 + e^{2x} \times 2x = e^{2x}(2x^2 + 2x)$

3. $\frac{dy}{dx} = (ax^2 + bx)(4cx^3 + d) + (2ax + b)(cx^4 + dx)$

$= 6acx^5 + 3adx^2 + 5bcx^4 + 2bdx$

4.2.3 Partial Derivatives

In mathematics, when dealing with functions of more than one variable, such as $f(x, y)$ or $f(x, y, z)$, we often want to know how the function changes when only one of the variables changes, while keeping the others constant. The derivative taken with respect to one variable, treating all other variables as constants, is called a partial derivative.

For example, if $z = f(x, y) = x^2 y + 3xy^2$, the partial derivative with respect to x measures the rate of change of z when x changes and y is fixed. It is denoted as $\frac{\partial z}{\partial x}$ or f_x. Similarly, $\frac{\partial z}{\partial y}$ or f_y represents the rate of change with respect to y while keeping x constant.

For example, If $f(x,y) = x^2y + y^3$, $f(x,y) = x^2y + y^3$, then

$\partial f/\partial x = 2xy$(treating y as constant)

$\partial f/\partial x = x^2 + 3y^2$ (treating y as constant)

If $z = f(x,y)$,then the partial derivatives f_x and f_y are also denoted by the symbols

$$f_x = \frac{\partial f}{\partial x} \quad and\ f_y = \frac{\partial f}{\partial y}$$

> Partial derivatives measure how a multivariable function changes as you vary one input variable while keeping the others constant

Illustration 4.2.5

Find the partial derivatives f_x and f_y of the function $f(x,y) = 2x^3y^2 + 2y + 4x$ at $(1,3)$.

Solution

$f(x,y) = 2x^3y^2 + 2y + 4x$

$f_x(x,y) = 6x^2y^2 + 4$

$f_x(1,3) = 54 + 4 = 58$

$f_y(x,y) = 4x^3y + 2$

$f_y(1,3) = 12 + 2 = 14$

Illustration 4.2.6

Find the partial derivatives z_x and z_y of the function.

a) $z = x^4 \sin(xy^3)$ b) $z = y^2e^x + y$

Solution

$z = x^4 \sin(xy^3)$

$$z_x = x^4 \frac{\partial}{\partial x}\sin(xy^3) + \sin(xy^3)\frac{\partial}{\partial x}x^4$$

$$= x^4 \cos(xy^3) \times y^3 + \sin(xy^3).4x^3$$

$$z_y = x^4 \frac{\partial}{\partial y}\sin(xy^3) + \sin(xy^3)\frac{\partial}{\partial y}x^4$$

$$= x^4 \cos(xy^3) \times 3xy^2$$

b) $z = y^2e^x + y$

$$z_x = y^2 \frac{\partial}{\partial x}\ e^x + e^x \frac{\partial}{\partial x} y^2 + \frac{\partial}{\partial x} y$$

$$= y^2\ e^x{}_y$$

$$z_y = y^2 \frac{\partial}{\partial y}\ e^x + e^x \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial y} y$$

$$= y^2 + e^x \times 2y + 1$$

Illustration 4.2.7

Find the partial derivatives f_x, f_y and f_z of the function.

a) $f(x, y, z) = x^3y^2z^4 + 2xy + z$

b) $f(x,y,z) = ye^z \sin\,(xz)$

Solution

a) $f(x, y, z) = x^3y^2z^4 + 2xy + z$

$$f_x = 3x^2y^2z^4 + 2y$$

$$f_y = 2x^3yz^4 + 2x$$

$$f_z = 4x^3y^2z^3 + 1$$

b) $f(x,y,z) = ye^z \sin\,(xz)$

$$f_x = ye^z \cos(xz)\, z = ye^z z \cos(xz)$$

$$f_y = e^z \sin\,(xz)$$

$$f_z = y[e^z \frac{\partial}{\partial z}(sinxz) + sinxz \frac{\partial}{\partial z} e^z]$$

$$= y[e^z(cosxz)x + sinxz. e^z]$$

4.2.4 Chain Rule

The chain rule is a fundamental concept in calculus used to find the derivative of a *composite function*, i.e., a function made by applying one function inside another. If a variable y depends on u, and u depends on x, then y also depends on x indirectly, and the chain rule helps compute $\frac{dy}{dx}$ in such cases.

Mathematically, if $y = f(u)$, $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

For example,

Let $y = \sin(x^2)$, $y = sinu$, $u = x^2$

$$\frac{dy}{du} = cosu, \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = cosu.2x = 2x\ cosx^2$$

If $z = f(x, y)$ is differentiable at the point

$(x, y) = (x(t), y(t))$ then

$z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Let $z = f(x, y)$ where $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v) and if z is differentiable at the point (x, y), then

$z = f(x(u, v), y(u, v))$ has first order partial derivatives at the point (u, v) is given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} \quad and$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

Illustration 4.2.8

Suppose $z = x^2y, \quad x = t^2, \quad y = t^3 \quad Find \ \frac{dz}{dt}$.

Solution

$x = t^2, \quad y = t^3$

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 3t^2$$

$z = x^2y$

$$\frac{\partial z}{\partial x} = 2xy, \qquad \frac{\partial z}{\partial y} = x^2$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= 2xy.2t + x^2.3t^2$$

$$= 2t^2t^3.2t + (t^2)^2.3t^2$$

$$= 2t^2t^3.2t + (t^2)^2.3t^2$$

$$= 4t^6 + 3t^6 = 7t^6$$

Illustration 4.2.9

Suppose $z = e^{xy}$, $x = 2u + v$, $y = \frac{u}{v}$. $Find\ \frac{\partial z}{\partial u}\ and\ \frac{\partial z}{\partial v}$.

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$z = e^{xy},$$

$$\frac{\partial z}{\partial x} = e^{xy}y, \quad \frac{\partial z}{\partial y} = e^{xy}x$$

$$x = 2u + v,$$

$$\frac{\partial x}{\partial u} = 2, \quad \frac{\partial x}{\partial v} = 1$$

$$y = \frac{u}{v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{v}, \quad \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{\partial z}{\partial u} = e^{xy}y.2 + e^{xy}x.\left(\frac{1}{v}\right) = e^{xy}\left(2y + \frac{x}{v}\right)$$

$$= e^{\frac{(2u+v,)u}{v}}\left(\frac{2u}{v} + \frac{(2u+v)}{v}\right)$$

$$= e^{\frac{(2u+v,)u}{v}}\left(\frac{4u}{v} + 1\right)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = e^{xy}y.1 + e^{xy}x.\left(-\frac{u}{v^2}\right) = e^{xy}\left(y - \frac{ux}{v^2}\right)$$

$$= e^{\frac{(2u+v,)u}{v}}\left(\frac{u}{v} - \frac{(2u+v)u}{v^2}\right)$$

$$= e^{\frac{(2u+v,)u}{v}}\left(-\frac{2u^2}{v^2}\right)$$

Illustration 4.2.10

Suppose $z = \frac{x}{y}$, $x = 2\ cosu$, $y = 5\ sinv$.

$Find\ \frac{\partial z}{\partial u}\ and\ \frac{\partial z}{\partial v}$.

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$z = \frac{x}{y}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y}, \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2}$$

$$x = 2cosu,$$

$$\frac{\partial x}{\partial u} = -2\ sinu, \qquad \frac{\partial x}{\partial v} = 0$$

$$y = 5\ sinv$$

$$\frac{\partial y}{\partial u} = 0, \ \frac{\partial y}{\partial v} = 5\ cosv$$

$$\frac{\partial z}{\partial u} = \frac{1}{y}(-2sinu) = -\frac{2sinu}{5sinv}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{1}{y}.0 + \left(-\frac{x}{y^2}\right)5cosv = -\frac{2cosucosv}{5\sin^2 v}$$

Illustration 4.2.11

Suppose $z = xy$, $x = t^{\frac{1}{4}}$, $y = t^4$ $Find\ \frac{dz}{dt}$.

Solution

$$x = t^{\frac{1}{4}}, \ y = t^4$$

$$\frac{dx}{dt} = \frac{1}{4}t^{-\frac{3}{4}}, \qquad \frac{dy}{dt} = 4t^3$$

$$z = xy$$

$$\frac{\partial z}{\partial x} = y, \qquad \frac{\partial z}{\partial y} = x$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= t^4.\frac{1}{4}t^{-\frac{3}{4}} + t^{\frac{1}{4}}.4t^3$$

$$= \frac{1}{4}t^{-\frac{13}{4}} + 4t^{\frac{13}{4}}$$

Illustration 4.2.12

Suppose $z = x^2 + y^2, \; x = a\,cost, \; y = b\,sint \quad Find \; \frac{dz}{dt}$.

Solution

$x = a\,cost, \; y = b\,sint$

$$\frac{dx}{dt} = -a\,sint, \qquad \frac{dy}{dt} = b\,cost$$

$$z = x^2 + y^2$$

$$\frac{\partial z}{\partial x} = 2x, \qquad \frac{\partial z}{\partial y} = 2y$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= 2x.-a\,sint + 2y.b\,cost$$

$$= 2acost.-a\,sint + 2bsint.b\,cost$$

$$= 2costsint(b^2 - a^2)$$

Summarized Overview

Differentiation is the process of finding the derivative, which represents the rate of change of a function. Derivatives of basic functions such as polynomials, trigonometric, exponential, and logarithmic functions follow standard rules. In functions involving multiple variables, partial derivatives are used to measure how the function changes with respect to one variable while keeping others constant. The chain rule allows us to differentiate composite functions by accounting for how variables depend on each other. These concepts are essential for analyzing change and solving problems in physics, engineering, and economics.

Assignments

1. If $y = (ax)^m + \left(\frac{b}{x}\right)^n$, find $\frac{dy}{dx}$
2. Determine the differential coefficient of $2\sqrt{x} + 8\log_e x + 2\log_a x$
3. If $y = x^y$ prove that $x\frac{dy}{dx} = \frac{y^2}{1-y\log x}$
4. If $x = a\cos\theta$, $y = b\sin\theta$ find $\frac{dy}{dx}$
5. If $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$, find $\frac{dy}{dx}$
6. If $ax^2 + 2hxy + by^2 = 0$, find $\frac{dy}{dx}$

Suggested Reading

1. Anton, Bivens,Davis: Calculus, John Wiley and Sons: 10th Edition
2. Thomas Jr. G.B Weir MD and Hass J R. Thomas' Calculus Pearson

Reference

1. James Stewart – *Calculus: Early Transcendentals*, 8th ed., Cengage Learning, 2015.
2. E. Kreyszig – *Advanced Engineering Mathematics*, 10th ed., Wiley, 2011.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

3 UNIT

Integration and Area Under Curve

Learning Outcomes

After completion of this unit, the learner will be able to:

- apply different techniques to integrate complex integrals
- understand and differentiate between definite and indefinite integrals
- apply various integration techniques
- analyze and evaluate the relevance of area under the curve (AUC) in Receiver Operating Characteristic (ROC) analysis

Background

Integration is a fundamental concept in calculus used to determine the accumulation of quantities and the area under curves, with applications spanning physics, engineering, and data science. An indefinite integral represents the general form of a function whose derivative is known, while a definite integral calculates the exact accumulated value between two limits, often interpreted as the signed area under a curve. Various techniques such as substitution, integration by parts, partial fractions, and numerical methods enable to solve a wide range of integrals. Beyond pure mathematics, integration finds powerful applications in Receiver Operating Characteristic (ROC) analysis in machine learning, where the Area Under the Curve (AUC) quantifies a model's ability to distinguish between classes. Here, the AUC is the definite integral of the True Positive Rate (TPR) with respect to the False Positive Rate (FPR) over all classification thresholds. For instance, in a medical test for detecting a disease, plotting the ROC curve, and calculating its AUC reveals how well the test separates diseased from healthy patients - a higher AUC indicating a more reliable diagnostic tool. Thus, integration bridges abstract mathematics with real-world decision-making, turning curves into actionable insights.

Keywords

Definite Integrals, Indefinite Integrals, Area Under Curves (ROC/AUC)

Discussion

4.3.1 Integration

Integration is a fundamental concept in calculus that deals with finding the total accumulation of quantities, often represented as the area under a curve. It is the reverse process of differentiation and can be classified into indefinite integrals, which represent a family of functions without specific limits, and definite integrals, which calculate a numerical value over a given interval.

For instance, if the speed of a car is described by the function $v(t) = 3t^2$ meters per second, then integrating this function over the interval from $t = 0$ to $t = 4$ seconds calculates the total distance traveled during that time. This illustrates how integration serves as a powerful tool across fields like physics, engineering, economics, and data analysis by helping measure cumulative quantities such as distance, area, volume, and probability.

Differential Calculus focuses on the concept of the derivative. It originally arose from the problem of defining tangent lines to the graphs of functions and determining their slopes. Integral Calculus, on the other hand, is motivated by the challenge of defining and calculating the area of the region enclosed by the graph of a function.

The functions that have a given function as their derivative are called the antiderivatives of that function. The formula that represents all these antiderivatives is known as the indefinite integral, and the process of finding antiderivatives is called integration. For example, if we know the instantaneous velocity of a moving object at every moment, integration allows us to determine its position over time. Integration is involved in many practical and theoretical problems. Integral calculus developed primarily to address two key problems: (a) finding a function when its derivative is known, and (b) calculating the area enclosed by the graph of a function under certain conditions. These problems give rise to two types of integrals indefinite and definite integrals which together form Integral Calculus. The Fundamental Theorem of Calculus establishes a powerful connection between indefinite and definite integrals, making definite integrals essential tools in science and engineering. In probability theory, definite integrals help to determine cumulative probabilities from probability density functions.

4.3.2 Indefinite Integral

An indefinite integral of a function $f(x)$ is the collection of all its antiderivatives. It represents the general form of all functions whose derivative is $f(x)$. The indefinite integral is written as: $\int f(x)\,dx = F(x) + C$

where $F(x)$ is any antiderivative of $f(x)$, i.e. $F'(x) = f(x)$, and C is an arbitrary constant called the constant of integration. This constant accounts for the fact that derivatives of constant terms are zero, so there are infinitely many antiderivatives differing by a constant.

4.3.3 Rules of Integration

The rules of integration depend on the rules of differentiation. In this section, we will discuss the basic rules of integration that will be obtained by reversing the corresponding rules of differentiation. Their accuracy is easily checked since the derivative of the integral must equal the integrand.

Rule 1. Constant Rule of Integration

The integral of a constant k is.

$$\int k\,dx = kx + c$$

For example, $\int 2\,dx = 2x + c$

Rule 2. Power Rule of Integration

The integral of a power function x^n, where $n \neq -1$ is given by:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad (n \neq -1)$$

For example, $\int x^5\,dx = \frac{x^{5+1}}{5+1} = \frac{x^6}{6} + c.$

Rule 3. Exponential Rule of Integration

The integral of an exponential function is.

$$\int a^{kx}\,dx = \frac{a^{kx}}{k\ln a} + c$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$$

For example,

$$\int 2^{3x}\,dx = \frac{2^{3x}}{3\ln 2} + c$$

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$$

Rule 4. Logarithmic Rule of Integration

The integral of x^{-1} is (or $\frac{1}{x}$) is.

$$\int x^{-1}\, dx = \ln x + c \quad (x > 0)$$

For example, $\int \frac{2}{x}\, dx = 2 \ln x + c$

Rule 5. The Constant Multiple Rule of Integration

The integral of a constant time a function equals the constant times the integral of the function.

$$\int k\, f(x)\, dx = k \int f(x)\, dx$$

For example, $\int 3\, x^2\, dx = 3 \int x^2\, dx + c = x^3 + c$

Rule 6. The Sum and subtraction Rule of Integration

The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

For example,

$$\int \left(x^6 + \frac{1}{x} + e^x - 1\right) dx = \int x^6\, dx + \int \frac{1}{x}\, dx + \int e^x\, dx - \int dx$$

$$= \frac{x^7}{7} + \log x + e^x - x + c$$

4.3.4 Integration By Substitution

In this operation, integration is done by changing the given integrand in standard form. For this any integration function of variable amount is put equal to another variable amount and substituting in integration, integration is changed in form of function of new variable amount in such a way that use of formula may be easily.

Illustration. 4.3.1

Determine the integral $\int \sqrt{1-5x} + 9e^{3x}\, dx$

Solution

$$\int \sqrt{1-5x} + 9e^{3x}\, dx = \frac{(1-5x)^{\frac{3}{2}}}{\frac{3}{2} \times -5} + \frac{9e^{3x}}{3} = -\frac{2}{15}(1-5x)^{\frac{3}{2}} + 3e^{3x} + c$$

Illustration 4.3.2

Find the value of $\int (3e^x - \frac{1}{5x} + \sec x \tan x)\, dx$

Solution

$$\int (3e^x - \frac{1}{5x} + secx\ tanx)\ dx = 3e^x - \frac{1}{5} logx + secx + c$$

Illustration. 4.3.3

Find $\int \left(x - \frac{1}{x}\right)^2\ dx$

Solution

$$\int \left(x - \frac{1}{x}\right)^2\ dx = \int x^2 - 2 + \frac{1}{x^2}\ dx = \frac{x^3}{3} - 2x - \frac{1}{x} + c$$

Illustration. 4.3.4

Determine the following integrals

1) $\int \frac{5x+7}{x} + e^{3x}\ dx$

2) $\int \frac{(x+2)(4x^2-5)}{x}\ dx$

Solution

1) $\int \frac{5x+7}{x} + e^{3x}\ dx = \int 5 + \frac{7}{x} + e^{3x}\ dx = 5x + 7\ logx + \frac{e^{3x}}{3} + c$

2) $\int \frac{(x+2)(4x^2-5)}{x}\ dx = \int \frac{4x^3 - 5x + 8x^2 - 10}{x}\ dx$

$$= \int (4x^2 - 5 + 8x - \frac{10}{x})\ dx$$

$$= \frac{4x^3}{3} - 5x + \frac{8x^2}{2} - 10\ logx$$

$$= \frac{4x^3}{3} - 5x + 4x^2 - 10\ logx + c$$

Illustration. 4.3.5

Determine the following integrals.

1) $\int \cos^2 x\ sinx\ dx$

2) $\int \frac{x^8}{(1-x^3)^{\frac{1}{3}}}\ dx$

3) $\int \frac{logx}{x}\ dx$

4) $\int \frac{x^7}{1+x^{16}}\ dx$

5) $\int \frac{6x^2}{1-2x^3}\ dx$

6) $\int \frac{\cos^2(logx)}{x}\ dx$

7) $\int \frac{3x^2}{(x^3+4)^5}\,dx$ 8) $\int \frac{e^x}{1+e^x}\,dx$

Solution

1) Put $\cos x = u, \quad -\sin x\,dx = du$

$$\int \cos^2 x \sin x\,dx = \int u^2(-du) = -\int u^2\,du = -\frac{u^3}{3} + c = -\frac{\cos^3 u}{3} + c$$

2) Put $1 - x^3 = u, \quad -3x^2\,dx = du, \quad x^2\,dx = -\frac{du}{3}$

$x^3 = 1 - u,$

$$\int \frac{x^8}{(1-x^3)^{\frac{1}{3}}}\,dx = \int \frac{(x^3)^2\, x^2}{(1-x^3)^{\frac{1}{3}}}\,dx$$

$$= \int \frac{(1-u)^2}{u^{\frac{1}{3}}} \times -\frac{du}{3}$$

$$= \int \frac{1-2u+u^2}{u^{\frac{1}{3}}} \times -\frac{du}{3}$$

$$= -\frac{1}{3}\left[\int u^{-\frac{1}{3}} - 2u^{\frac{2}{3}} + u^{\frac{5}{3}}\right] du$$

$$= -\frac{1}{3}\left[\frac{u^{\frac{2}{3}}}{\frac{2}{3}} - 2\frac{u^{\frac{5}{3}}}{\frac{5}{3}} + \frac{u^{\frac{8}{3}}}{\frac{8}{3}}\right] + c$$

$$= -\frac{1}{2}u^{\frac{2}{3}} + \frac{2}{5}u^{\frac{5}{3}} - \frac{1}{8}u^{\frac{8}{3}}$$

$$= -\frac{1}{2}(1-x^3)^{\frac{2}{3}} + \frac{2}{5}(1-x^3)^{\frac{5}{3}} - \frac{1}{8}(1-x^3)^{\frac{8}{3}}$$

3) Put $\log x = u, \quad \frac{1}{x}\,dx = du$

$$\int \frac{\log x}{x}\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{(\log x)^2}{2} + c$$

4) Put $x^8 = u, \quad 8x^7\,dx = du, \quad x^7\,dx = \frac{du}{8}$

$$\int \frac{x^7}{1+x^{16}}\,dx = \int \frac{1}{1+u^2}\frac{du}{8} = \frac{1}{8}\tan^{-1} u + c = \frac{1}{8}\tan^{-1} x^8 + c$$

5) Put $1 - 2x^3 = u, \quad -6x^2\,dx = du,$

$$\int \frac{6\,x^2}{1-2x^3}\,dx = \int \frac{du}{u} = \log u + c = \log(1-2x^3) + c$$

6) Put $\log x = u$, $\quad \frac{dx}{x} = du$,

$$\int \frac{\cos^2(\log x)}{x}\,dx = \int \cos^2 u\;du = \int \frac{1+\cos 2u}{2}\,du$$

$$= \frac{1}{2}\left[u + \frac{\sin 2u}{2}\right\} + c$$

$$= \frac{1}{2}\left[\log x + \frac{\sin 2(\log x)}{2}\right\} + c$$

7) Put $x^3 + 4 = u$, $\;3x^2\,dx = du$,

$$\int \frac{3\,x^2}{(x^3+4)^5}\,dx = \int \frac{du}{(u)^5} = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + c$$

$$= -\frac{1}{4}(x^3+4)^{-4} + c$$

8) Put $1 + e^x = u$, $\;e^x\,dx = du$,

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{du}{u} = \log u + c = \log e^x + c = x + c$$

4.3.5 Integration by Parts

If $f(x)$ and $g(x)$ are two functions of x, then the integration of $f(x).g(x)$ is done by integration by parts.

Consider $f(x)$ is the first function and $g(x)$ is the second function.

$$\int f(x).g(x) = f(x)\int g(x) - \int\left[\frac{df(x)}{dx}\int g(x)dx\right]dx$$

i.e., $\int f(x).g(x) = f(x)\int g(x) - \int[f'(x)\int g(x)dx\,]dx$

Integration of multiplication of two function

= First function x integral of second-integral of (differential of first x integral of second)

Illustration. 4.3.6

Determine the following integral.

1) $\int \log x\;dx$ $\qquad$ 2) $\int x\sin x\;dx$

3) $\int x^2 \ logx \ dx$ 4) $\int x \ e^{2x} \ dx$

5) $\int \frac{2x}{(x-8)^3} \ dx$ 6) $\int 6x \ e^{(x+3)} \ dx$

7) $\int x(x+1)^5 \ dx$ 8) $\int 15x \ (x+4)^{\frac{3}{2}} \ dx$

Solution

1) Let $f(x) = logx, \quad g(x) = 1$

$$\int logx \ dx = \int logx.1 \ dx$$

$$= logx \int 1 \ dx - \int \left(\frac{d}{dx}(logx) \times \int 1.dx\right) dx$$

$$= logx \times x - \left(\int \tfrac{1}{x} \times x \ \right) dx$$

$$= x \ logx - \left(\int \ dx\right)$$

$$= x \ logx - x + c$$

2) Let $f(x) = x, \quad g(x) = sinx$

$$\int x \, sinx \ dx = x \int sinx - \int \left(\frac{d}{dx}(x) \int sinx \, dx\right) dx$$

$$= x(-cosx) - \int (1 \times (-cosx)) \, dx$$

$$= -x(cosx) + sinx + c$$

3) Let $f(x) = x^2, \quad g(x) = logx$

$$\int x^2 \ logx \ dx = x^2 \int logx - \int \left(\frac{d}{dx}(x^2) \int logx \ dx\right) dx$$

$$= x^2 \times \tfrac{1}{x} - \int 2x \times \tfrac{1}{x} \ dx$$

$$= x - \int 2 \ dx$$

$$= x - 2x + c = -x + c$$

4) Let $f(x) = x, \quad g(x) = e^{2x}$

$$\int x \ e^{2x} \ dx = x \times \int e^{2x} \, dx - \int \left(\frac{d}{dx}(x) \int e^{2x} \ dx\right) dx$$

$$= x \times \frac{e^{2x}}{2} - \int 1 \times \frac{e^{2x}}{2} \ dx$$

$$= x \times \frac{e^{2x}}{2} - \frac{e^{4x}}{4} + c$$

5) Let $f(x) = 2x, \quad g(x) = \frac{1}{(x-8)^3} = (x-8)^{-3}$

$$\int \frac{2x}{(x-8)^3}\,dx = \int 2x \times (x-8)^{-3}\,dx$$

$$= 2x \int (x-8)^{-3}$$

$$- \int \left(\frac{d}{dx}(2x) \int (x-8)^{-3} dx\right) dx$$

$$= 2x \times \frac{(x-8)^{-2}}{-2} - \int 2 \times \frac{(x-8)^{-2}}{-2} dx$$

$$= -x \times (x-8)^{-2} - (x-8)^{-1} + c$$

6) Let $f(x) = 6x, \quad g(x) = e^{x+3}$

$$\int 6x\ e^{x+3}\ dx = 6x \int e^{x+3}\ dx - \int \left(\frac{d}{dx}(6x) \int e^{x+3}\ dx\right) dx$$

$$= 6x \times e^{x+3} - \int 6 \times e^{x+3}\ dx$$

$$= 6xe^{x+3} - 6e^{x+3} + c$$

7) Let $f(x) = x, \quad g(x) = (x+1)^5$

$$\int x(x+1)^5\ dx$$

$$= x \int (x+1)^5\ dx - \int \left(\frac{d}{dx}(x) \int (x+1)^5\ dx\right) dx$$

$$= x \times \frac{(x+1)^6}{6} - \int 1 \times \frac{(x+1)^6}{6}\ dx$$

$$= x \times \frac{(x+1)^6}{6} - \frac{1}{6}\frac{(x+1)^7}{7} + c$$

$$= \frac{x\,(x+1)^6}{6} - \frac{(x+1)^7}{42} + c$$

8) Let $f(x) = 15x, \quad g(x) = \ (x+4)^{\frac{3}{2}}$

$$\int 15x\ (x+4)^{\frac{3}{2}}\ dx = 15x \times \int\ (x+4)^{\frac{3}{2}} dx -$$
$$\int \left(\frac{d}{dx}(15x) \int\ (x+4)^{\frac{3}{2}}\ dx\right) dx$$

$$= 15\,x \times \frac{(x+4)^{\frac{5}{2}}}{\frac{5}{2}} - \int 15 \times \frac{(x+4)^{\frac{5}{2}}}{\frac{5}{2}}\,dx$$

$$= 6x \times (x+4)^{\frac{5}{2}} - 6 \times \frac{(x+4)^{\frac{7}{2}}}{\frac{7}{2}} + c$$

$$= 6x \times (x+4)^{\frac{5}{2}} - \frac{12}{7}(x+4)^{\frac{7}{2}} + c$$

4.3.6 Definite integral

A definite integral is denoted by $\int_a^b f(x)dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral.

If a function is continuous and integrable in the interval [a, b] then the area between the graph of f and the interval [a, b] is

$$A = \int_a^b f(x)dx$$

Properties of definite integral

1. If a is in the domain of f then $\int_a^a f(x)dx = 0$

2. If f is integrable in $[a, b]$, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

The Fundamental Theorem of Calculus

It states that if a function f is continuous on an interval $[a, b]$, then the definite integral of f from a to b can be computed using any of its antiderivatives F. In other words,

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where $F'(x) = f(x)$. This theorem shows that integration and differentiation are inverse processes and provides a practical way to evaluate definite integrals.

Illustration. 4.3.7

Evaluate the following integrals.

a) $\int_0^3 (2x+3)\ dx$ b) $\int_1^9 \sqrt{x}\ dx$

c) $\int_0^{\frac{\pi}{2}} 2\ sinx\ dx$ d) $\int_{-1}^{2} 4x(1-x^2)\ dx$

e) $\int_4^9 3x\sqrt{x}\ dx$ f) $\int_0^3 f(x)\,dx\ \ if\ \ f(x) = x^2,\ \ x \le 2 = \ 3x-2\,, x \ge 2$

Solution

a. $\int_0^3 (2x+3)\,dx = 2\left(\frac{x^2}{2}\right)_0^3 + (3x)_0^3 = (3^2-0)+(9-0) = 9+9 = 18$

b) $\int_1^9 \sqrt{x}\;dx = \int_1^9 x^{\frac{1}{2}}\;dx = \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^9 = \frac{2}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{2}{3}(27-1) = \frac{2}{3}.26 = \frac{52}{3}$

c. $\int_0^{\frac{\pi}{2}} 2\,sinx\,dx = 2(-cosx)_0^{\frac{\pi}{2}} = -2(\cos\frac{\pi}{2} - cos0) = -2(0-1) = 2$

d .

$$\int_{-1}^2 4x(1-x^2)\;dx = 4\int_{-1}^2 (x-x^3)\;dx = 4\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_{-1}^2 = 4\left(\left(\frac{4}{2}-\frac{16}{4}\right)-\left(\frac{1}{2}-\frac{1}{4}\right)\right)$$

$$= 4\left(\left(\frac{4}{2}-\frac{16}{4}\right)-\left(\frac{1}{2}-\frac{1}{4}\right)\right) = 4\left(-2-\frac{1}{4}\right) = -9$$

e. $\int_4^9 3x\sqrt{x}\;dx = 3\int_4^9 x^{\frac{3}{2}}\;dx = 3\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_4^9 = \frac{6}{5}\left(9^{\frac{5}{2}} - 4^{\frac{5}{2}}\right) = \frac{6}{5}(3^5-2^5) = \frac{1266}{5}$

f. $\int_0^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$

$$= \left(\frac{x^3}{3}\right)_0^2 + \left(\frac{3x^2}{2} - 2x\right)_2^3$$

$$= \left(\frac{8}{3}-0\right) + \left(\frac{15}{2}-0\right) = \frac{49}{6}$$

4.3.7 Area Under the Curve (ROC/AUC)

The Area Under the Curve (AUC), particularly under the Receiver Operating Characteristic (ROC) curve, plays a crucial role in evaluating the performance of classification models. ROC is a graphical representation of the performance of a binary classification model across all classification thresholds.

For example, suppose we want to classify whether a person has cancer based on a screening test that measures certain blood values, where higher values indicate a higher likelihood of cancer. We collect data from ten individuals, including their blood test values and whether they have the disease. Choose, a class threshold of thirty. In one case of the five disease individuals, we would correctly classify four are diseased and one incorrectly as healthy. i.e., we correctly classify four out of five are disease, this value is called true positive rate (TPR). On the other hand, of the five healthy individuals we misclassified two as deceased and three as healthy. So, we misclassified two out of five as deceased. This value is called the False Positive Rate (FPR).

So, the threshold of 5 we get TPR of 4/5=0,8 and FPR of 3/5 0.6.

Thus, $\boldsymbol{TPR} = \frac{\boldsymbol{True\ Positives}}{\boldsymbol{True\ Positives+False\ Negatives}}$

$$\boldsymbol{FPR} = \frac{\boldsymbol{FalsePositives}}{\boldsymbol{False\ Positives + True\ Negatives}}$$

True Positives correctly means classifieds as deceased,

False Negatives means incorrectly classifieds as healthy,

False Positives means healthy misclassified as deceased.

True Negatives correctly means classifieds as healthy.

We can now calculate for each threshold the TPR and FPR then we plot these values in ROC curve.

The ROC curve plots the True Positive Rate (sensitivity) against the False Positive Rate (1-specificity) at various threshold settings. This graphical representation helps in understanding how well the model can distinguish between classes. The AUC quantifies this performance into a single number ranging from 0 to 1, where an AUC of 0.5 indicates no discriminative power (equivalent to random guessing), and an AUC of 1.0 indicates perfect classification. The higher the AUC, the better the model is at predicting positive instances as positive and negative instances as negative.

AUC-ROC is particularly useful in scenarios where class imbalance exists or when the cost of false positives and false negatives is not the same. It allows practitioners to compare multiple models without fixing a specific threshold, thus offering a threshold-independent performance measure. For example, in medical diagnostics, where false negatives could be critical, a model with a higher AUC ensures better sensitivity while keeping the false alarm rate low. The area under the ROC curve thus serves as a comprehensive indicator of model quality, guiding model selection, tuning, and evaluation across diverse application domains like fraud detection, spam classification, and disease prediction.

Summarized Overview

Integration is the mathematical process of finding the area under a curve or determining the accumulation of a quantity, with *indefinite integrals* representing the general antiderivative of a function and *definite integrals* giving the exact area between a curve and the x-axis over a specified interval. Various integration techniques—such as substitution, integration by parts, and partial fractions—are used to evaluate these integrals. In statistics and machine learning, integration plays a key role in calculating the *Area Under the Curve (AUC)* in *Receiver Operating Characteristic (ROC)* analysis, where the integral of the ROC curve quantifies a model's ability to distinguish between classes, with higher AUC indicating better classification performance.

Assignments

1. Find the integral

   i. $\int x^3 e^{2x}$ dx

   ii. $\int x^{\frac{1}{2}} + 3x^{-\left(\frac{1}{2}\right)}\, dx$

   iii. $\int x^4 \left(2x^5 - 5\right)^4 dx$

   iv. $\int \frac{6x^2+4x+10}{(x^3+x^2+5x)^3}\, dx$

   v. $\int \frac{3x^2+2}{4x^3+8x}\, dx$

   vi. $\int \frac{2x}{(x-8)^3}\, dx$

   vii. $\int x^2 e^{x+3}\, dx$

   viii. $\int x(x^3+1)^5\, dx$

2. Evaluate the following.

   i. $\int_1^3 (2x+3)\,dx$

   ii. $\int_0^2 (x^2+3x^4)\,dx$

iii. $\int_0^3 \left(\sqrt{x} + 3x\right)^2 dx$

iv. $\int_0^{\frac{\pi}{2}} (cosx + sinx)\, dx$

v. $\int_0^{\frac{\pi}{2}} \left(cosecx + \frac{1}{x}\right) dx$

Suggested Reading

1. Anton, Bivens,Davis: Calculus, John Wiley, and Sons: 10th Edition
2. Thomas Jr. G.B Weir MD and Hass J R. Thomas' Calculus Pearson

Reference

1. James Stewart – *Calculus: Early Transcendentals*, 8th ed., Cengage Learning, 2015.
2. E. Kreyszig – *Advanced Engineering Mathematics*, 10th ed., Wiley, 2011

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

4 UNIT Optimization Techniques

Learning Outcomes

After completion of this unit, the learner will be able to:

- explain the core principle of gradient descent and distinguish between Batch, Stochastic, and Mini-Batch variants
- differentiate between convex, concave, and non-convex functions
- relate function characteristics to optimization: Explain why convexity is important in optimization
- define and differentiate between local minima, global minima, and saddle points
- understand how the presence of local minima and saddle points can complicate finding the optimal solution

Background

In many real-world applications, especially in data science and machine learning, we often need to find the best possible solution by minimizing errors or maximizing performance, and these problems can be represented using mathematical functions. To understand how to locate such optimal values, it is important to study how a function behaves, which is explained through concepts like local maxima and minima, global extrema, and saddle points. A local minimum represents the lowest value within a small neighborhood, while a global minimum is the lowest value over the entire domain, and saddle points indicate positions where the function changes direction without being a clear maximum or minimum. The behavior of functions is further understood using convexity and concavity, where convex functions make optimization easier because any local minimum is also a global minimum, whereas non-convex functions may have multiple local minima, making it harder to find the best solution. Based on this understanding, optimization techniques such as gradient descent are used, which iteratively move in the direction of the steepest decrease using gradients, thereby helping in efficiently finding the minimum value of a function and forming the basis for many machine learning algorithms.

Keywords

Gradient Descent, Convexity, Concavity, Local Minima, Global Minima Saddle Points

Discussion

4.4.1 Local Maxima and Minima

Local Maxima and Minima refer to the points in the domain of a function that define the highest and lowest values of that function. The derivatives of the function are helpful in identifying and calculating the Local Maxima and Local Minima. The Local Maxima and Minima can be found through the use of both the First derivative test and the Second derivative test in the case of a single variable function.

As an example, the highest point on peak is a local maximum point and a point at the lowest point in a river is a local minimum. Local Minima and Local Maxima are also called Local Extrema.

4.4.1.1 Local Maxima

A Local Maxima is a point in the domain of a function where the function attains its maximum value. A point is said to be a local maximum for a function f(x) if the value of the function at that point is greater than its values corresponding to all points in an interval containing that point i.e., point $x = a$ is called a Local maximum if the value of f(a) is greater than or equal to all the values of f(x) in a small interval around a.

Mathematically, $f(a) \geq f(a - h)$ and $f(a) \geq f(a + h)$ where $h > 0$, then a is called the Local maximum point.

4.4.1.2 Local Minima

In the case of Local Minima, at the minimum point, the values of a function close to that point are always greater than the values of the function i.e.,a point $x = a$ is called a Local minimum if the value of f(a) is lesser than or equal to all the values of f(x) in an interval around a.

Mathematically, $f(a) \leq f(a - h)$ and $f(a) \leq f(a + h)$ where $h > 0$, then a is called the Local minimum point.

4.4.1.3 Absolute Maximum and minimum

The point at which highest value of a function in the whole domain is attained is called absolute maximum. The point at which lowest value of a function in the whole domain is attained is called absolute minimum. They are sometimes called global maximum and minimum.

4.4.2 Determination of Local Maxima and Minima

Following is a method to calculate the Local Maxima and Minima:

- In first step, we find the first order derivative of the function.
- In second step, we find critical points where the first order derivative is zero.
- In third step, we use **Second derivative test** to determine the Local Maxima and Local Minima .

If c is a critical point and if $f''(c) < 0$, the point $x= c$ will be the Local Maximum and $f(c)$ will be the Local maximum value of $f(x)$.

If c is a critical point and if $f''(c) >0$, the point $x= c$ will be the Local Minimum and $f(c)$ will be the Local maximum value of $f(x)$.

If the first derivative $f'(c) = 0$, and the second derivative $f''(c) = 0$, then the point c is called Point of Inflection (Saddle Point).

Illustration 4.4.1

Find the local maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution

$f'(x) = 3x^2 - 12x + 9$

Set $f'(x) = 0$: $3x^2 - 12x + 9 = 0$

$(3x - 3)(x - 3) = 0$

$x = 1$ or $x = 3$

$f''(x) = 6x - 12$

At $x = 1$: $f''(1) = -6 < 0$, so local maximum

At $x = 3$: $f''(3) = 6 > 0$, so local minimum

Local maximum at $x = 1$, $f(1) = 5$

Local minimum at $x = 3$, $f(3) = 1$

4.4.3 Global Maxima and Minima

The global maximum also called the absolute maximum is the highest value in the entire domain of the function. The global minimum also called the absolute minimum is the lowest value in the entire domain of the function.

The point x = a where a is in domain D is called global maximum of f(x) if f(x) ≤ f(a) for all x ∈ D. The point x = a where a is in domain D is called global minimum of f(x), if f(x) ≥ f(a) for all x ∈ D.

4.4.4 Maximum and minimum for multi-variable functions

To find the critical points in multivariate functions we follow the below steps.

- Find the first order partial derivative of the function with respect to all the input variables.
- Find all critical points by equating them to zero.

For two variable function f(x,y),

$calculate\ f_{xx}(a,b), f_{yy}(a,b)\ and\ f_{xy}(a,b)$ for all critical points (a,b).

Let $D = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$.

$Then$ (1) $D > o\ and\ f_{xx}(a,b) > 0,\ (a,b)$ is a relative minimum

(2)$D > o\ and\ f_{xx}(a,b) < 0,\ (a,b)$ is a relative maximum

(3) $D < 0,\ then\ (a,b)$ is a saddle point

(4) $D = 0$, no conclusion

This method can be generalised to functions having more than two variables using matrices.

4.4.5 Convexity and concavity

While discussing gradient descent method, the goal of attaining a global minimum was achieved through the gradient of an objective function. A tacit assumption was if every condition is satisfied, a global minimum is attained. Actually it depends on the nature of the objective function also. If the function is not convex, this may not happen. For a non-convex function, a local minima may be the end point.

Suppose x and y are two pints in [a,b]. A point in the interval between x and y is given by $\lambda x + (1-\lambda) y, 0 \le \lambda \le 1$

A function f(x) is said to be concave if for every pair of points (x,y) $f\left(\lambda x + (1-\lambda) y\right) \ge \lambda f(x) + (1-\lambda) f(y), 0 \le \lambda \le 1$. A function f(x) is said to be convex if

$$f\left(\lambda x + (1-\lambda) y\right) \le \lambda f(x) + (1-\lambda) f(y), 0 \le \lambda \le 1 \cdot$$

But when multi variable functions are considered, the extension of this definition becomes difficult. So the definition is modified based on derivative of the function.

A function f(x) is said to be convex if $f''(x) > 0$ for all x and concave if $f''(x) < 0$ for all x .For multi variable functions, determine all stationary points from ∇ if, gradient is zero. Then calculate the eigen values of the Hessian matrix. If all of them are positive(then matrix is called positive definite), then the function is convex where Hessian matrix is defined for $f(x_1,x_2)$ as $H=\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$.

Convex functions are important because for a convex function a local minimum is also a global minimum.

$f(x)= x^2$ and $f(x)=e^x$ are convex functions while $f(x)=\log x$ is a concave function.

Illustration 4.4.2

Show that $f(x) = e^x$ is a convex function.

Solution:

First, compute the second derivative of f(x):

$f'(x) = e^x$ and $f''(x) = e^x$

Since $f''(x) = e^x \geq 0$ for all x , f(x) is convex.

Illustration 4.4.3

Show that $f(x) = \sin x$ is neither a convex nor concave.

Solution:

First, compute the second derivative of f(x):

$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x$

$f''(x)$ is positive in some parts of the domain and negative in some other parts, so neither convex nor concave.

Illustration 4.4.4

Find the critical points and determine if $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$ is convex or not at these points.

Solution:

$f(x) = 12x^5 - 45x^4 + 40x^3 + 5,\ f'(x) = 60x^4 - 180x^3 + 120x^2,$

$f''(x) = 240x^3 - 540x^2 + 240x$

To find stationary points:

$f'(x) = 60x^4 - 180x^3 + 120x^2 = 0 \Rightarrow 60x^2\left(x^2 - 3x + 2\right) = 0$

$\Rightarrow x=0, x=1, x=2$

$f''(x) = 240x^3 - 540x^2 + 240x$

At $x=0, f''(x) = 0 \Rightarrow x = 0$ is a point of inflexion

At $x=1, f''(x) < 0 \Rightarrow x = 1$ is a local maximum point

At $x=2, f''(x) > 0 \Rightarrow x = 2$ is a local minimum point

So, f(x) is convex at x=2.

4.4.6 Gradient

Gradient is a basic concept which gives a measure of the steepest change and also the direction of that change. In geometry, it is a way to measure how steep a line or a surface is. For a straight line, the gradient tells how much the line goes up or down as you move along the x-axis. Gradients are also called "slopes," especially in geometry and graphing. You'll find gradient used in coordinate geometry, calculus (as derivatives), and real-world contexts like speed or rate of change. Here's the standard formula for the gradient of a straight line between two points (x_1,y_1) and (x_2,y_2) :

$$\text{Gradient} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

4.4.6.1 Multivariable functions and gradients

Geometrically multi variable functions represent curves and surfaces. The gradient of the function is expressed in terms of the derivatives of the function. When represented as a vector, gradient gives the direction of the steepest change and its magnitude gives the rate of change.

The find the gradient (also called the gradient vector) of a two variable function, we'll use the formula $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

This gives a vector-valued function that describes the function's gradient everywhere. If we want to find the gradient at a particular point (x,y),

$\nabla f(x,y) = \langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \rangle$

evaluate at that point. The maximal directional derivative is given by the magnitude of the gradient $\|\nabla f\| = \|(a,b)\|$ where *a* and *b* are components of $\nabla f(x,y) = \langle a,b \rangle$.

For three variable functions f(x,y,z), gradient at (x,y,z) is

$\nabla f(x,y,z) = \langle \frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \rangle$. Now this can be generalised to n variables.

Illustration 4.4.5

Find the gradient vector of the function and the maximal directional derivative.

$f(x,y) = x^3 + 2x^2y + 4y^2$ at $P(1,1)$

Solution

We'll start with the partial derivatives of the given function *f*.

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy$$

$$\frac{\partial f}{\partial y} = 2x^2 + 8y$$

The gradient of the function in general is

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$$

$\nabla f = \langle 3x^2 + 4xy\ ,\ 2x^2 + 8y \rangle$

To find the gradient at the point $P(1,1)$, substitute x=1, y=1.

$\nabla f = \langle 3(1)^2 + 4.(1).(1)\ ,\ 2.(1)^2 + 8.(1) \rangle = \langle 7,10 \rangle$

To find the maximal directional derivative, we take the magnitude of the gradient that we found.

$\|\nabla f\| = \|a,b\| = \sqrt{a^2 + b^2} = \sqrt{7^2 + 10^2} = \sqrt{149}$

Illustration 4.4.6

Find the gradient vector of the function $f(x_1,x_2,x_3) = e^{x_1x_2x_3} + \sin(x_1x_2)$ at (1,Π/2,0)

Solution

$$f(x_1,x_2,x_3) = e^{x_1x_2x_3} + \sin(x_1x_2),$$

$$\frac{\partial f}{\partial x_1} = x_2x_3e^{x_1x_2x_3} + x_2\cos(x_1x_2), \frac{\partial f}{\partial x_2} = x_1x_3e^{x_1x_2x_3} + x_1\cos(x_1x_2),$$

$$\frac{\partial f}{\partial x_3} = x_1x_2e^{x_1x_2x_3}$$

$$Gradient\ \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

$$\Rightarrow \left(x_2x_3e^{x_1x_2x_3} + x_2\cos(x_1x_2),\ x_1x_3e^{x_1x_2x_3} + x_1\cos(x_1x_2), x_1x_2e^{x_1x_2x_3}\right)$$

At $(1,\pi/2,0), \nabla f = (0,\ 0,\ \pi/2)$

Illustration 4.4.7

Find the stationary points and the Hessian matrix if

$$f(x_1,x_2) = \frac{2x_1^3}{3} - 2x_1x_2 - 5x_1 + 2x_2^2 + 4x_2 + 5 \quad .$$

Solution

$$f(x_1,x_2) = \frac{2x_1^3}{3} - 2x_1x_2 - 5x_1 + 2x_2^2 + 4x_2 + 5,$$

$$\frac{\partial f}{\partial x_1} = 2x_1^2 - 2x_2 - 5, \frac{\partial f}{\partial x_2} = -2x_1 + 4x_2 + 4$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4x_1, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -2, \quad \frac{\partial^2 f}{\partial x_2^2} = 4$$

To find stationary points:

$$Gradient\ \nabla f = 0 \Rightarrow \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = 0$$

$$\Rightarrow \left(2x_1^2 - 2x_2 - 5,\ -2x_1 + 4x_2 + 4\right) = (0,\ 0)$$

$\Rightarrow (-1,-3/2),\ (3/2,-1/4)$ are the stationary points.

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial {x_1}^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial {x_2}^2} \end{bmatrix} = \begin{bmatrix} 4x_1 & -2 \\ -2 & 4 \end{bmatrix}$$

Note that for (-1,-3/2) eigen valus are not all positive and for (3/2,-1/4) both eigen values are positive. So at the latter point the function is convex.

4.4.7 Gradient descent

Gradient descent is an iterative optimisation algorithm which is widely used in applications like machine learning. It is used to find the local minimum of an objective function. Models use training data to update their parameters learning through time. An appropriate objective function called cost or loss function is defined and at each step its value defines how the parameters in the model need to be updated. This function calculates an error at each step between the actual and predicted value when learning is done by training data. It's important as fine-tuning parameters helps us to reduce prediction errors. For this gradient descent method needs a direction and a rate of change called learning rate. Both these are obtained when gradient is calculated at a step which consists of the partial derivatives with respect to the input variables.

The basic concept can be seen in the equation

Suppose θ is the model parameter to be updated and L(θ) is the Loss function. If θ_t denotes the parameter value at time t and γ is the learning rate, then $\theta_{t+1} = \theta_t - \gamma\ \nabla L(\theta)$ where $\nabla L(\theta)$ denotes the gradient.

Gradient measures the change in the model parameters with respect to the change in the error. When the gradient is higher, the model can learn faster. Learning rate describes the size of the steps and usually it is taken as a small quantity. This may increase the number of steps required but a large learning rate risks overshooting the minimum required. Direction of the changes in the parameters indicate whether the model moves away from the actual minimum value. The loss function calculates the difference between actual and predicted value at each step continuously moving along the steepest descent. It halts when cost function is close to zero within a prescribed limit. The parameter values at this step give the optimal values.

4.4.8 Variants of gradient descent

Gradient descent is simple to implement. But it has some limitations. When gradient is gentle in a region it takes much time to move from that. It was also found to have instability in high dimensional data. So variants of the basic methods are suggested. Mainly three variants are discussed:

1. Batch gradient descent:

In this method, error is calculated for each example in the training data but the parameter is updated only after the error is summed up for all the examples. This calculation done cyclically and the duration is called an epoch. For smaller data sets, it gives good result. It has computational efficiency producing stable error gradient and convergence but for large data sets it needs more space as it has to store all training data at each step for updating. Also, it may be trapped in local minima.

2. Stochastic gradient descent:

In this method, only one example of the training data is used to compute the gradient and parameters are updated one by one in each epoch. These frequent updates provide a detailed picture of the changes in the parameters but it may also lead to noisy gradients resulting in jump changes rather than slow decrease. It may also be computationally more expensive.

3. Mini- batch gradient descent:

It is a combination of both the above methods. In this method, training data set is split into small subsets or batches and updates are performed based on the computation from these batches .It strikes a balance between the two previous methods taking advantage of the small size for efficiency present in batch gradient method and robustness of gradient which is the advantage of stochastic gradient method. Batch size can be between 50 and 256 depending on the particular problem.

4. Momentum based gradient descent method:

The basic gradient descent method is usually slow and when there are many local maxima and minima, it may converge slowly to the global minimum. In momentum based method to overcome this disadvantage, a fraction of the previous update is also added to the current update. This momentum maintains a consistent update speeding up the convergence. Geometrically, in the steep regions of the loss function, this helps to smooth out the oscillations.

There are other variants like Adam(Adaptive Moment estimation), AdaGrade (Adaptive Gradient Algorithm), RMSprop(Root Mean Squared Propagation), etc. which are usually called adaptive methods. They consider the history of updates to resolve variable gradient, oscillations and nonstationarity in the data.For instance, in AdaGrad, if there were more updates, the learning rate is decayed.

One major challenge in gradient descent method is the choice of learning rate. It is critical for successful convergence to the optimal value. A large learning rate can cause divergence, oscillations or overshooting of the algorithm. A low learning rate will increase computation time and slows down convergence. In ideal cases, a large learning rate for small gradient-regions and small learning rate for high-gradient regions are required. For large data sets and in non convex optimisation methods, it may be necessary to choose different learning rates for different epochs.

There are other challenges also. There are local minima points which mimic global minimum near which the gradient of the cost function increases on one side and decreases on the other side. There are saddle points at which on one side local minimum is attained and on the other, a local maximum. In both these cases, the gradient is close to zero which is the halting condition and hence the algorithm may stop learning.

To overcome these challenges, one method is to plot the graph of loss function by taking number of iterations on the horizontal axis and the value of loss function on the vertical axis. This allows a visualisation of the cost function after each step allowing modification of learning rate after each training. The plot also shows whether the cost function is increasing.

Summarized Overview

This unit provides a foundational understanding of optimization techniques essential for various scientific and engineering disciplines. It begins by introducing the core concept of the gradient as a tool for navigating a function's landscape to find its minimum. The primary focus is on gradient descent, a fundamental algorithm that iteratively adjusts parameters by moving in the direction opposite to the gradient.

The unit then delves into the nuances of the optimization landscape itself, defining convex and concave functions and explaining why a convex function is desirable as it guarantees that any local minimum is also the global minimum. Conversely, it addresses the challenges posed by non-convex functions, which can contain multiple local minima and saddle points, where the algorithm can get stuck.

Finally, the unit explores the practical variations of the gradient descent algorithm, including Batch, Stochastic, and Mini-Batch Gradient Descent, highlighting the trade-offs in computational cost, convergence speed, and stability that make each variant suitable for different applications and datasets.

Assignments

1. What is gradient descent?
2. What are the main variants of gradient descent algorithms?
3. Explain the importance of the learning rate in gradient descent.
4. How does gradient descent help in finding the local minimum of a function?
5. .What challenges arise when using gradient descent on non-convex functions?
6. Explain the purpose of using gradient descent in machine learning models.
7. Describe the concept of the cost function and its role in gradient descent.

8. Explain what a derivative tells us about the cost function in the context of gradient descent.
9. What is batch gradient descent, and when would you use it?
10. Discuss the concept of stochastic gradient descent (SGD) and its advantages and disadvantages.
11. What is mini-batch gradient descent, and how does it differ from other variants?
12. What is the role of second-order derivative methods in gradient descent?
13. Locate and classify all critical points of $f(x) = x^{2/3} - 2x$.
14. Find the absolute maximum and minimum values of

    $h(x) = 2x^3 - 3x^2 - 36x + 7$ on the closed interval [-3, 4].
15. Find the global maxima and minima of function $f(x) = 4e^x + 3$ in the interval [0, 3].
16. Find the local extrema of $g(x) = xe^{-(x^2)}$ for $x \geq 0$
17. Find the inflection points of $h(x) = x^4 - 4x^3 + 6x^2$.
18. Find and analyze the Local Maxima and Local Minima of the function $f(x) = -x^2 + 4x - 5$ by using the second derivative test.
19. Determine if $f(x)=x^4$ is convex or concave.
20. Identify whether the function $f(x)=\log(x)$ is convex or concave.
21. Find the critical points and the nature of the function

    $f(x) = x^3 + 3x^2 - 9x + 1$

Suggested Reading

1. Mathematics for Machine Learning, Marc Peter Deisenroth A. Aldo Faisal Cheng Soon Ong
2. James Stewart – Calculus: Early Transcendentals, 8th ed., Cengage Learning, 2015

Reference

1. E. Kreyszig – Advanced Engineering Mathematics, 10th ed., Wiley, 2011.
2. Stephen Boyd, Lieven Vandenberghe – Convex Optimization, Cambridge UniversityPress,2004.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

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QP CODE: ……… Reg. No : ....

Name : ……...........

Model Question Paper- set-I

End Semester Examination

M25CA02DC: Mathematical Foundation for Computer Applications

(CBCS - UG)

2025-26 - Admission Onwards

Time: 3 Hours Max Marks: 70

Section A

Answer any ten of the following questions. Each answer should be in two or three valid points. Each question carries 2 marks. (10x2=20)

1. State if the following statement is true or false and justify the answer: If the row-echelon form of the augmented matrix of a system of linear equations contains the row [1 0 0 0 0], then the original system is inconsistent.
2. Find algebraic multiplicity and geometric multiplicity of eigen values of the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.
3. Find if $S = \{(-2, 2), (3, 5)\}$ is linearly independent.
4. Explain if $W = \{(x_1, x_2, 1): x_1$ and x_2 are real numbers$\}$ can be a subspace of R^3.
5. Express in symbols and write its negation in sentence: There is an honest politician.
6. A simple graph G has 44 edges. It has 4 vertices of degree 5, 5 vertices of degree 4, and the remaining vertices have degree 2. Find the total number of vertices
7. Define Hamiltonian circuit and Hamiltonian path. Draw a graph that has a Hamiltonian path, but does not have a Hamiltonian circuit.
8. Discuss the steps in Singular Value Decomposistion.
9. The number of typing mistakes that Revathy makes on a given page has a Poisson distribution with a mean of 3 mistakes.

   a. What is the probability that she makes exactly 7 mistakes on a given page?

b. What is the probability that she makes no mistake on a given page?

10. Describe two merits and two demerits of mean and mode.

11. Explain skewness and Hurtosis and discuss their measures.

12. Let μ denote the true average tread life of a certain type of tire. Test μ=30,000 against μ>30,000 based on a sample of size 16 from a normal population distribution with σ=1500 at level of significance 0.01.

13. Find the first order derivative with respect to x and second order derivative with respect to z for $f(x, y, z) = ye^z \sin(xz)$

14. Explain the significance of area under a curve in Receiver Operating Characteristic curve.

15. How a convex function is identified from its graph? Find if $f(x) = x^4$ is convex or concave.

Section B

Answer any six of the following questions. Provide adequate valid points with clear explanations, proportionate to the marks allotted. Each question carries 5 marks.

(5x6=30)

16. Solve by Gaussian elimination method: x-2y+3z=9, -x+3y=-4, 2x-5y+5z=17.

17. Examine if {(2, −4, 2), (0, 2, 4), (−10, −4, 2)} is orthogonal. Is it a basis for R^3? Find the L^1 norm of the vectors.

18. Define binary tree in a graph. Prove that a graph G has a spanning tree if and only if G is connected.

19. From the following table, find the area bounded by the curve and the x-axis, from x=7.47 to x=7.52 using Simpson's one-third rule:

7.47	7.48	7.49	7.50	7.51	7.52
1.93	1.95	1.98	2.01	2.03	2.06

20. Find the LU factorization of A= $\begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix}$

21. The discrete random variable X has the following PMF.

$$p_X(x) = \begin{cases} b & \text{for } x = 0 \\ 2b & \text{for } x = 1 \\ 3b & \text{for } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of b? (b) Determine the values of (i) $P[X < 2]$ (ii) $P[X \leq 2]$, and (iii)$P[0 < X < 2]$

22. 20 % of a large consignment of apples are found to be bad. Find the probability that at least 25% apples are bad in a sample of size 400 drawn from it. (Given: for standard normal variable z, area between (i) z = 0 to z = 2.5 is 0.4938

23. Suppose $z = e^{xy}, \quad x = 2u + v, \quad y = \frac{u}{v}. \quad Find \ \frac{\partial z}{\partial u} \ and \ \frac{\partial z}{\partial v}$

24. Find the local maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 1$

Section C

Answer any two of the following questions. Provide adequate valid points with clear explanations, proportionate to the marks allotted. Each question carries 10 marks.

(10X2=20)

25. Find the eigen vectors of the matrix $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. If possible diagonalise it.

26. From the following table of values of x and y, using difference table,find dy/dx and d^2y/dx^2 :

`x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

27. Suppose that a laboratory test on a blood sample yields one of two results, positive or negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without the disease will also produce a positive result (a false positive). Suppose that 1% of the population actually has the disease. What is the probability that a person chosen at random from the population will have the disease, given that the person's blood yields a positive result?

28. Discuss gradient descent method and its variants.

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Model Question Paper- set-II

End Semester Examination

M25CA02DC: Mathematical Foundation for Computer Applications

(CBCS - UG)

2025-26 - Admission Onwards

Time: 3 Hours Max Marks: 70

Section A

Answer any ten of the following questions. Each answer should be in two or three valid points. Each question carries 2 marks. (10x2=20)

1. Find if the following matrix is in echelon form and give reasons:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

2. Find the eigen values of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

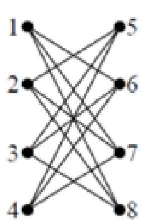
3. Show that the given set of vectors $\{(1, 1), (1,-1)\}$ in R^2 is linearly independent.

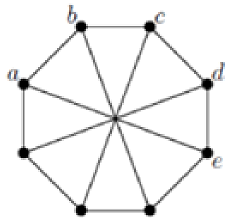
4. Describe the additive inverse of a vector in the vector space of matrices of order 2x3.

5. Find the converse and contrapositive of the statement: If Ilia can afford the car, then Ilia is buying the car.

6. Draw the graph as a diagram and find if it is connected : given V = {1, 2, 3, 4, 5} and E = {{1, 2}, {1, 5}, {1, 3}, {2, 4}, {2, 5}, {3, 4}, {4, 5}, {5}}.

7. Determine whether the following graphs are isomorphic or not? Give reason.





8. Discuss the steps in Singular Value Decomposistion.

9. For the discrete probability distribution

x :	0	1	2	3	4	5	6	7
f(x) :	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

Determine the following:

i. The value of K (ii) mean of X.

10. Compare mean, median and mode as measures of central tendency.

11. What is the meaning of skewness and Kurtosis? How they are measured?

12. Test H0: p=0.55 against H1: p>0.55 for significance level 0.05 and sample proportion 0.60.

13. Find the first order derivative with respect to x and second order derivative with respect to z for $f(x, y, z) = ye^z sin\,(xz)$

14. Explain the significance of area under a curve in Receiver Operating Characteristic curve.

15. Prove that the function f(x) = x^2 is convex using the definition of convexity.

Section B

Answer any six of the following questions. Provide adequate valid points with clear explanations, proportionate to the marks allotted. Each question carries 5 marks.

(5x6=30)

16. Solve the system of equations: $2x_1 + 4x_2 + 3x_3 - 6x_4 = 0$, $x_1 + 2x_2 + 2x_3 - 5x_4 = 0$, $3x_1 + 6x_2 + 5x_3 - 11x_4 = 0$

17. Find the inner product of the vectors u = (0, 7, 2) and v = (9, −3, −2), find their L^2 norms and distance between them. Are they orthogonal?

18. Give an example of an Euler graph. Prove: Every connected graph has at least one spanning tree.

19. Evaluate using Simpson's one-third rule and Trapezoidal rule: $\int_0^6 \frac{dx}{1+x^2}$

20. Find the LU factorization of A= $\begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix}$

21. A random variable X takes values 1,2,3 and 4 such that 2P(X=1)=3P(X=2)=P(X=3)=5P(X=4). Find its probability distribution, mean and variance.

22. 20 % of a large consignment of apples are found to be bad. Find the probability that at least 25% apples are bad in a sample of size 400 drawn from it. (Given: for standard normal variable z, area between (i) z = 0 to z = 2.5 is 0.4938

23. Suppose $z = e^{xy}, \quad x = 2u + v, \quad y = \frac{u}{v}. \quad Find \ \frac{\partial z}{\partial u} \ and \ \frac{\partial z}{\partial v}$

24. Find the local maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 1$

Section C

Answer any two of the following questions. Provide adequate valid points with clear explanations, proportionate to the marks allotted. Each question carries 10 marks. (10X2=20)

25. Show that $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ is diagonalizable.

26. From the following table of values of x and y, using difference table, find dy/dx and d^2y/dx^2 :

x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

27. Suppose that a laboratory test on a blood sample yields one of two results, positive or negative. It is found that 95% of people with a particular disease produce a positive result. But 2% of people without the disease will also produce a positive result (a false positive). Suppose that 1% of the population actually has the disease. What is the probability that a person chosen at random from the population will have the disease, given that the person's blood yields a positive result?

28. What are the important variants of gradient Descent method? Compare them.

സർവ്വകലാശാലാഗീതം

വിദ്യയാൽ സ്വതന്ത്രരാകണം
വിശ്വപൗരരായി മാറണം
ഗ്രഹപ്രസാദമായ് വിളങ്ങണം
ഗുരുപ്രകാശമേ നയിക്കണേ

കൂരിരുട്ടിൽ നിന്നു ഞങ്ങളെ
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നീതിവൈജയന്തി പാറണം

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ബോധരശ്മിയിൽ തിളങ്ങുവാൻ
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