MANAGEMENT OPTIMISATION TECHNIQUES

COURSE CODE: M21CM11DC

Postgraduate Programme in Commerce

Discipline Core Course

Self Learning Material

SREENARAYANAGURU OPEN UNIVERSITY

The State University for Education, Training and Research in Blended Format, Kerala

SREENARAYANAGURU OPEN UNIVERSITY

Vision

To increase access of potential learners of all categories to higher education, research and training, and ensure equity through delivery of high quality processes and outcomes fostering inclusive educational empowerment for social advancement.

Mission

To be benchmarked as a model for conservation and dissemination of knowledge and skill on blended and virtual mode in education, training and research for normal, continuing, and adult learners.

Pathway

Access and Quality define Equity.

Management Optimisation Technique

Course Code: M21CM11DC

Semester-IV

Master of Commerce
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OPEN UNIVERSITY

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MESSAGE FROM VICE CHANCELLOR

Dear learner,

I extend my heartfelt greetings and profound enthusiasm as I warmly welcome you to Sreenarayanaguru Open University. Established in September 2020 as a state-led endeavour to promote higher education through open and distance learning modes, our institution was shaped by the guiding principle that access and quality are the cornerstones of equity. We have firmly resolved to uphold the highest standards of education, setting the benchmark and charting the course.

The courses offered by the Sreenarayanaguru Open University aim to strike a quality balance, ensuring students are equipped for both personal growth and professional excellence. The University embraces the widely acclaimed "blended format," a practical framework that harmoniously integrates Self-Learning Materials. Classroom Counselling and Virtual modes, fostering a dynamic and enriching experience for both learners and instructors.

The University aims to provide learners with an engaging and intellectually stimulating educational experience. This Self-Learning Material, designed for various MCom courses, builds upon the foundational knowledge acquired at the undergraduate level. It deepens the understanding of business and management concepts while integrating theoretical perspectives with practical applications. Each course has been carefully structured to combine academic insights with real-world relevance through the use of case studies, illustrations, and examples. These materials foster critical thinking, analytical skills, and informed decision-making—essential skills for success in the dynamic world of commerce and management.

The Self-Learning Material has been thoughtfully prepared to ensure clarity and comprehensive understanding, guiding learners towards academic excellence and professional competence.Rest assured, the university's student support services will be at your dis-posal throughout your academic journey, readily available to address any concerns or grievances you may encounter. We encourage you to reach out to us freely regarding any matter about your academic programme. It is our sincere wish that you achieve the utmost success.

Regards,
Dr. Jagathy Raj V. P.

01-10-2025

Contents

01 BLOCK

Operations Research

Block Content

Unit 1

Introduction to Operations Research

Learning Outcomes

After completing this unit, the learner will be able to:

- define the concept of Operations Research
- understand approaches of Operations Research
- explain the OR Models

Background

Britain's military capabilities were severely limited during World War II. It became crucial to properly organize every military operation and make prudent use of those resources. In order to accomplish this, the United States and Britain assembled specialists such as statisticians, mathematicians, psychologists, and physicists. These professionals used scientific methods to help tackle war-related issues. Their efforts were crucial to the victory of significant conflicts, including the 'Air Battle of Britain,' the Pacific War, and the 'Battle of the North Atlantic.' Due to the success of these initiatives, nations such as the United States, Canada, and France began implementing similar strategies. In Britain, research conducted to support military operations was referred to as 'Operations Research' (OR). Later, this method was also referred to as 'Cost Benefit Analysis,' 'System Analysis', 'Operational Analysis', 'Decision Science,' and 'Management Science'.

Following the war, industries began utilizing OR to address business issues after realizing its value. Businesses required more effective methods for problem-solving and decision-making as they grew more complicated. This is the reason why business and industry started using OR techniques.

Keywords

Operations Research, OR Models, Approaches of OR

Discussion

1.1.1 Introduction to Operations Research

1.1.1.1 Meaning

- QR is the application of advanced analytical method to help make better decisions

Operations Research is a systematic, scientific, and analytical approach to decision-making. It involves studying complex real-world problems and developing models that help in finding optimal or near-optimal solutions. In simple terms, OR is used to find the best possible way to use limited resources like time, money, materials, machines, and people to achieve desired goals. This can be illustrated with an example.

A bakery faces the challenge of deciding how many cakes, bread, and cookies to produce daily using limited ingredients and oven time. To solve this, the bakery applies Operations Research techniques. First, they identify the problem and gather data on ingredient usage, baking time, and profit margins. Then, they build a simple mathematical model to find the most profitable combination of products within their resource limits. After analysing the model, they test the solution on a small scale. Once proven effective, they implement it into daily operations. They continue monitoring and updating the plan as conditions change, ensuring optimal performance and profit. This practical use of OR helps the bakery make smarter, data-driven decisions.

The significance of Operations Research can be better understood through the following example:

Hospitals worldwide faced a significant dilemma during the COVID-19 pandemic: how to fairly and effectively distribute a small number of intensive care unit beds among a high number of severely ill patients. Many hospitals employed Operations Research (OR) methods, such as simulation, priority matrices, and optimization models, to help them make decisions in this life-or-death circumstance. In order to save as many lives as possible, healthcare management used these OR procedures to determine who should be admitted, how to schedule staff and how to use ventilators.

This case demonstrates the effectiveness of operations research in using data, logic, and mathematics to solve challenging, high-stakes situations.

1.1.1.2 Definition

Organizations today deal with difficult issues on a daily basis, such as how to efficiently organize deliveries, schedule employees, manage resources, and cut expenses. It's not always simple to make the best choices in these circumstances, particularly when time, money, and manpower are scarce. Operations Research (OR) can help with this.

The definitions stressed by various experts and Societies on the subject together enable us to know what OR. is, and what it does. They are as follows:

Committee on OR National Research Council, USA defines Operations Research as

'The application of the scientific method to the study of operations of large complex organizations or activities. It provides top level administrators with a quantitative basis for decisions that will increase the effectiveness of such organizations in carrying out their basic purposes'.

Miller and Starr state, 'OR. is applied decision theory, which uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problem'.

Saaty considers OR. as tool of improving quality of answers. He says, 'OR. is the art of giving bad answers to problems which otherwise have worse answers'.

1.1.2 Nature of Operations Research

1. Scientific and Systematic Approach

The scientific method is the foundation of OR. In order to identify the best feasible solution, it investigates real-world issues using observation, data collection, model construction, and logical analysis.

2. Quantitative in Nature

OR works with statistics, numbers, and quantifiable elements. It makes decision-making more objective by representing issues and identifying solutions using statistical and mathematical methods.

3. **Decision-Oriented**

The primary goal of OR is to assist decision-makers in selecting the best course of action. It offers the best decision under given constraints, not just an analysis of the issue.

4. **Logical and Rational Thinking**

OR calls for rational, well-organized, and transparent thinking. To arrive at the best choice it avoids guesswork.

5. **Dynamic and Flexible**

In a variety of industries, including manufacturing, transportation, healthcare, banking, and even military operations, OR can be applied to solve a wide range of issues.

1.1.3 Features of Operations Research

1. **Decision-Making Focused**

The purpose of OR is to assist decision-makers in making better decisions. Based on information and sound reasoning, it provides the best option among several options.

2. **Scientific and Systematic Approach**

OR employs a methodical, systematic approach, defining the issue, creating models, evaluating them, and offering solutions. Because of this, the procedure is scientific rather than speculative.

3. **Quantitative and Analytical**

It analyses issues mostly using mathematical models and quantitative data. This eliminates personal prejudice and adds objectivity to judgments.

4. **Interdisciplinary in Nature**

OR integrates methods and instruments from computer science, engineering, economics, statistics, mathematics, and management. It is not restricted to any particular field.

5. **Based on Real-Life Problems**

OR is application-focused and pragmatic. Real-world organizational issues like staff scheduling, inventory control, and production planning are resolved with its help.

6. **Optimization**

One of the core components of OR is optimization. The best way to solve an issue is determined by OR practitioners using their models. The answer could be to reduce expenses, increase earnings, or shorten the amount of time needed to finish a task. To find the optimal solution, OR practitioners employ optimization strategies like integer, dynamic, and linear programming.

1.1.4 Phases of Operations Research Study

Operations Research (OR) is both a science and an art when used for decision-making. It is a science because it uses mathematical methods and techniques. At the same time, it is an art because solving problems effectively also depends on the creativity, judgment, and experience of the people applying those methods. The following are the main stages of putting OR into practice:

- **The problem's definition**

◆ Understand and define the real world problem

Problem definition involves outlining the extent of the issue being studied. The entire OR team should do this function. The goal is to define the three main components of the decision problem: (1) a description of the options for making a decision, (2) the study's goal, and (3) a description of the constraints that the modelled system must work within.

- **Model construction**

◆ Build a mathematical or logical representation of the problem

Model construction involves translating the defined problem into mathematical relationships. If the resulting model matches a standard mathematical type, such as linear programming, a solution can usually be found using existing algorithms.

- **Model solution**

Solving the model is usually the easiest step in Operations Research because it uses clear and proven methods to find the best solution. A key part of this step is called sensitivity analysis, which helps understand how the solution might change if some numbers or conditions in the model are slightly different. This is especially useful when the exact values of certain inputs are uncertain or hard to estimate. In such cases, it's important to see how small changes in those values might affect the final result.

- **Model verification**

- Solve the mathematical model using appropriate methods

Model validity is about checking whether the model works as expected and gives reliable results. The first step is to make sure the results are reasonable and not surprising. In simple terms, the solution should make sense and feel right based on experience. A common way to check this is by comparing the model's results with actual data from the past. If the model gives similar results when given the same conditions, it is considered valid.

- **Putting the solution into practice**

- Ensure the solution remains effective over time

When a validated model's solution is implemented, the findings must be converted into clear operational instructions that can be given to those in charge of running the suggested system. This task is mostly the responsibility of the OR group.

Advantages of Operations Research

1. **Improved Efficiency:** OR helps improve operations by optimizing tasks like managing resources, planning production, and controlling inventory, leading to cost reduction and higher productivity.
2. **Better Decision-Making:** It offers data-driven tools and models that support managers in making smart, well-informed choices, especially in complex or uncertain situations.
3. **Multidisciplinary Nature:** OR combines knowledge from fields like mathematics, statistics, and computer science, enabling detailed and well-rounded problem analysis.
4. **Wide Range of Uses:** OR techniques are effectively used in many sectors such as supply chain, healthcare, banking, and transport, proving its value in solving real-life problems.

Limitations of Operations Research

1. **Data Dependence:** OR models need accurate and complete data. If the data is wrong or missing, the solutions may not be effective.
2. **Model Complexity:** Creating and solving OR models can be difficult, time-consuming, and require a lot of resources.

3. **Uncertainty and Risk:** OR may not account for all possible risks and unpredictable factors, which can lead to unexpected results.
4. **Implementation Issues:** Applying OR solutions in real life can be challenging, especially in big organisations with many stakeholders.
5. **Unrealistic Assumptions:** OR models often make assumptions to simplify problems, but these may not reflect real-world conditions.
6. **Limits of Human Judgment:** OR provides support for decision-making but cannot replace human intuition and experience.

Despite these issues OR is still a useful tool. When combined with other methods and used wisely, it helps organisations make better decisions.

1.1.5 Approaches of OR

Operations research employs different approaches to investigate issues and methodically create solutions. The following are some of the most widely used strategies:

1. **Scientific Method Approach**

In OR, this is the most basic strategy. Observation, hypothesis development, data gathering, model construction, testing, and implementation are all part of this scientific approach to decision-making.

The following are the steps involved in scientific approach:

- Scientific method to investigate and solve problems

- Determine the issue
- Gather information
- Construct a model
- Examine and evaluate the model
- Put the solution into practice
- Monitor and update

For example, a chain of supermarkets employs OR to examine customer movement and cut down on checkout wait times by applying scientific analysis of historical sales and footfall data.

2. **System Approach**

This method views the entire company as a system composed of interdependent pieces. It emphasizes how modifications to one component impact the system as a whole. When creating OR models, all variables and their interactions are taken into account.

- All variables and their interactions are taken into account

The main idea is to maximize system performance rather than just departmental performance. For example to increase overall efficiency, inventory, production, and transportation are coordinated at a manufacturing facility using OR.

3. **Mathematical Modelling Approach**

This method entails creating mathematical models of actual issues. Computational or mathematical methods can be used to solve these models. Types of models used under this approach are:

- To represent real-world Problems

- Linear programming
- Queuing models
- Simulation models
- Inventory models

For example, a business uses linear programming to figure out how to optimally distribute scarce raw materials among several products in order to maximize profit.

4. **Simulation Approach**

Simulation is used to build virtual models of real-world systems and examine how they respond in various scenarios when issues are too complicated for precise mathematical solutions. Imagine an airport is facing problems like long lines at the check-in counters and delays at security checks. Instead of changing everything physically right away (like building new counters or hiring more staff), the airport creates a computer simulation—a kind of virtual model of how passengers move through the airport.

- To study behaviour under various conditions

In this simulation, they input data like:

- Number of passengers arriving at different times
- How long check-in and security usually take
- Number of counters available

- Staff schedules - Then, using this model, they test different ideas like:
- What if we add 2 more security staff?
- What if we open more check-in counters during peak hours?

This helps them see the results without spending money or making real-world changes.

1.1.6 Operations Research Models

- Mathematical depiction of real-world system

A logical or mathematical depiction of a real-world issue is called an operations research model. It assists decision-makers in examining intricate circumstances, experimenting with various approaches, and determining the optimal course of action.

They assist in identifying high-quality or ideal solutions and also take into account limitations and uncertainties

Table 1.1.1 **Types of OR Models with Examples**

Type of OR Model	Description	Example
1. Descriptive Model	Describes how a system works, but doesn't give decisions	A flowchart showing how raw materials move in a factory
2. Prescriptive Model	Suggests the best solution among different alternatives	Linear programming model to maximize profit
3. Predictive Model	Predicts future outcomes based on past data	Forecasting sales using time series analysis
4. Deterministic Model	All input values are known with certainty	A transportation model where costs and supplies are fixed
5. Probabilistic (Stochastic) Model	Includes uncertainty or randomness	A queuing model for customer service with random arrival times
6. Simulation Model	A virtual system is built to test different scenarios	Simulating traffic flow at a busy intersection to plan signal timings
7. Dynamic Model	Problem changes over time and is analysed in stages	Inventory control model updated every month
8. Static Model	Problem is studied at one point in time	Assignment problem for allocating workers to jobs once

Purpose of OR Models

- To comprehend difficult problems.
- To forecast results in various scenarios
- To maximize performance (save expenses, save time, boost productivity)
- To encourage improved decision-making

1.1.7 Operations Research Models in Practice

There isn't just one kind of problem that Operations Research (OR) can solve. OR uses different methods to handle different types of real-world decision-making problems. Here's a simple explanation of the main types of OR models used in practice:

1. Allocation Models

- Allocate resources in an optimum way

These models help in deciding how to best use limited resources (like money, time, materials, or labour) across different tasks or departments. The goal is to get the best possible result (like maximum profit or minimum cost).

2. Inventory Models

- Optimize stock level and orders

These models help businesses decide how much to order and when to order items to avoid running out of stock or having too much inventory. They aim to reduce three types of costs: storage costs, shortage costs, and ordering/setup costs. They are also useful when dealing with bulk discounts or managing different types of inventories.

3. Waiting Line (Queuing) Models

- Analyses waiting lines and service times

These models are used to manage queues or waiting lines (like in hospitals, banks, or customer service). They help balance the cost of providing service (like hiring more staff) with the time customers have to wait. The model looks at how people arrive, wait, and get served.

4. Competitive (Game Theory) Models

- Competitive Situation with multiple decision makers

These models study situations where two or more people or businesses compete. Each one wants to do better than the other. The model helps figure out the best strategy depending on how the others might act. It applies to business, military, sports, and economics.

5. **Network Models**

- Flow through a network of nodes

Used in large projects (like construction or product launches), these models help plan and schedule tasks. Techniques like PERT and CPM are used to find out which tasks might cause delays and how to use time and resources efficiently.

6. **Sequencing Models**

- Minimize total time and cost

These models decide the order in which tasks should be done (for example, in a factory or hospital) so that the total time or cost is minimized.

7. **Replacement Models**

- Minimize total time and cost

These models help figure out the best time to replace equipment or machinery—either when its performance drops over time or when it fails suddenly.

8. **Dynamic Programming Models**

- Solve multi stage decision problems

Used for complex problems with several stages or steps, these models break the problem into smaller parts and solve each one step by step to find the best overall solution.

9. **Markov Chain Models**

- Used to represent under go transition from one state to another

These models are useful when a system changes from one condition to another over time (like customer behaviour or weather patterns). They use probabilities to predict future changes based on current conditions.

10. **Simulation Models**

- Initiate the operations of real world system

When a problem is too complex for exact solutions, simulation models help by mimicking real-life situations on a computer. Repeating the simulation many times helps in understanding what decisions work best.

11. **Decision Analysis Models**

- Choice under uncertainty by evaluating alternatives

These models help in choosing the best option when outcomes are uncertain. They consider different possible results and their likelihood to guide decision-making under risk.

Table 1.1.2 : Operations Research Models: Chart Overview

Model Type	Main Purpose	Examples
1. Allocation Models	Distribute limited resources for the best outcome	Budget allocation, staff assignment
2. Inventory Models	Decide how much and when to order	Stock control in warehouses
3. Waiting Line Models	Balance customer wait time and service efficiency	Bank queues, hospital patient flow
4. Game Theory Models	Compete smartly against rivals	Business pricing, military strategies
5. Network Models	Plan and schedule large tasks efficiently	Project management, road planning
6. Sequencing Models	Decide the best order to perform tasks	Job scheduling in factories
7. Replacement Models	Find the right time to replace worn-out equipment	Machinery or vehicle replacement
8. Dynamic Programming	Solve problems with multiple steps or decisions	Route planning, financial decisions
9. Markov Chain Models	Predict changes in systems over time using probabilities	Customer behavior, brand switching
10. Simulation Models	Imitate real-life processes to test different scenarios	Traffic systems, business process testing
11. Decision Analysis	Choose the best option when outcomes are uncertain	Investment decisions, risk-based choices

1.1.8 Principles of Modelling

In operations research (OR), modelling is the process of employing simulation, logic, or mathematics to create a simplified version of a real-world situation. Modelling is used to better understand the issue, evaluate potential solutions, and assist in decision-making to identify the best course of action. The following are the main principles that direct modelling process of OR:

1. **Principle of Simplicity**

- Create simplified version of real-world problems

The model should be as easy as it can be while yet retaining the key elements of the actual issue. It can be challenging to solve and interpret models that are too complicated.

2. **Principle of Relevance**

- Contains pertained variables and relationships

Make sure the model only contains pertinent variables and relationships. Extraneous information may divert attention from the primary decision issue.

3. The Accuracy Principle

- Use of trustworthy assumptions and accurate data

For the model to yield useful results, it must accurately depict the real-world system. Make use of trustworthy assumptions and accurate data.

4. The Validity Principle

- Comparing actual historical data from the real world

To ensure the model gives logical and realistic results in expected situations, it needs to be tested. This is often done by comparing the model's results with actual historical data from the real world.

5. Principle of Cost-Effectiveness

The benefits the model provides for decision-making should be greater than the time, cost, and effort needed to create and apply it.

Advantages of Model Building

1. **Easy to Understand:** Models show how different parts of a system are related, which is easier to understand than just explaining it with words.
2. **Complete View:** They help us see the entire problem at once, so we don't miss any important part.
3. **Better Communication:** Models make it easier for people in a company to share ideas and understand each other—for example, showing a better work process using a simple chart.
4. **Safe Testing:** We can test and improve complex systems using models without affecting the real ones—like testing a satellite launch without actually sending one into space.
5. **Predict the Future:** Models help us guess what might happen in the future, which helps in making smarter decisions.

1.1.9 Scope of Operations Research

1. Inventory Control

- Maintaining the right amount of stock by balancing supply with customer demand

Operations Research helps in maintaining the right amount of stock by balancing supply with customer demand. It focuses on reducing excess inventory, avoiding shortages, and keeping costs low.

2. Transportation and Logistics

- Finding the shortest or least cast route for delivery

It assists in planning the most efficient travel routes, vehicle scheduling, and improving the flow of goods in the supply chain. The aim is to cut down on transport expenses and ensure timely deliveries.

3. Production Scheduling

- Create effective production time table

OR is used to create effective production timetables, allocate machinery and labor, and manage factory capacity. This helps increase output while reducing production costs.

4. Project Scheduling and Management

- Complete the projects on time within the budget

It supports the planning of project timelines, resource assignment, and risk handling. The goal is to complete projects on time and within the allocated budget.

5. Financial Analysis

- Selecting the most profitable investment portfolios

OR methods help in studying financial data, predicting future trends, and supporting smart investment planning and decision-making.

6. Healthcare Systems

- Improvents in different areas in healthcare

In the healthcare field, OR improves patient appointment systems, allocates medical staff and equipment effectively, and helps in planning treatments, all to enhance service quality and cut down operational costs.

7. Network Planning

- Low costs and improve service performance

OR helps in designing efficient systems such as transport routes, communication lines, or distribution networks. The objective is to lower costs and improve service performance.

8. Resource Distribution

- Achiever overall outcome

It helps in wisely dividing limited resources (like funds, manpower, or equipment) among various competing needs to achieve the best overall outcome or benefit.

Summarised Overview

Operations Research (OR) is a multidisciplinary field that solves challenging decision-making issues by applying statistical analysis, mathematical models, and optimization strategies. It entails the methodical examination of activities in a range of industries, including manufacturing, logistics, and services, with the goal of increasing efficacy and efficiency. Linear programming, simulation, queuing theory, and inventory models are examples of OR techniques that are used to solve practical issues including production scheduling, transportation management, and resource allocation. Phases such as problem identification, model development, solution finding, and implementation are all part of the process. OR has applications in a variety of sectors, including manufacturing, healthcare, and finance. Its fundamental principles are practicality, logic, and optimization.

Self-Assessment Question

1. Discuss the advantage and limitations of operations research.
2. Explain OR as an interdisciplinary approach.
3. What is Mathematical Modelling approach?
4. What are the phases of Operations Research study?
5. What are the advantages of model building?
6. What are the scope of Operations Research?

Assignments

1. Explain various types of OR models and indicate their applications to production, inventory and distribution systems.
2. Describe the role of Operations Research in decision-making in modern businesses.
3. What are the limitations of Operations Research? How can they be addressed in real-world applications?
4. Explain the various stages of operations reserach from problem formulation to implementation.
5. 'Operations Research has a wide scope in modern management'. Comment on this tatement with suitable examples.

Reference

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Suggested Reading

1. Taha, H. A. (2017). Operations Research: An Introduction (10th ed.). Pearson Education.
2. Sharma, J.K., Operations Research: Theory and Application (6th ed.).Trinity press

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

Unit 2

Techniques of OR

Learning Outcomes

After completing this unit, the learner will be able to:

- understand the concept of decision making
- explain the relation between decision making and OR
- narrate steps in decision making
- elucidate techniques of OR

Background

Operations Research originated during World War II when scientific research was used to solve military operational problems. In order to create models that replicate actual business processes, it has now developed into a multidisciplinary topic that integrates mathematics, statistics, economics, computer science, and engineering. In many fields, including manufacturing, logistics, supply chain management, finance, healthcare, transportation, and project management, OR provides a methodical and quantitative approach to problem-solving.

By using methods like network analysis, simulation, inventory models, queuing theory, game theory, linear programming, decision trees, and network analysis, operations research aims to assist decision-makers in choosing the optimal course of action among available options. These methods give managers the ability to assess the possible results of various methods, allocate resources optimally, reduce expenses, and increase efficiency or profits.

Operations research is essential to business firms for decision-making in both strategic and operational contexts. OR offers a systematic approach to extracting useful insights from data, whether it is for monitoring inventory levels, planning production activities, choosing the best product mix, or creating distribution networks. Knowing how operations research is applied in business is not only pertinent, but also crucial, given the growing emphasis on data-driven and evidence-based decision making.

Keywords

Operations Research, Decision making

Discussion

1.2.1 Techniques used in OR

Operations Research (OR) is a field that uses advanced analytical methods to help organizations make better decisions and improve efficiency. It's about finding the 'best' way to operate given limited resources and complex situations. The following are re some key techniques used in Operations Research:

- **Linear Programming (LP):** Used when the objective and constraints can be expressed as linear mathematical relationships. It's widely applied for resource allocation, production planning, and transportation.
- **Integer Programming (IP):** A variant of LP where some or all decision variables must be whole numbers (e.g., you can't have half a truck).
- **Nonlinear Programming (NLP):** Used when the objective function or constraints have nonlinear relationships.
- **Dynamic Programming (DP):** Solves complex problems by breaking them down into smaller, overlapping subproblems and solving each subproblem only once. Useful for multi-stage decision-making.
- **Simulation:** When a system is too complex to be modelled mathematically, simulation is used. It involves creating a computer model of a real-world system and running experiments on it to observe its behaviour over time.
- **Queueing Theory (Waiting Line Models):** Analyzes and optimizes waiting lines or queues (e.g., in call centers, supermarkets, or manufacturing plants) to minimize waiting times and maximize service efficiency.
- **Network Analysis:** Used for planning, scheduling, and controlling complex projects.
- **Inventory Control Models:** Helps in managing inventory levels to balance costs associated with holding inventory,

ordering, and potential shortages. Examples include Economic Order Quantity (EOQ).

- **Game Theory:** Provides a framework to analyze decision-making in competitive situations where the outcome for one player depends on the actions of others (e.g., pricing strategies between rival companies).
- **Decision Analysis:** Helps in making decisions under uncertainty by evaluating different alternatives and their potential outcomes, often using tools like decision trees.
- **Forecasting:** Uses historical data and statistical methods to predict future events or trends, aiding in planning and resource allocation.
- **Heuristic Methods:** Provide 'good enough' solutions to complex problems in a reasonable amount of time, especially when finding an optimal solution is computationally too expensive or impossible.

1.2.2 Operations Research and Business Decision Making

- Using mathematical models to choose the best decision for an organisation

Operations Research (OR) is a method of solving problems and making decisions using numbers and analysis. It helps businesses work better by using mathematical models to study complicated situations, find issues, and suggest the best solutions to improve efficiency, profits, and performance. Decision making in Operations Research means using mathematical models and analysis to choose the best option for an organization. It involves picking the most effective solution from different choices to meet goals like reducing costs, using resources wisely, improving efficiency, or satisfying customer needs, all while staying within limits like time, budget, or available resources.

Decision making may be defined as 'a process which result in the selection from a set of different alternative courses of action, that course of action which is considered to meet the objectives of the decision problem more satisfactorily than others as judged by the decision maker.'

1.2.2.1 Advantages of Operations Research in Decision Making

- Spot problems and manage operations

Efficient Control

Operations Research helps in better control and supervision of an organization's activities. Since managing a business is complex and costly, it needs proper planning and regular monitoring. OR gives managers a clear, logical, and data-based way to spot problems and improve operations. It is especially useful in areas like production, manufacturing, construction scheduling, and inventory management.

Effective Systems

- Analyse the problem of decision making

Operations research techniques are to analyse the problem of decision making such as good site for plant, whether to open a new storehouse, etc. It also helps in assortment of cost-effective means of transportation, jobs sequencing, production scheduling, replacement of old machinery, etc

Better and Analytical Decisions

- Making accurate decisions

Operations Research (OR) techniques help in making smarter and more accurate decisions. By using these methods, managers can reduce the chances of making mistakes and get a clearer understanding of the possible outcomes before taking action.

Good Coordination and Management

- Better planning and coordination

OR methods support better planning and coordination across different departments of a company. This helps improve teamwork, ensures smooth operations, and leads to more efficient management.

Maximize Profits & Minimize Losses

- Choose the best possible option

Operations Research (OR) techniques use numerical data to support decision-making. This helps decision-makers choose the best possible option, leading to increased profits and reduced losses for the organization.

1.2.2.2 Steps in Decision Making

1. **Identify the Problem:**

- Analysing the situations

Every decision-making process begins with clearly understanding the core issue or objective that needs to be addressed. This involves analyzing the situation, recognizing the constraints, and defining the decision goal in measurable terms. A well-defined problem sets the foundation for effective analysis and solution.

2. **List Possible Actions (Alternatives):**

- Determine all feasible course of action

Once the problem is identified, the next step is to determine all feasible courses of action. These are the available alternatives or strategies that the decision maker can choose from. Each action represents a possible solution or response to the problem.

3. **Identify Future Events (States of Nature):**

These refer to external conditions or events that may occur and are beyond the control of the decision maker. States of nature represent the uncertainty in the environment, such as market demand, competitor behavior, or weather conditions. Each state affects the outcome of a chosen action differently.

4. **Construct a Pay-off Table:**

A payoff table presents the possible outcomes (profits, costs, or other measures of value) for each combination of an action and a state of nature. It helps to visualize and compare the consequences of different decisions under various future scenarios.

5. **Select an Optimum Decision Criterion:**

Based on the data in the payoff table, a suitable decision rule is applied to select the best course of action. Common decision criteria include maximin (choosing the best of the worst outcomes), minimax regret (minimizing potential regret), and expected value (selecting the action with the highest average payoff based on probabilities).

1.2.2.3 Limitations of Operations Research in Decision Making

- Dependence on computers and software

1. **Operated by Technical & Electronic Devices:** These days operational research techniques obtain an optimal solution using various computational systems, model and techniques. In this scenario, these factors are colossal and expressing them in quantity and establish the relationships among these require calculations that can only be handled by computers.

- Deals with numerical data

2. **Solve only quantitative problems:** O.R. techniques provide a solution only when all the elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors that cannot be quantified find no place in O.R. models.

- Communication gap between decision makers and researchers

3. **Widen the Gap between executive and Researcher:** O.R. being specialist's job requires a mathematician or a statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of O.R. Thus, there is a gap between the two.

- High cost and time consuming

4. **More time and cost consuming:** When the basic data are subjected to frequent changes, incorporating them into the O.R. models is a costly affair. Moreover, a fairly good solution at present may be more desirable than a perfect O.R. solution available after sometime.

Summarised Overview

The crucial function of decision-making in the context of operations research is the main topic of this chapter. It examines the methodical and scientific approach that OR provides to solving challenging business issues. Operations Research (OR) has increasingly contributed to improving the quality of decisions in the field of management. It serves as a valuable analytical approach that supports decision-makers in resolving complex managerial problems. By incorporating scientific methods, OR allows for better clarity and understanding of such issues. OR techniques use a mathematical framework to represent decision-making problems, helping managers make more logical and structured choices. This method relies on quantifying the elements of a decision—such as objectives, alternatives, and influencing factors—using equations and models. OR is especially beneficial in areas like marketing, production, inventory, and distribution, where it helps identify the most efficient choices from the available options. For example, in logistics and distribution, OR assists in determining the best routes, warehouse locations, and delivery schedules to minimize cost and improve service. Similarly, in marketing, it can help optimize product mixes or advertising strategies based on customer data. Ultimately, by applying OR techniques across various departments, organizations can enhance coordination, improve efficiency, and support strategic business decisions.

Self-Assessment Question

1. What is the role of Operations Research (OR) in managerial decision-making?
2. How does OR improve the quality of decisions in management?
3. What are the steps involved in decision making?
4. Explain the limitations of operations research in decision making.
5. Describe the techniques used in OR.

Assignments

1. What are the essential components involved in the OR approach to decision-making?
2. What are the advantages of Operations Research in Decision Making?

Reference

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Suggested Reading

1. Kapoor, V. K. (2000). Operations research. New Delhi: Sultan Chand & Sons.
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Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

Unit 3

Linear Programming

Learning Outcomes

After completing this unit, the learner will be able to:

- Understand the concept of Linear Programming
- List the components of Linear Programming Problem (LPP)
- Formulate LPP
- List the steps to solve a Linear Programming Problem
- Solve LPP graphically

Background

Linear Programming (LP) emerged as a significant mathematical approach for solving optimization problems involving limited resources and competing activities. It was first introduced by Soviet mathematician and economist Leonid Kantorovich in 1939, primarily to address transportation, scheduling, and resource allocation challenges during World War II. The technique quickly gained popularity due to its ability to deliver optimal solutions for real-world decision-making problems in industries such as manufacturing, finance, logistics, and operations management.

LP deals with the allocation of scarce resources — like materials, labour, time, or money — among various competing alternatives in a way that optimizes a particular outcome, such as maximizing profit or minimizing cost. The term 'linear' refers to the straight-line relationships between variables, and 'programming' signifies planning or selecting the best course of action based on these relationships.

This technique relies on a set of assumptions, such as linearity, certainty, divisibility, and non-negativity, which define its applicability. Over time, linear programming has become a foundational tool in operations research and decision sciences, shaping systematic problem-solving approaches across multiple sectors.

Keywords

Linear Programming, Objective Function, Linearity, Optimal Solution, Unbounded Problem.

Discussion

1.3.1 Meaning of Liner Programming

A student has 6 hours to prepare for two subjects: Maths and English.

- Maths needs more practice but gives good marks.
- English needs reading time but is also important.
- The student cannot spend all 6 hours on one subject because time is limited.

Question: How should the student divide 6 hours between Maths and English to score maximum marks?

A transport company has two types of trucks: small trucks and big trucks.

- Small trucks carry fewer goods but cost less to run.
- Big trucks carry more goods but cost more to run.
- The company has limited fuel and drivers.

Question: How many small trucks and big trucks should be used so that the company can deliver all goods at the lowest cost?

These are all Linear Programming problems. In each case, we want the best result (better marks, lower cost) but we also face limits (time, fuel, resources). Linear Programming helps us find the best balance.

- Find the all best possible outcome

Linear Programming is a simple mathematical method used to find the best possible outcome when you have limited resources and several conditions or restrictions to consider. Linear programming is a widely used method for determining the optimal allocation of resources. Linear Programming (LP) is a popular method used to make better decisions. It helps find the best way to use limited resources when there are several needs that may compete or conflict with each other.

The first major use of linear programming was made by Soviet mathematician and economist Leonid Kantorovich in 1939 during World War II. He used it to plan transportation, scheduling, and resource allocation while considering limits like cost and availability.

The term 'linear' indicates a straight-line relationship between variables in a model, meaning that a change in one variable leads to a directly proportional change in another. For instance, if you buy 1 notebook for ₹20, buying 2 notebooks will cost ₹40. The cost increases in direct proportion to the number of notebooks — this is a linear relationship. On the other hand, 'programming' refers to the process of using mathematical models to solve problems involving limited resources. It involves selecting the most suitable option from a set of available choices to achieve a specific goal or objective.

Imagine you run a small factory that makes two products: chairs and tables. You have a limited amount of wood and labour hours available each day. You earn profit from both products, but you can't make as many as you want because of those limits.

So, your questions are:

- How many chairs and tables should I make every day to earn the most money?
- But without using more wood and labour than I have?

Linear programming helps you answer this question. It finds the combination of chairs and tables that gives you the maximum profit, while still staying within your limits for wood and labour.

1.3.2 Definition

Samuelson and Slow defines linear programming as 'the analysis of problems in which a linear function of a number of variables is to be maximized (minimized) when these variables are subject to number of restraints in the form of linear inequalities.'

According to Loomba, 'linear programming is only one aspect of what has been called a system approach to management, where involved programs are designed and evaluated in the terms of their ultimate effects in the realization of business objectives.'

1.3.3 Components of Linear Programming

The basic components of the LP are as follows:

- Decision Variables
- Constraints
- Data
- Objective Function

Requirements of LPP

The fundamental conditions that must be met in order for a problem to be articulated and addressed using linear programming techniques are known as the requirements (or assumptions) of a linear programming problem (LPP). These are the fundamental components that define and limit the use of LPP in situations involving decision-making.

Objective Function

There must be a clearly defined objective to be maximized (e.g., profit) or minimized (e.g., cost).The objective function should be expressed as a linear equation.

Decision variables

In order to optimize the objective function, the issue must have easily identifiable decision variables whose values must be ascertained. These variables need to be continuous.

Constraints

One or more linear constraints on the decision variables, such as time, money, resource availability, etc., must be included in the problem. These constraints must be expressible as linear inequalities or equations.

1.3.4 Assumptions of LPP

1.Proportionality / Linearity

- Changes in input lead to directly proportional changes in the output

Linear programming is based on the idea that changes in the input (constraints) lead to directly proportional changes in the output (objective function). If the input increases or decreases, the result will increase or decrease in the same ratio.For example, if a machine produces 2 shirts in one hour, then according to the proportionality assumption, it will produce 10 shirts in 5 hours, assuming the production rate stays constant and there are no changes in efficiency or conditions.

2. Certainty

- Certainty in all data

In linear programming, certainty means that all the numbers used in the problem are known exactly and do not change. This includes things like how much profit you earn from each product or how many hours of labor or raw materials are available — everything is fixed and clear.

3. Additivity

- Combined effect of multiple variables are the sum of individual effect

In linear programming, additivity means that the total result (like profit or resource usage) is just the sum of the individual results from each item or activity. For example, if you make two products, the total profit is simply the profit from Product A plus the profit from Product B. In the same way, the total amount of a material used is the amount used for A plus the amount used for B.

For example, if you earn ₹200 profit from selling each unit of Product A and ₹300 from each unit of Product B, then selling one unit of each will give you a total profit of ₹200 + ₹300 = ₹500. Similarly, if making A uses 4 litters of paint and B uses 6 litters, the total paint needed for one unit of each is 4 + 6 = 10 litters.

4. Non-NegativeVariable

- All decision variables cannot be negative

In linear programming, it is assumed that all decision variables (like quantity produced, time used, or money spent) must be zero or more — they cannot be negative. This makes sense because in real life, you can't produce or use a negative number of items.

For example, if a company is planning how many tables and chairs to make, it can't decide to make -5 chairs or -10 tables. The number of items produced must be zero or a positive number.

5. Finiteness

- Resources are available are finite and limited

In linear programming, finiteness means that there is a limited number of possible choices or activities to consider. The method works only when the options and resources are countable and clearly defined. If there are unlimited or endless possibilities, then it becomes impossible to find the best (optimal) solution.

For example, if a company has 3 machines and can produce 4 types of products, linear programming can be used to decide how much of each product to make. But if there were an unlimited number of product types or machine settings, it would not be possible to use linear programming effectively.

6. Divisibility

In linear programming, the divisibility assumption means that the values of decision variables can be continuous — they don't need to be whole numbers. For example, a milk dairy can collect 6.254 thousand litres of milk, which is perfectly fine because milk is a divisible quantity.

- Decision variables in a linear programming model can take any non negative fractional value

However, this doesn't work for all situations. For instance, you can't produce 2.5 machines — machines must be made in whole numbers. In such cases, where values must be integers or limited to specific whole numbers, linear programming alone isn't suitable. Instead, we need a different method called Integer Programming.

Advantages of LPP

- **Efficient use of resources:** Linear programming helps make the best use of available resources by guiding how to allocate them effectively.
- **Better decision-making:** It improves the quality of decisions by making them more logical and less based on personal judgment
- **Practical solutions:** While linear programming provides solutions, it also allows for real-world adjustments, recognizing that not everything planned can always be carried out exactly as calculated.
- **Identifies bottlenecks:** It helps spot issues in production, like when some machines are overworked while others are underused.
- **Flexible adjustments:** The solutions can be changed or modified easily if needed.
- **Adaptability:** It allows updating the plan when situations or conditions change.

Disadvantages of LPP

- **Not always realistic:** Linear programming assumes straight-line (linear) relationships between variables, but in real life, things often don't work that way.
- **Non-integer solutions can be impractical:** Sometimes the solution includes fractions, like 2.5 machines or 3.7 work-

ers, which doesn't make sense. Simply rounding off may not give the best result.

- **Ignores time and uncertainty:** The model assumes that everything stays the same over time, but in real situations, conditions often change and are uncertain.
- **Only one goal at a time:** Linear programming focuses on a single objective, but real-world problems often have multiple, sometimes conflicting, goals.

1.3.6 General model of LPP

The general model of LPP aims to maximize or minimize a result (like profit or cost), subject to certain limitations, while keeping the values of variables non-negative. To use linear programming effectively, the first step is to create a mathematical model of the problem. Following are the basic steps:

1. Identify the Decision Variables

- Determine the variables that represent the decisions to be made

This step includes:

- Finding out what decisions need to be made
- Identifying the input data related to these decisions.
- Writing down the constraints and the objective clearly.

How to do this:

a. Write each constraint in simple words. Check if it is a 'greater than or equal to' (≥), 'less than or equal to' (≤), or 'equal to' (=) type of condition.

b. Write the objective in words. Are you trying to maximize profit or minimize cost?

c. From the above, identify what decisions need to be made. Ask: 'What do I need to decide in order to reach my goal?'

After identifying the decision variables, assign symbols (like x_1, x_2) and mention units (e.g., units produced, hours used). This helps in understanding the final solution.

2. Identify the Problem Data

- Collecting and defining all the relevant inforamtion

List all the fixed data, like available resources, costs, profits, etc.

Remember: You can control the decision variables, but you cannot control the data—they are already given.

3. Write the Constraints

Take the verbal constraints (like limits on resources) and convert them into mathematical inequalities or equations using the decision variables.

Each constraint must be written correctly so that the solution meets all the requirements. A wrong formulation may give a wrong or unusable result.

Write the Objective Function

- Define the goal of the problem

Clearly decide whether the goal is to maximize or minimize something. Then write the objective function as a mathematical expression using the decision variables and the profits/costs related to them.

Example: A Factory Producing Tables and Chairs

A furniture company makes tables and chairs.

Each table gives a profit of ₹200 and each chair gives a profit of ₹100.

To make a table, it takes 3 hours of labour and 4 units of wood.

To make a chair, it takes 2 hours of labour and 3 units of wood.

The company has 60 hours of labour and 84 units of wood available.

The company wants to maximize profit.

Step 1: Identify the Decision Variables

Let:

x_1 = number of tables to produce

x_2 = number of chairs to produce

These are the decisions to be made.

Step 2: Identify the Problem Data

Resource	Table (per unit)	Chair (per unit)	Available
Labour (hrs)	3	2	60
Wood (units)	4	3	84
Profit (₹)	200	100	-

Step 3: Formulate the Constraints

Labor constraint:

$3x_1 + 2x_2 \leq 60$

Wood constraint:

$4x_1 + 3x_2 \leq 84$

Non-negativity condition:

$x_1 \geq 0, x_2 \geq 0$ (You can't produce a negative number of products)

Step 4: Formulate the Objective Function

We want to maximize profit:

Maximize $Z = 200\, x_1 + 100\, x_2$

Final LP Model

Maximize $Z = 200\, x_1 + 100\, x_2$

Subject to:

$3x_1 + 2x_2 \leq 60$ (Labour)

$4x_1 + 3x_2 \leq 8$ (Wood)

$x_1 \geq 0, x_2 \geq 0$ (Non-negativity)

Steps to Solve a Linear Programming Problem

To solve a linear programming problem, the first step is to create a mathematical model based on the situation. This model helps to find the best possible solution by applying logical steps.

When the problem involves only two variables, the constraints and objective function can be drawn on a graph. To find the solution, iso-profit or iso-cost lines (lines showing the same level of profit or cost) are used. The best solution is found at the point on the feasible region that lies farthest in the direction that improves the objective (either to maximize profit or to minimize cost). This point shows the most effective decision for the given problem.

In case there are more than two variables, Simplex method is considered to find the solution. Three stages of the solution are as follows:

Feasible Solution

A feasible solution is any set of values for the decision variables (like X_1, X_2, ..X_n.) that satisfies all the constraints of the linear programming problem, including non-negativity conditions (i.e., values must be zero or positive).

Basic Feasible Solution

A basic feasible solution is a special type of feasible solution where:

A few variables (equal to the number of constraints) are chosen as basic variables, and

These variables are linearly independent (not combinations of each other).

The rest of the variables are set to zero.

Optimal Solution

An optimal solution is a feasible solution that gives the best possible value (maximum or minimum) for the objective function.In any linear programming problem, if an optimal solution exists, it can be found through systematic steps starting from a basic feasible solution.

Illustration 1.3.1

ABC Promotions Private Ltd. has been approached to assist a client in designing an advertising budget for launching a new product. The advertising budget is to be allocated between television commercials and magazine ads. The client has specified the following conditions:

1. A maximum of ₹2, 00, 000 can be spent on television advertisements.
2. A minimum of ₹1, 00, 000 must be allocated for magazine advertisements.
3. The cost of one advertisement is ₹5, 000 for television and ₹2, 000 for magazines.

The total expenditure on advertising should not exceed ₹5,00,000. Based on past market research, it is estimated that ₹1 spent on television advertising reaches 300 people, while ₹1 spent on magazine advertising reaches 200 people.

The task is to determine how much should be spent on television and magazine advertising in order to maximize the total number of people reached by the campaign.

Note: Only formulate the linear programming model. Do not solve the problem.

Solution

Decision Variables:

Let

x = amount spent on television advertisements

y = amount spent on magazine advertisements

Objective Function:

We want to maximize the number of people reached:

- ₹1 spent on TV reaches 300 people
- ₹1 spent on Magazine reaches 200 people

So,

Maximize $Z=300x+200y$

Subject to Constraints:

1. TV ad spending should not exceed ₹2, 00, 000

   $\rightarrow x \leq 2,00,000$

2. At least ₹1, 00, 000 should be spent on magazines

$\rightarrow y \geq 1,00,000$

3. Cost of advertisement:

- TV ad costs ₹5, 000
- Magazine ad costs ₹2, 000
- Total cost must not exceed ₹5, 00, 000

So,

$5000x+2000y \leq 5,00,000$

Non-negativity condition:

$x \geq 0, y \geq 0$

Illustration 1.3.2

A company produces three products P, Q, and R from three raw materials A, B, and C. One unit of product P requires two units of A and three units of B. One unit of product Q requires two units of B and five units of C. One unit of product R requires three units of A, two units of B, and four units of C. The company has 8 units of material A, 10 units of material B, and 15 units of material C available to it. Profits per unit of products P, Q, and R are Rs. 3, Rs. 5, and Rs. 4 respectively. Formulate the LPP problem.

Solution

Variables:

- x_1 = units of Product P
- x_2 = units of Product Q
- x_3 = units of Product R

Objective Function:

- Maximize $Z=3x_1+5x_2+4x_3$

Constraints:

Material A:

$2x_1+3x_3 \leq 8$

Material B:

$3x_1+2x_2+2x_3 \leq 10$

Material C:

$5x_2+4x_3 \leq 15$

Non-negativity:

$x_1, x_2. x_3 \geq 0$

The given problem formulated as LPP model as follows:

Maximize $Z=3x_1+5x_2+4x_3$

Subject to the constraints:

$2x_1+3x_3 \leq 8$

$3x_1+2x_2+2x_3 \leq 10$

$5x_2+4x_3 \leq 15$

$x_1, x_2. x_3 \geq 0$

Illustration 1.3.3

Vitamins V and W are found in two different foods F1 and F2. One unit of food F1 contains two units of vitamin V and three units of vitamin W. One unit of food F2 contains four units of vitamin V and two units of vitamin W. One unit of food F1 and F2 cost Rs. 5 and Rs. 2.5 respectively. The minimum daily requirements (for a person) of vitamin V and W is 40 and 50 units respectively. Assuming that anything in excess of daily minimum requirement of vitamin V and W is not harmful, find out the optimal mixture of food F1 and F2 at the minimum cost which meets the daily minimum requirement of vitamins V and Formulate this as a linear programming problem.

Solution

Decision Variables:

Let

x_1 = number of units of food F1

x_2 = number of units of food F2

Objective Function:

Minimise the total cost:

Minimise $Z=5x_1+2.5x_2$

Constraints:

Vitamin V requirement (at least 40 units):

$2x_1+4x_2 \geq 40$

Vitamin W requirement (at least 50 units):

$3x_1+2x_2 \geq 50$

Non-negativity constraints:

$x_1 \geq 0, x_2 \geq 0$

The given problem formulated as LPP model as follows:

Minimise $Z=5x_1+2.5x_2$

Subject to the constraints:

$2x_1+4x_2 \geq 40$

$3x_1+2x_2 \geq 50$

$x_1 \geq 0, x_2 \geq 0$

Illustration 1.3.4

Mr.A, a retired government officer, has recently received his retirement benefits, viz., provident fund, gratuity, etc. He is contemplating how much money he should invest in various alternatives open to him so as to maximize return on investment. The investment alternatives are Government securities, fixed deposits of a public limited company, equity shares, time deposits in a bank, and house construction. He has made a subjective estimate of the risk involved on a five-point scale. The data on the return on investment, the number of years for which the funds will be blocked to earn this return on investment and the subjective risk involved are as follows;

Investment	Return %	No. of Years	Risk
Government securities	6	15	1
Company deposits	13	3	3
Time deposits	10	5	2
Equity shares	20	6	5
House construction	25	10	1

He was wondering as to what percentage of funds he should invest in each alternative so as to maximize the return on investment. He decided that the risk should not be more than 4, and funds should not be locked up for more than 15 years. He would necessarily invest at least 25% in house construction. Formulate this problem as an LP model.

Solution

Decision Variables:

Let:

- x_1 = proportion of total funds invested in Government securities
- x_2 = proportion of funds invested in Company deposits
- x_3 = proportion of funds invested in Time deposits
- x_4 = proportion of funds invested in Equity shares
- x_5= proportion of funds invested in House construction

Objective Function (Maximize Return):

Return per ₹1 invested:

- Gov. Sec. = 6%

- Company Deposits = 13%
- Time Deposits = 10%
- Equity Shares = 20%
- House Construction = 25%

Maximize $Z=6x_1+13x_2+10x_3+20x_4+25x_5$

Subject to Constraints:

1. Total Investment = 100%:(100% of Mr.A's investment money must be fully allocated among all 5 available investment options)

   $x_1+x_2+x_3+x_4+x_5=1$

2. Average risk should not exceed 4 (weighted by proportion):

   $1x_1+3x_2+2x_3+5x_4+1x_5 \leq 4$

3. Maximum lock-in period allowed is 15 years (weighted average years ≤ 15):

   $15x_1+3x_2+5x_3+6x_4+10x_5 \leq 15$

4. Minimum 25% in House construction:

   $x_5 \geq 0.25$

5. Non-negativity:

   $x_1,x_2,x_3,x_4,x_5 \geq 0$

Final LPP Formulation:

Maximize:

$6x_1+13x_2+10x_3+20x_4+25x_5$

Subject to:

$x_1+x_2+x_3+x_4+x_5=1$

$1x_1+3x_2+2x_3+5x_4+1x_5 \leq 4$

$15x_1+3x_2+5x_3+6x_4+10x_5 \leq 15$

$x_5 \geq 0.25$

$x_1,x_2,x_3,x_4,x_5 \geq 0$

Illustration 1.3.5

The manager of an oil refinery must decide on the optimal mix of 2 possible blending processes of which the inputs and outputs per production run are as follows:

Process	Input Crude A	Input Crude B	Output Gasoline X	Output Gasoline Y
1	6	3	6	9
2	5	6	5	5

The maximum availability of crude A & B are 250 units and 200 units respectively. The market requirement shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and 2 are ₹40 and ₹50 respectively. Formulate the problem for maximising the profit.

Solution

Decision Variables:

Let

x_1 = number of production runs of Process 1

x_2= number of production runs of Process 2

Objective Function (Profit Maximization):

Each run of Process 1 gives ₹40 profit

Each run of Process 2 gives ₹50 profit

So,

Maximize $Z = 40x_1 + 50x_2$

Constraints:

1. Crude A availability:

- Process 1 uses 6 units of Crude A per run
- Process 2 uses 5 units of Crude A per run
- Maximum available = 250 units

So,

$6x_1 + 5x_2 \leq 250$

2. Crude B availability:

- Process 1 uses 3 units per run
- Process 2 uses 6 units per run
- Maximum available = 200 units

So,

$3x_1 + 6x_2 \leq 200$

3. Gasoline X production requirement:

- Process 1 produces 6 units per run
- Process 2 produces 5 units per run
- At least 150 units needed

So,

$6x_1 + 5x_2 \geq 150$

4. Gasoline Y production requirement:

Process 1 produces 9 units per run

Process 2 produces 5 units per run

At least 130 units needed

So,

$9x_1 + 5x_2 \geq 130$

Non-negativity constraints:

$x_1 \geq 0$

$x_2 \geq 0$

Final LPP Model:

Maximize $Z = 40x_1 + 50x_2$

Subject to:

$6x_1 + 5x_2 \leq 250$

$3x_1 + 6x_2 \leq 200$

$6x_1 + 5x_2 \geq 150$

$9x_1 + 5x_2 \geq 130$

Whereas: $x_1 \geq 0$

$x_2 \geq 0$

1.3.7 Graphical Method to solve an LPP

- Simple and visual approach used to solve LPP with two decision variables

The Graphical Method is among the simplest methods for resolving linear programming problems. When working with two variable problems, this approach works especially well since it helps us see the answer and comprehend how the constraints relate to one another. It requires plotting the constraints, locating the feasible region formed by them, and then finding the point within that region that gives the maximum or minimum value of the objective function.

Following are the steps involved in the graphical method for solving a linear programming problem:

1. **Define the Linear Programming Problem:** Clearly state the objective function and the associated constraints.
2. **Convert Inequalities to Equations:** Rewrite each constraint inequality as an equation to prepare for graphing.
3. **Graph the Constraint Equations:** Plot the lines or boundaries represented by these equations on a graph.
4. **Determine the Feasible Region:** Identify the area on the graph that satisfies all constraints simultaneously. This region includes all feasible solutions. If any constraint is not met, the solution is considered infeasible.
5. **Find the Corner Points:** Locate the vertices (or corner points) of the feasible region formed by the intersection of constraint lines.
6. **Evaluate the Objective Function:** Calculate the value of the objective function at each of these corner points.
7. **Select the Optimal Solution:** Identify the corner point that gives the best result for the objective function—either the highest value for maximisation problems or the lowest value for minimisation problems. This point represents the optimal solution.

Illustration 1.3.6

Maximise $Z = 20x_1 + 30x_2$ Profit function

Subject to $2x_1 + 5x_2 \leq 50$ Material constraint

$4x_1 + 3x_2 \leq 60$ Labour constraint

Where $x_1, x_2 \geq 0$ Non-negativitySolution

Step 1: Convert inequalities to equations for graphing

Constraint 1: $2x_1 + 5x_2 = 50$

Constraint 2: $4x_1 + 3x_2 = 60$

Step 2: Find intercepts to plot each constraint line

For $2x_1 + 5x_2 = 50$

When $x_1 = 0 \rightarrow 5x_2 = 50 \rightarrow x_2 = 10$ ($x_2 = 50 \div 5 = 10$)

When $x_2 = 0 \rightarrow 2x_1 = 50 \rightarrow x_1 = 25$ ($x_1 = 50 \div 2 = 25$)

For $4x_1 + 3x_2 = 60$

When $x_1 = 0 \rightarrow 3x_2 = 60 \rightarrow x_2 = 20$ ($x_2 = 60 \div 3 = 20$)

When $x_2 = 0 \rightarrow 4x_1 = 60 \rightarrow x_1 = 15$ ($x_1 = 60 \div 4 = 15$)

Step 3: Plot the constraint lines and identify feasible region

Draw the lines for each constraint.

Shade the region that satisfies all inequalities including $x_1 \geq 0$ and $x_2 \geq 0$.

The overlapping shaded area is the feasible region.

Step 4: Identify corner points of the feasible region

Let's find the corner points by solving equations pairwise.

Intersection of $2x_1 + 5x_2 = 50$ and $4x_1 + 3x_2 = 60$

$2x_1 + 5x_2 = 50$ (Equation A)

$4x_1 + 3x_2 = 60$ (Equation B)

To eliminate x_1, we need both equations to have the same x_1 coefficient.

Multiply Equation A by 2:

$2\times(2x_1+5x_2)=2\times50 \Rightarrow 4x_1+10x_2=100$ (EquationA)

Now subtract Equation B from Equation A:

$(4x_1+10x_2)-(4x_1+3x_2)=100-60$

$4x_1-4x_1+10x_2-3x_2=40 \Rightarrow 7x_2=40 \Rightarrow x_2=740\approx5.71$

Substitute x_2 back into one of the original equations

We'll use Equation A:

$2x_1+5x_2=50$

Substitute $x_2 = \frac{40}{7}$

$$2x_1 + 5x_2 = 50 \Rightarrow x_1 + \frac{200}{7} = 50$$

Now subtract $\frac{20}{7}$ from both sides:

$$2x_1 + 5\left(\frac{40}{7}\right) = 50 \Rightarrow 2x_1 \frac{350-200}{7} = \frac{150}{7}$$

Now divide both sides by 2:

$$x_1 = \frac{150}{14} = \frac{75}{7} \approx 10.71$$

The point where the two lines intersect is:

$$x_1 = \frac{75}{7} \approx 10.71,\ x_2 = \frac{40}{7} \approx 5.71$$

x_1-intercepts and x_2 -intercepts from earlier:

- From Constraint 1: (0, 10) and (25, 0)
- From Constraint 2: (0, 20) and (15, 0)

Now check which of these points lie in the feasible region by testing if they satisfy both inequalities.

Feasible corner points:

- (0, 0)
- (0, 10)
- (10.71, 5.71)
- (15, 0) → Check:

Constraint 1: 2(15) + 5(0) = 30 ≤ 50

Constraint 2: 4(15) + 3(0) = 60 ≤ 60

So final feasible corner points:

(0, 0), (0, 10), (10.71, 5.71), (15, 0)

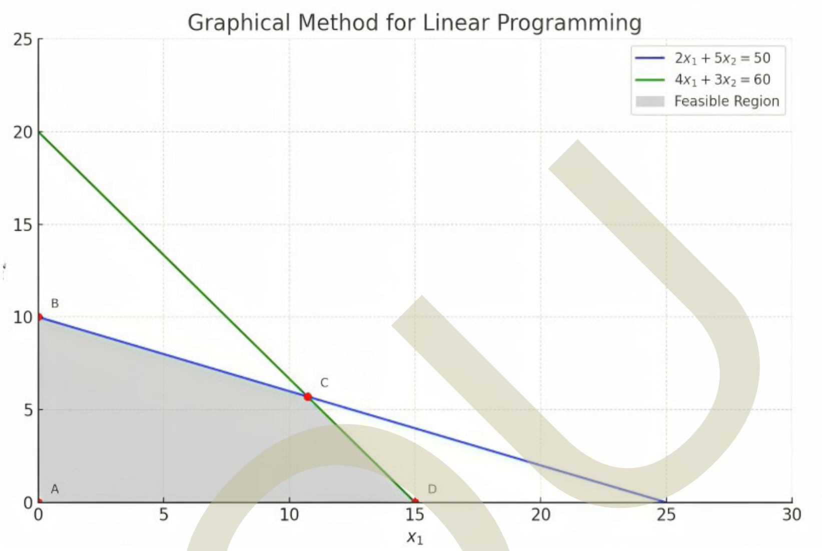
Step 5: Calculate Z = $20x_1 + 30x_2$ at each point

Corner Point	Z = 20 + $30x_2$
(0, 0)	0
(0, 10)	300
(10.71, 5.71)	≈ 20×10.71 + 30×5.71 = 214.2 + 171.3 = 385.5
(15, 0)	300

Optimal Solution:

Maximum Z = 385.5 at $(x_1, x_2) \approx (10.71, 5.71)$

This is the point where the profit function reaches its maximum value under the given constraints.



Unbounded Problem

In a maximization problem, if the shaded feasible region is open-ended, it indicates that the objective function can increase indefinitely. This means there is no maximum value, and the linear programming problem does not have a finite solution. Therefore, the problem is considered to have an unbounded solution.

- Feasible region is open or unlimited in the direction of optimization

In some problems, the feasible region created by the constraints doesn't stay within the 1st quadrant alone, t may extend into the 2nd, 3rd, or 4th quadrants. In such cases, the objective function (like maximizing profit) might keep increasing without any limit because it never hits a boundary that stops it. Simply put, if the constraints go into other quadrants and there's no common area that satisfies all of them, then the problem is considered unbounded.

Illustration 1.3.7

Solve the LPP by Graphic Method

Maximise $Z= 5x_1+ 4x_2$

Subject to $2x_1- 4x_2 \leq 1$

$2x_1+ 4x_2 \geq 3$

$x_1, x_2 \geq 0$

Solution

Step 1: Convert constraints into equalities to graph them

1. L1: $2x_1 - 4x_2 = 1$
2. L2: $2x_1 + 4x_2 = 3$

Step 2: Find the intercepts for each line.

For L1: $2x_1 - 4x_2 = 1$

- If $x_1 = 0$, then $-4x_2 = 1 \Rightarrow x_2 = -1/4 = -0.25$. Point: (0, −0.25)
- If $x_2 = 0$, then $2x_1 = 1 \Rightarrow x_1 = 1/2 = 0.5$. Point: (0.5, 0)

For L2: $2x_1 + 4x_2 = 3$

- If $x_1 = 0$, then $4x_2 = 3 \Rightarrow x_2 = 3/4 = 0.75$. Point: (0, 0.75)
- If $x_2 = 0$, then $2x_1 = 3 \Rightarrow x_1 = 3/2 = 1.5$. Point: (1.5, 0)

Step 3: Identify the feasible region

- From constraint (1), shade below the line $2x_1 - 4x_2 = 1$
- From constraint (2), shade above the line $2x_1 + 4x_2 = 3$

Also consider $x_1 \geq 0$, $x_2 \geq 0$

The feasible region lies in the first quadrant where these inequalities overlap.

Step 4: Check if the region is bounded

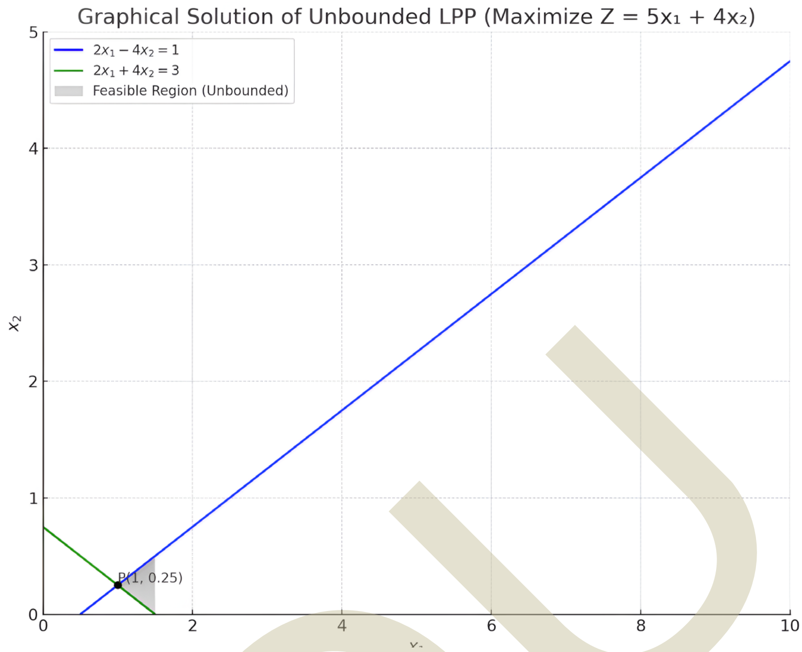
Look at the plot again. You'll see:

- The region extends infinitely toward the upper-right side.
- There is no boundary stopping the objective function from increasing indefinitely in that direction.

This means $Z = 5x_1 + 4x_2$ can grow without bound as $x_1, x_2 \to \infty$ while still satisfying all constraints.

This is an unbounded linear programming problem.

- A point like (1, 0.25) (1, 0.25) (1, 0.25) satisfies all constraints and gives Z=6
- But higher values of x_1, x_2 still satisfy all constraints and yield higher Z
- Hence, Z has no maximum value — it can be increased without limit.



Infeasible Problem

- No solution satisfies all the constraints simultaneously

In linear programming, the feasible region is the set of all possible solutions that satisfy every constraint of the problem. Essentially, it's the area on a graph where all the conditions imposed by the inequalities are met. On the other hand, the infeasible region comprises all points that do not satisfy at least one of the problem's constraints. The linear programming equation (or objective function) is only 'defined' or relevant within this feasible region, as any solution outside it violates one or more problem requirements.

Illustration 1.3.8

Maximise $Z= -5x_2$

Subject to $x_1+x_2 \leq 1$

$-0.5x_1 - 5x_2 \leq -10$

$x_1, x_2 \geq 0$

Solution

Step 1: Convert inequalities to equations for graphing

First Constraint: $x_1+x_2=1$

- If $x_1=0$, $x_2=1$
- If $x_2=0$, $x_1=1$

Second Constraint: $-0.5x_1 - 5x_2 \leq -10$

To avoid negative signs on the left side, multiply the entire inequality by −1.

$(-1)\times(-0.5x_1-5x_2)\leq(-1)\times(-10)$

Now, perform the multiplication:

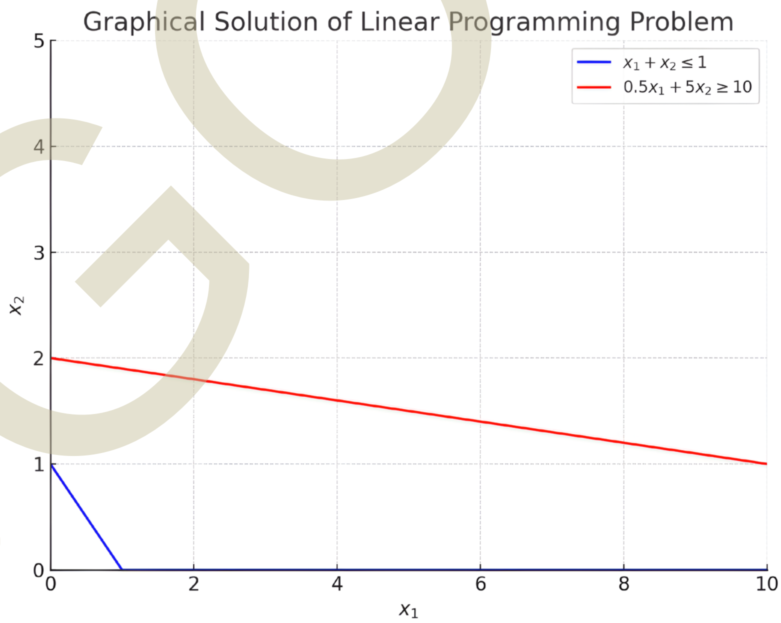
- $(-1)\times(-0.5x_1)=+0.5x_1$
- $(-1)\times(-5x_2)=+5x_2$
- $(-1)\times(-10)=+10$

Because we multiplied by a negative number (which is −1), we reverse the inequality sign from '≤' to '≥'.

So, the inequality becomes: $0.5x_1+5x_2 \geq 10$

$0.5x_1+5x_2=10$

- If $x_1=0$, $5x_2=10 \Rightarrow x_2=2$
- If $x_2=0$, $0.5x_1=10 \Rightarrow x_1=20$



Intersection of $x_1+x_2=1$ and $0.5x_1+5x_2=10$. No feasible intersection exists within the bounds (x_1, $x_2 \geq 0$). Since there is no common region that satisfies all constraints simultaneously, the problem is infeasible. This means there is no solution to this linear programming problem.

Multiple Optimal Solution

When the objective function has more than one optimal point, the problem is said to have multiple solutions. This happens only

- More than one solution gives the same otimal value of the objective function

when the slope of the objective function matches the slope of one of the constraints. In such cases, the solution lies along a boundary made up of connected straight-line segments.

Illustration 1.3.9

Maximise $Z= 4x_1+2x_2$

Subject to $x_1+x_2 \leq 4$

$2x_1+ x_2 \leq 6$

Whereas $x_1,\ x_2 \geq 0$

Solution

Convert each inequality into an equation

$x_1+ x_2=4$

If $x_1=0$, then $x_2=4$. Point: (0, 4)

f $x_2=0$, then $x_1=4$. Point: (4, 0)

$2x_1+ x_2=6$

If $x_1=0$, then $x_2=6$. Point: (0, 6)

If $x_2=0$, then $2x_1=6 \Rightarrow x_1=3$. Point: (3, 0)

Identify the Feasible Region:

- Shade the region where all inequalities hold true.
- The feasible region is bounded by the vertices (0, 0), (0, 4) (2, 2) (2, 2)and (3, 0).

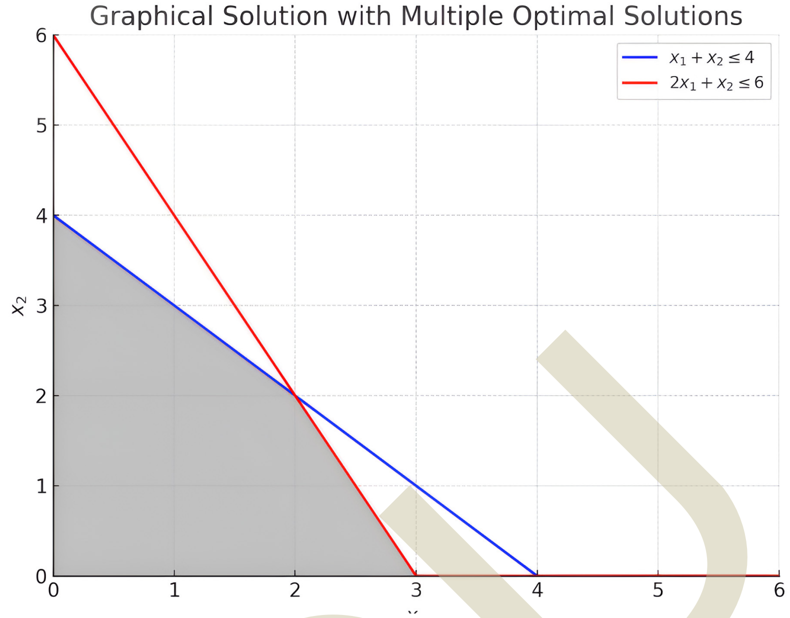
Calculate the Objective Function $Z=4x_1+2x_2$

- Substitute the vertices into :Z

1. (0, 0): Z=4(0)+2(0)=0
2. (0, 4): Z=4(0)+2(4)=8
3. (2, 2): Z=4(2)+2(2)=12 (Maximum),
4. (3, 0): Z=4(3)+2(0)=12 (Maximum)

Confirm Multiple Optimal Solutions:

- Since Z=12 occurs at both (2, 2) and (3, 0), the objective function Z has multiple optimal solutions.
- The objective function is parallel to the constraint $x_1+ x_2$=-4making every point along the edge (2, 2) to (3, 0) optimal.



Summarised Overview

Linear Programming (LP) is a mathematical method used to find the best possible solution, such as maximizing profit or minimizing cost, when dealing with limited resources and competing needs. It involves decision variables, a linear objective function, and constraints that represent resource limitations. LP works under key assumptions like proportionality (linear relationships), certainty (fixed data), additivity (total equals the sum of parts), non-negativity (no negative values), finiteness (limited choices), and divisibility (variables can take fractional values). It can be solved using graphical methods for simple problems or the simplex method for complex ones. While LP is highly useful in areas like production, finance, and logistics, it has limitations—such as not handling changing conditions, multiple goals, or situations requiring whole number solutions. The document includes practical examples and explains special cases like unbounded and infeasible solutions.

Self-Assessment Question

1. What is linear programming? Briefly explain its purpose.
2. What are the main components of a linear programming problem (LPP)?
3. What are constraints? Give two examples.
4. Differentiate between bounded and unbounded solutions.

Assignments

1. In a graphical method, what does the feasible region represent?
2. A manufacturing company is engaged in producing three types of products: A, B and C. The production department produces, each day, components sufficient to make 50 units of A, 25 units of B and 30 units of C. The management is confronted with the problem of optimizing the daily production of the products in the assembly department, where only 100 man-hours are available daily for assembling the products. The following additional information is available:

Type of Product	Profit Contribution per Unit (₹)	Assembly Time per Product (hrs)
A	12	0.8
B	20	1.7
C	45	2.5

The company has a daily order commitment for 20 units of products A and a total of 15 units of products B and C. Formulate this problem as an LP model so as to maximize the total profit.

3. A company sells two different products A and B, making a profit of Rs 40 and Rs 30 per unit, respectively. They are both produced with the help of a common production process and are sold in two different markets. The production process has a total capacity of 30, 000 man-hours. It takes three hours to produce a unit of A and one hour to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8, 000 units and that of B is 12, 000 units. Subject to these limitations, products can be sold in any combination. Formulate this problem as an LP model to maximize profit.
4. Use the graphical method to solve the following LP problem.

Maximize $Z = 15\,x_1 + 10\,x_2$

subject to the constraints $4\,x_1 + 6x_2 \leq 360$,

$3x_1 + 0x_2 \leq 180$,

$0x_1 + 5x_2 \leq 200$

and $x_1 .\ x_2 \geq 0$

5. What is meant by a feasible solution?

Reference

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2. Kapoor, V. K. (2000). Operations research. New Delhi: Sultan Chand & Sons.
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4. Taha, H. A. (2017). Operations Research: An Introduction (10th ed.). Pearson Education.

Suggested Reading

1. Sharma, J. K. (2014). Operations Research: Theory and Applications (5th ed.). Macmillan Publishers India.
2. Kapoor, V. K. (2000). Operations research. New Delhi: Sultan Chand & Sons.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

Unit 4

Simplex Method

Learning Outcomes

After completing this unit, the learner will be able to:

- Identify the concept of Simplex Method
- Describe the steps of solving Simplex Method
- Solve an LPP through Simplex Method
- Solve LPP having duality

Background

George B. Dantzig created the Simplex Method, an excellent mathematical method for resolving Linear Programming Problems (LPP) with more than two variables, where graphical approaches are inadequate. This iterative approach, which was developed to address intricate real-world optimization issues in industries, gradually improves the objective function to find the best answer. Even when there are limits, such as equalities, inequalities, or mixed constraints, it guarantees viability and effectiveness.

Keywords

Linear Programming Problems, Simplex Method, Basic Variable, Slack Variable, Pivot Element, Duality of LPP, Big M Method

Discussion

A bakery makes cakes and cookies.

- Each cake gives a profit of ₹40.
- Each cookie gives a profit of ₹30.
- The bakery has limited flour and sugar.

The goal is to decide how many cakes and cookies to make for maximum profit.

If there were just two products, we could draw a graph. But imagine if the bakery produced 5–6 products — then the graph method won't work.

Instead, we use the Simplex Method to solve it step by step using tables.

- When there are more than 2 decision variables

The graphical method of solving Linear Programming Problems (LPP) is limited to two-variable cases and is not suitable for complex or large-scale industrial problems. To handle such cases, the Simplex Method, developed by G.B. Dantzig, is widely used. It is an iterative approach that starts with a feasible solution and moves step by step toward the optimal solution by improving the objective function. The method is efficient, can identify unbounded or infeasible cases, and ensures the best result after a finite number of steps.

In the Simplex Method:

1. We set up a table (called a simplex tableau).
2. We start with an initial feasible solution.
3. We check if profit can be improved.
4. Step by step, we improve the solution until profit cannot increase anymore.
5. That final point gives us the maximum profit and the number of cakes and cookies to produce.'

The standard form is the basic structure required before solving any linear programming problem. It must meet three conditions: (1) the objective function should be a maximization, (2) all constraints must be written as 'less-than-or-equal-to' inequalities, and (3) all variables must be non-negative. Any linear program can be converted into this form using simple algebraic steps. This standardized format ensures an efficient starting point for solving problems using the Simplex method or other optimization techniques. The standard form of Simplex Method is given below:

Maximise $Z=C_1X_1+C_2X2+\ldots\ldots.+C_n X_n$

Subject to constraints:

$a_11\ x_1+a_{12}\ x_2+\ldots+\ a_{1n}\ x_n\ (\leq = \leq)\ b_1$

$a_21\ x_1+a_{22}\ \ x_2+\ldots+a_{2n}\ x_n\ (\leq = \leq)\ b_2$

……………………………………

……………………………………

$a_{m1}\ x_1+\ a_{m2}\ x_2+\ \ldots\ldots\ldots.\ a_{mn}\ \ x_n\ (\leq = \leq)\ b_m$

Let $x_1 \geq 0,\ x_2 \geq 0\ldots\ldots\ldots\ldots\ldots\ x_n \geq 0$

1.4.1 Basic Terminology

Basic Variables

Variables in the current solution that are non-zero. These correspond to columns forming the identity matrix in the simplex tableau.

Slack Variables

A slack variable is introduced to the left-hand side of a 'less than or equal to' constraint to transform it into an equation. Economically, it indicates the unused or remaining capacity in a given resource.

Surplus Variables

A surplus variable is subtracted from the left-hand side of a 'greater than or equal to' constraint to change it into an equation. It is also referred to as a negative slack variable, representing the amount by which the actual value exceeds the constraint limit.

Artificial variable

An artificial variable is a non-negative variable introduced to help compute an initial basic feasible solution. Specifically, it is added to the left-hand side of a 'greater than or equal to' (or 'equal to') constraint to convert it into an equation, enabling the application of the simplex method.

Pivot Element

An element used in each iteration to update the tableau and move towards the optimal solution.

Entering Variable

A non-basic variable that enters the basis (used to improve the value of the objective function).

Leaving Variable

A basic variable that leaves the basis in exchange for the entering variable.

Maximisation Problems

When the objective is to maximize and all constraints are of '≤' type, we follow these steps:

1. The problem should have a maximization objective function.
2. All constraints must be 'less than or equal to (≤)' type.
3. Slack variables are added to the constraints. These have a coefficient of 0 in the objective function and a coefficient of 1 in the constraint to form a unit matrix.
4. Calculate Z_j (total contribution from basic variables) and $C_j - Z_j$ (net evaluation row).
5. Choose the entering variable by finding the highest positive value of $C_j - Z_j$.
6. Compute the minimum ratio by dividing the RHS (solution value) by the values in the key column (positive entries only).
7. The variable corresponding to the row with the smallest positive ratio becomes the leaving variable.
8. Repeat steps 4 to 7 until all $C_j - Z_j$ (values are zero or negative, indicating the optimal solution is reached.)

Illustration 1.4.1

Solve the following LPP by Simplex method

Maximize $Z = 6X_1 + 8X_2$

Subject to $30x_1 + 20\,x_2 \leq 300$

$5x_1 + 10\,x_2 \leq 110$

$x_1, x_2 \geq 0$

Solution:

Convert inequalities to equations using slack variables

Introduce slack variables S_1 and S_2:

$30x_1 + 20\,x_2 + S_1 = 300$

$5x_1 + 10\,x_2 + S_2 = 110$

In the simplex method, every variable that appears in one equation must also be present in all the other equations. If a variable doesn't influence a particular equation, it is included with a coefficient of zero. For instance, slack variables $S_1S_1S_1$ and $S_2S_2S_2$, which indicate unused resources and do not contribute to profit, are added to the objective function with zero coefficients.

So, the objective function becomes:

Maximize $Z = 6x_1 + 8\,x_2 + 0.S1 + 0.S_2$

Subject to $30x_1 + 20\,x_2 + S1 + 0.S_2 = 300$

$5x_1 + 10\,x_2 + 0.S_1 + S_2 = 110$

$x_1, x_2, S_1, S_2 \geq 0$

In the given problem:

- $S_1S_1S_1$ and $S_2S_2S_2$ are added to the constraints.
- The variables that start with a value of zero in the solution are called non-basic variables.
- The variables with non-zero initial values are called basic variables.
- This setup gives an initial basic feasible solution.

Simplex Table 1

		Cj	6	8	0	0	
CB	Basic Variables	Solution Values	2		S_1	S_2	Minimum Ratio
0	S1	300	30	20	1	0	300/20=15
0	S2	110	5	10*	0	1	110/10=11
	Zj	0	0	0	0	0	Key Raw (Leaving Variables)
	Cj - Zj		6	8	0	0	

Explanation of the elements in the table:

- C_j → Contribution Per Unit: These are the coefficients of the variables in the objective function.
- Basic Variable: These are the variables currently in the basis (i.e., non-zero values).
- Solution Values (RHS): The current values of the basic variables.
- x_1 x_2 S_1 S_2: The coefficients of the variables in the constraints.
- Z row: Represents the coefficients of the objective function.
- S_1, S_2 rows: Represent the constraint equations.
- Minimum Ratio: Calculated to determine the leaving variable.
- * (asterisk): Indicates the pivot element (10 in this case).
- →(arrow): Points to the leaving variable's row.
- ↑ (arrow): Points to the entering variable's column.
- C_j - Z_j: This row indicates the net contribution to the objective function if a non-basic variable enters the basis. A negative value here indicates potential for improvement in a maximization problem.
- The Z_j row helps in evaluating whether including a new variable (like x_1 or x_2) into the solution will improve the total profit. It shows how much the objective function (profit) would change if one unit of a particular variable were added to the current solution.
- To calculate Z_j values, we multiply the profit contribution (C_j) of each basic variable (like S_1 or S_2) by its corresponding coefficient in the constraint equations (i.e., from the column of the variable in question), and then sum them up.

Formula:

$Z_j = \sum$ (coefficient of Cj column × corresponding coefficients of constraint set)

To find Z_j for x_1 :

$Z_j = (0\times30) + (0\times5) = 0$

Repeat similarly for other variables like X_2, S_1, and S_2.

Net Profit Contribution ($C_j – Z_j$)

To decide which variable should enter the solution next, we use the value of : $C_j – Z_j$

- If it's positive, including that variable can increase total profit.
- If it's zero, the total profit will remain unchanged.
- If it's negative, including that variable would reduce the total profit.

Variable	Profit per unit (C_j)	Net Loss per unit (Z_j)	Net Contribution $C_j – Z_j$
x_1	6	0	6
x2	8	0	8
S_1	0	0	0
S_2	0	0	0

From this table, we can observe:

- Adding x_1 increases profit by ₹6 per unit.
- Adding x_2 increases profit by ₹8 per unit.

Thus, as long as there are positive values in $C_j – Z_j$, the solution can still be improved. The optimal solution is reached only when all $C_j – Z_j$ values are zero or negative.

Steps to Develop the Second Simplex Table

Step I: Choose the entering variable

- Look at the $C_j – Z_j$ row (profit row).
- Choose the column with the highest positive value.
- This tells us which variable (say X_2) should enter the solution mix to improve profit.

Step II: Choose the leaving variable

- Now choose which current variable will leave the solution (which row to replace).
- For this, divide the solution value in each row by the number in the column of the entering variable.

For S_1 row 300/20 = 15

For S_2 row 110/10 = 11

Select the smallest positive ratio.

- Since S_2 has the smaller positive ratio, that row is called the key row or replaced raw, and the variable leaving is called the departing variable.

Step III: Form the new key row

- Take the key row and make the element at the intersection of the key row and column into 1.
- Do this by dividing every number in the key row (S_2) by the key element.(10)

110/10=11; 5/10=1/2; 10/10=1; 0/10=0; 1/10

The new x2 raw -- 11, 1/2, 1, 0, 1/10

Step IV: Update other rows

- Use the formula:

New Row= [Element in old raw]- [(Intersectional element of old raw) × (Corresponding elements in replacing raw)]

New S1 row:

Solution value column = 300-20×11 =80

x_1 column = 30-20×5/10 = 20

x_2 column = 20-20×1 = 0

S_1 column = 1-20×0 = 1

S_2 column = 0-20 ×1/10 = -2

Step V: Compute new $C_j - Z_j$

Z_J Calculation

Z_J (for total profit) = (0)× 80 =(8) × 11= 88

Z_J (x_1 column) = (0)× 20 + (8) × 5/10

Z_J (x2 column) = (0)× 0 + (8) × 1

Z_J (S1 column) = (0)×1 + (8) ×0

Z_J (S2 column) = (0)×(-2) + (8) ×1/100

Calculation of Cj – Zj :

Variable	Profit/Unit Cj	-	Profit/Loss per unit ZJ	= Net profit/unit Cj – Zj
2	6	-	4	= 2
S1	8	-	8	= 0
S2	0	-	0	= 0
	0	-	8/10	= -8/10

Simplex Table 2

	Cj		6	8	0	0	
CB	Basic Variables	Solution Values		X2	S1	S2	Minimum Ratio
0	S1	80	20*	0	1	-2	80/20=4 →
8	2	11	5/10	1	0	1/10	11÷ 5/10 =22
	Zj	88	4	8	0	8/10	Key Raw (Leaving Variables)
		Cj - Zj	2	0	0	- 8/10	

As long as there's a positive value in the Cj – Zj row, keep repeating the steps to improve the solution.

Simplex Table 3

		Cj	6	8	0	0
CB	Basic Variables	Solution Values		2	S1	S2
6		4	1	0	1/20	-1/1 0
8	2	9	0	1	-1/40	3/20
	Zj	96	6	8	1/10	6/10
		Cj - Zj	0	0	-1/10	-3/5

Answer: Cj - Zj ≤ 0 hence optimal solution

Where Zj = 96, x_1 =4, x2 =9

Minimization Case – Big M Method (Artificial Variables)

In minimization problems where all constraints are of greater than or equal to (≥) type, we use:

- Surplus variables (subtracted from constraints)
- Artificial variables (added to constraints)

These help convert the problem into standard form so we can apply the Simplex Method.

In the objective function:

- Surplus variables are added with a coefficient of 0
- Artificial variables are added with a coefficient of M (a large number, like 10, 000)

Steps to Solve Using Big M Method (Minimization Case)

1. The goal is to minimize the objective function.
2. All constraints must be of the ≥ type.
3. Add both Surplus and Artificial variables:
4. Surplus variables are subtracted in constraints and given a 0 in the objective function.
5. Artificial variables are added in constraints and given a +M in the objective function.
6. Calculate Zj and Cj – Zj.
7. Choose the most negative value in Cj – Zj as the entering variable.
8. Calculate ratios: divide each solution value by the corresponding element in the key column.
9. The row with the smallest positive ratio gives the leaving variable.
10. Repeat steps 4 to 7 until all values in Cj – Zj ≤ 0
11. If any Artificial variable remains in the final solution, it means the problem has no feasible solution.

Illustration 1.4.2

Minimize $Z = 3x_1 + 2.25\ x_2$

Subject to $2x_1 + 4\ x_2 \geq 40$

$5x_1 + 2\ x_2 \geq 50$

$x_1, x_2 \geq 0$

Solution

Given:

- $2x_1 + 4\ x_2 \geq 40$
- $5x_1 + 2\ x_2 \geq 50$

These are 'greater than or equal to' (≥) constraints.To solve using the Simplex Method, we must convert them into equalities (=). But we can't just change ≥ to =.

The first step is to convert these inequalities into equalities by introducing surplus variables. These variables represent the extra quantity beyond the required level.

When the constraint is '≥', we subtract something (a surplus variable) to make it an equation.

1. $2x_1 + 4\,x_2 - S_1 = 40$
2. $5x_1 + 2\,x_2 - S_2 = 50$

Subtracting S_1, S_2 might lead to negative basic variables, which is not allowed. So, we introduce artificial variables to fix this. They act like temporary helpers to start the simplex method. They are assigned a very high penalty cost (+M) in the objective function (since it's a minimization problem), to ensure they are eliminated from the final solution.

So, after adjusting with surplus and artificial variables:

Modified Objective Function:

Minimize $Z=3x_1 + 2.25X_2+0S_1+0S_2+MA_1+MA_2$

Subject to:

$2x_1+4x_2-S_1+A_1=40$

$5x_1+2x_2-S_2+A_2=50$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Simplex Table 1

CB	Cj Basic Variables	3 2.25 0 0 M M Solution Values	3 2.25 0 0 M M 2 S1 S2 A1 A2	Minimum Ratio
M	A1	40	2 4 -1 0 1 0	40/2=20
M	A2	50	5* 2 0 -1 0 1	50/5=10→
	Zj	90M	7M 6M -M -M M M	
		Cj - Zj	3-7M 2.25-6M M M 0 0	

Simplex Table 2

Cj Basic Variables	Solution Values	3 2.25 0 0 M 2 S1 S2 A1	Minimum Ratio
A1	20	0 16/5* -1 2/5 1	20/16/5=6.25→
X1	10	1 2 /5 0 -1/5 0	10 /2/5=25
Zj	30+20M	3 -M 0	
	Cj - Zj	0 M 0	

Cj →

↓ M

3

↑

Since our goal is to minimize cost, we need to choose the variable that gives the most negative value of Cj - Zj. This is why X_2 is selected as the entering variable, because its Cj - Zj value is the most negative. The next step is to identify the variable that will leave the basis, called the departing variable. To do this, we compute the ratio of the solution value to the corresponding positive element in the column of the entering variable. The smallest ratio determines the departing variable. The element at the intersection of the entering column and the departing row is called the pivot (or key) element. Once the pivot is found, we proceed with the same steps used in the maximization case. The improved (updated) solution is shown in Table 3.

Simplex Table 3

	Cj		3	2.25	0	0
CB	Basic Variables	Solution Values		2	S_1	S_2
3	X_1	6.25	0	1	-5/16	1/8
2.25	X_2	7.50	1	0	1/8	-1/4
	Zj	36.5625	3	0	-21/64	-15/32
		Cj - Zj	0	0	21/64	15/32

- Basic Variables: x_1, x_2
- Solution Values: 36.5625 and 6.25

All Cj−Zj values are ≥ 0 → Optimal solution reached

Mixed Constraints

If the problem contains a combination of different types of constraints such as less than or equal to (≤), greater than or equal to (≥), and equal to (=), it is referred to as a mixed constraint problem.

- If any constraint has a negative RHS, multiply both sides by -1 to make RHS positive (this will also flip the inequality sign).
- Use:

Slack variables for '≤' constraints.

Surplus and Artificial variables for '≥' constraints.

Artificial variables for '=' constraints.

For '=' constraints and '≥' type, use artificial variables to ensure a starting feasible solution.

- Assign 0 as the cost to slack and surplus variables in the objective function.

- Assign:
- +M (a very large number) for minimization problems
- −M for maximization problems to artificial variables in the objective function.
- Stop:

In maximization, when all Cj−Zj values are ≤ 0

In minimization, when all Cj−Zj values are ≥ 0

Constraint	Adjustment	Max. Problem Coefficient	Min. Problem Coefficient
≤	Add a slack variable	0	0
=	Add an artificial variable	-M	+M
≥	Subtract a surplus variable and add an artificial variable	0, -M	0, +M

Illustration 1.4.3

An animal feed company must produce 200 kg of a mixture consisting of ingredient x_1 and X_2 daily. X_1 costs Rs. 3 per kg and X_2 Rs. 8 per kg. No more than 80 kg of X_1 can be used and at least 60 kg of X_2 must be used. Find how much of each ingredient should be used if the company wants to minimize cost.

Solution

Step 1 : Mathematical formulation

Let:

- x_1 = kg of ingredient X_1, to be used
- x_2 = kg of ingredient X_2 to be used

We are asked to minimize cost:

- Cost of x_1 = Rs. 3/kg
- Cost of = Rs. 8/kg

So the objective function becomes:

Minimize $Z=3\ x_1 +8\ x_2$

Constraints:

1. Total amount required:

   $x_1 + {}_{x2} = 200$

2. No more than 80 kg of :

   $x1 \leq 80$

3. At least 60 kg of X_2:

   $x2 \geq 60$

4. Non-negativity:

   $x_1 \geq 0 \quad x_2 \geq 0$

Minimize

$Z = 3\,x_1 + 8\,x_2$

Subject to:

$x_1 + x_2 = 200$

$x1 \leq 80$

$x2 \geq 60$

$x1 \geq 0 \quad x2 \geq 0$

Step 2: Convert inequalities into equalities using slack, surplus, and artificial variables.

$x_1 + x_2 = 200$: add artificial variable $A_1 \Rightarrow x_1 + x_2 + A_1 = 200$

$x_1 \leq 80$ add slack variable $S_1 \Rightarrow x_1 + S_1 = 80$

$x_2 \geq 60$ subtract surplus variable S_2 add artificial variable A_2 $\Rightarrow x_2 - S_2 + A_2 = 60$

Standard form

Minimize

- $0S_1 + 0S_2$: Added to include slack variables without changing the goal

They do not contribute to cost, so their coefficient in the objective function is 0.

- $MA_1 + MA_2$: Artificial variables with large penalty M (Big M method)

Artificial variables are used to start the simplex method when a basic feasible solution is not obvious—that is, when constraints are of type:

- $\geq$
- = (equality)

These types of constraints do not automatically give slack variables that can serve as starting basic variables. So we introduce artificial variables (A_1, A_2...) just to begin the solution process.

Subject to $x_1 + x_2 + A_1 = 200$

$x_1+S_1=80$

$x_2 - S_2 + A_2 = 60$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Step 3 : Initial Simplex Table

Simplex Table 1

Cj			2	8	M	0	0	M	
CB	Basic Variables	Solution Values	x1	x2	S_1	S_2	A_1	A_2	Min. Ratio
M	A_1	200	1	1	0	0	1	0	200/1 = 200
0	S_1	80	1	0	1	0	0	0	
M	A_2	60	0	1*	0	-1	0	1	60/1 = 60 →
	Z = 260 M	Zj	M	2M	M	0	-M	M	
	Cj – Zj		3−M	8−2 M	0	0	M	0	

The key column is x_2 (most negative Cj – Zj)

The minimum ratio is smallest for A_2 row: 60/1 = 60, so A_2 is the key row.

This means A_2 will leave the basis, and x2 will enter the basis in the next iteration.

x_2 has the most negative value in the row Cj – Zj : 8−2 M

In minimization with Big M, we aim to eliminate artificial variables and reduce Z as fast as possible.

Hence, x_2 enters the basis (key column) to improve the solution.

Simplex Table 2

CB	Basic Variables	Cj	3	8	M	0	0	Min. Ratio
CB	Basic Variables	Solution Values (XB)	x1	x2	S_1	S_2	A_1	Min. Ratio
M	A_1	140	1	0	0	1	1	140/1
0	S_1	80	1*	0	1	0	0	80/1 →
8	x2	60	0	1	0	-1	0	
		Zj	M	8	0	M-8	M	
	Cj – Zj	3−M	0	0	-M+8	0		

The Zj is computed as:

Zj = ∑(CB × corresponding column value)

x_1 column:

=M(1)+0(1)+8(0)=M

x_2 column:

=M (0) + 0 (0) + 8(1) = 8

S_1 column:

=M (0) +0 (1) +8(0) =0

S_2 column:

=M (1) +0 (0) + 8(−1) =M−8

A_1 column:

= M(1)+0(0)+8(0)=M

Computation of Cj - Zj for each column:

x1 : 3 − M

x2 : 8 − 8 = 0

S_1: 0 − 0 = 0

S_2: 0 − (M − 8) = -M + 8

A_1: 0 − M = -M

To determine the leaving variable, we do:

Minimum Ratio = For x_1 column (as pivot):

- Row 1 (A_1): 140 / 1 = 140
- Row 2 (S_1): 80 / 1 = 80 smallest

So S_1 leaves, x_1 enters.

Simplex Table 3

CB	Basic Variables	Cj Solution Values (XB)	3 x1	8 x2	M S_1	0 S_2	0 A_1	M Min. Ratio
M	A_1	60	0	0	-1	1*	1	60/1 →
3	x1	80	1	0	1	0	0	--
8	x2	60	0	1	0	-1	0	
		Zj	3	8	-M+3	M-8	M	
	Cj – Zj	0	0	M-3	-M+8	0		

Since the value of $C_5 - Z_5$ in row Cj − Zj is the most negative, the column for x2 is chosen as the key (or pivot) column. The smallest positive ratio is in the first row (A_1), so that becomes the key (pivot) row. We ignore other ratios because dividing by zero or a negative number is not allowed. After doing the necessary row operations, we get the new solution shown in Table 4.

Simplex Table 4

CB	Basic Variables	Cj Solution Values (XB)	3 x1	8 x2	M S_1	0 S_2
0	S_2	60	0	0	-1	1
3	x_1	80	1	0	1	0
8	x_2	120	0	1	-1	0
	Z=1200	Zj	3	8	-5	0
		Cj – Zj	0	0	5	0

Zj is calculated as:

Zj for a column=∑(CB× corresponding column values)

- For : x1

$(0\times0)+(3\times1)+(8\times0)=3$

- For x^2 :

$(0\times0)+(3\times0)+(8\times1)=8$

- For S_1:

$(0\times-1)+(3\times1)+(8\times-1)=3-8=-5$

- For S_2:

$(0\times1)+(3\times0)+(8\times0)=0$

Cj – Zj Row:

- For x_1: 3 − 3 = 0
- For x_2 : 8 − 8 = 0
- For S_1: 0 − (−5) = +5
- For S_2: 0 − 0 = 0

▶ Since all values in the Cj – Zj row are ≥ 0, this solution is optimal

$x_1 = 80, x_2 = 120$

Min Z = 80 × 3 + 120 × 8 = Rs.1200

Maximisation Problem (Mixed Constraint)

Illustration 1.4.5

- Maximise $Z = 4x_1 + 5x_2 - 3x_3$

Subject to the Constraint $x1 + x_2 + x_3 = 10$

$x_1 - x_2 \geq 1$

$2x_1 + 3x_2 + x_3 \leq 30$

$x1, x_2\ x_3 \geq 0$

Solution

- Convert to Standard Form

Constraint 1:

$x_1 + x_2 + x_3 + A_1 = 10$ (add artificial variable A_1)

Constraint 2:

$x_1 - x_2 - x_3 - S_2 + A_2 = 1$ (subtract surplus, add artificial)

Constraint 3:

$2x_1 + 3x_2 + x_3 + S_3 = 30$ (add slack variable S_3)

- Rewrite Objective Function in Standard Form (Big M Method)

Maximise $Z = 4x_1 + 5x_2 - 3x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to the constraints $x_1 + x_2 + x_3 + A_1 = 10$

$x_1 - x_2 - x_3 - S_1 + A_2 = 1$

$2x_1 + 3x_2 + x_3 + S_2 = 30$

$x_1, x_2, x_3\ S_1, S_2, A_1, A_2 \geq 0$

Simplex Table 1

		Cj	4	5	-3	0	0	-M	-M	
CB	Basic Variables	Solution Values (XB)	x_1	x_2	x_3	S_2	S_2	A_1	A_2	Min. Ratio
-M	A_1	10	1	1	1	0	0	1	0	10/1
-M	A_2	1	1*	-1	0	-1	0	0	1	1/1 →
0	S_2	30	2	3	1	0	1	0	0	30/2
	Zj	-2M	0	-M	M	0	-M	-M		
	Cj – Zj	4+2M	5	-3+M	-M	0	0	0		

calculate Zj by multiplying the CB values by the entries in that column and summing across rows.

Zj Calculations:

$x1 = (-M\times1)+(-M\times1)+(0\times2) = -2M$

$x2 = (-M\times1)+(-M\times-1)+(0\times3) = 0$

$x_3 = (-M\times1)+(-M\times0)+(0\times1) = -M$

$S1 = (-M\times0)+(-M\times-1)+(0\times0) = M$

$A_1 = -M$, $A_2 = -M$

Calculation of Cj – Zj Row

$x_1 = 4-(-2M) = 4+2M$

$x_2 = 5-0 = 5$

$x_3 = -3-(-M) = -3+M$

$S_1 = 0-M = -M$

$S_2 = 0-0 = 0$

Key Column & Key Row:

- Look for most positive Cj – Zj to choose entering variable:

Here, 4+2M is the most positive → so x1 enters.

- Find minimum ratio (only positive pivot entries considered):

Row 2 (A_2) gives 1 / 1 = 1 (minimum) → A_2 leaves.

- Pivot element: 1 (at intersection of A_2 row and x1 column)

Simplex Table 2

CB	Basic variable	Solution values	x1	x2	x_3	S1	S2	A1	MinRatio
	Cj		4	5	-3	0	0	-M	
-M	A1	9	0	2*	1	1	0	1	9/2→
4	x1	1	1	-1	0	-1	0	0	1/-1
0	S2	28	0	5	1	2	1	0	28/5
Z=-9M+4		Zj	4	-2M-4	-M	-M-4	0	-M	
	Zj-Cj	0	9+2M	-3+M	M+4	0	0		

↑

Entering Variable: The most positive Zj−Cj value is 1+2M (assuming M is a very large positive number). This corresponds to the variable in the column labeled '2' (likely x_2). So, variable x_2 is the entering variable.

Leaving Variable: As indicated by the '9/2→' and the pivot element 2*, A1 is the leaving variable. This is good, as artificial variables should ideally leave the basis in an optimal solution.

The objective function value is Z=−9M+4. Since M is a very large number, this Z value is highly negative, indicating that the solution is not yet optimal for a minimization problem where we want to drive Z towards a smaller, positive value or zero

Simplex Table 3

CB	Basic variables	Solution values	1	2		S1	S2
		Cj	4	5	-3	0	0
5	2	9/2	0	1	1/2	½	0
4	1	11/2	1	0	1/2	-1/2	0
0	S2	11/2	0	0	-3/2	-1/2	1
		Zj	4	5	9/2	5/2 – 4/2	0
		Zj-Cj	0	0	-3 – 9/2	-1/2	0

Since there is no positive Δ j the solution is optimal.

The solution X_1 X_2 and S_2 to be produced

X_1 = 1 1/2, X_2 = 9/2, S_2 is not in actual production

Duality of LPP

In linear programming, duality implies that each linear pro-

- Any LP problem can be stated in another equivalent form based on the same data

gramming problem can be analyzed in two different ways but would have equivalent solutions. Any LP problem (either maximization and minimization) can be stated in another equivalent form based on the same data. The new LP problem is called dual linear programming problem or in short dual. The first way of stating linear programming problem is called the Primal of the problem. The second way of stating the same problem is called the dual.

The optimal solution for the primer and the dual are equivalent but they are derived through alternative procedures the dual contains economic information useful to management and it may also be easier to solved than the Primal problem. Generally if the LP Primal involves maxmasing a profit function subject to less than or equal to resource constraint, the dual will involve minimising total opportunity cost subject to greater than or equal to product profit constraint. Formulating the dual problem from a given primary is not difficult. And once it is formulated the solution procedure is exactly the same as for any LP problem.

Formulation of Dual Problem

Let's consider a standard maximization primal problem:

Objective Function:

Maximize $Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Subject to Constraints:

$a_{11} x_1 + a_{12} x_2 + \cdots + a_1 n\, x_n \le b_1$

$a_{21} x_1 + a_{22} x_2 + \cdots + a_2 n\, x_n \le b_2$

$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m$

Non-negativity Constraints:

$x_1, x_2 \ldots\ldots x_n$

Here:

$x_1, x_2 \ldots\ldots x_n$ are primal decision variables

c_1, c_2, c_n….. are the coefficients of the objective function.

b_1 b_2… bm, are the right-hand side values (resource limits).

The dual of this problem is expressed as:

Objective Function:

Minimize $Z^* = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$

subject to $a_{11}\ y_1 + a_{21}\ y_2 + \cdots + y_{m1}\ y_m \geq c_2$

$a_{12}\ y_1 + a_{22}\ y_2 + \cdots + a_{m2}\ y_m \geq c$

………………………

$a_1n\ y_1 + a_2n\ \ y_2 + \ldots\ldots + a_mn + y_m \geq c_n$

$y_1\ y_2 \ldots\ldots\ldots\ldots y_m \geq 0$

$y_1\ y_2 \ldots\ldots\ldots\ldots y_m$ are the dual decision variables.

Rules for constructing the dual problems

Dual of a Maximization Problem is a Minimization Problem

- If the primal is a maximization, the dual will be a minimization, and vice versa

Transpose the Coefficient Matrix

- The coefficients of the variables in the primal constraints become the coefficients of the dual constraints. This is essentially a transpose of the matrix of coefficients a_{ij}

1.4.2 Right-hand Side (RHS) Becomes Objective Coefficients

- The RHS constants b_i of the primal constraints become the objective coefficients of the dual.

Objective Coefficients Become RHS

- The objective function coefficients c_j in the primal become the RHS values in the dual.

Number of Constraints and Variables Are Switched

- If the primal has m constraints and n variables, then the dual will have n constraints and m variables.

Inequality Direction is Reversed

- A '≤' constraint in the primal corresponds to a '≥' constraint in the dual.
- A '≥' constraint in the primal corresponds to a '≤' constraint in the dual.
- An '=' constraint stays as '=' in both.

Non-negativity and Free Variables

- If a primal variable xj≥0, then the corresponding dual constraint is of type ≥.

- If a primal constraint is =, then the corresponding dual variable is unrestricted in sign.
- If a primal variable is unrestricted in sign, then the corresponding dual constraint is equality (=).

Dual of Dual is the Primal

Constructing the dual of a dual LP problem will bring you back to the original primal.

- First convert the minimization LPP into maximization form, if it is given in the minimization form.
- Convert the '≥' type inequalities into '≤' type by multiplying by –1.
- Introduce slack variables in the constraints of the given problem and obtain an initial basic solution.
- Test the nature of in C_J -Z_J the starting table:
- If all CJ -ZJ and XB are non-negative, then an optimal basic feasible solution has been attained.

Table No. 1.4.1.1 Dual Construction Rules Table

Aspect	Primal (Maximization)	Dual (Minimization)
Objective Type	Maximize Z	Minimize Z
Constraint Type	≤ (less than or equal to)	≥ (greater than or equal to)
Decision Variables	x1, x2, ..., xn	y1, y2, ..., ym
Number of Constraints	M	n (becomes number of dual constraints)
Number of Variables	N	m (becomes number of dual variables)
RHS of Constraints	Becomes coefficients in the dual objective function	Comes from primal constraint RHS values
Coefficients of Objective Function	Becomes RHS of dual constraints	Comes from primal objective coefficients
Constraint Coefficients Matrix	Rows	Columns (transposed)
Variable Non-negativity	xj≥	yi≥0

Illustration 1.4.6

Obtain the dual of :

Maximise $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \leq 2$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

and $x_1, x_2 \geq 0$

Solution

Let the dual variables be:

- y_1 for Constraint 1
- y_2 for Constraint 2
- y_3 for Constraint 3

Dual Rules :

- The dual objective is a minimization problem.
- Coefficients of the dual objective come from the RHS values of the primal constraints.
- The dual constraints are constructed from the columns of the primal constraint matrix, and the coefficients of the primal objective become the RHS of the dual constraints.

Apply the dual rules:

Rule	From Primal		To Dual	Values
1. Type	Maximize Z		Minimize Z*	
2. RHS of constraints	2, 10, 12		Coefficients in dual objective	$Z^* = 2y_1 + 10y_2 + 12y_3$
3. Coefficients of objective function	5(from x_1), 3 (from x_2)		RHS of dual constraints	Constraints become ≥ 5 ≥ 3
4. Coefficient matrix	Rows:		Columns of dual constraints	Dual Constraints:
	1	1		$y_1 + 5y_2 + 3y_3 \geq 5$
	5	2		$y_1\ 2y_2 + 8y_3 \geq 3$
	3	8		
5. Non-negativity	$x_1, x_2 \geq 0$		$y_1, y_2 \geq 0$	

Minimise $Z^* = 2y_1 + 10y_2 + 12y_3$

Subject to $y_1 + 5y_2 + 3y_3 \geq 5$

$y_1 \; 2\,y_2 + 8\,y_3 \geq 3$

$y_j \geq 0 \; j = (1, 2, 3\ldots.)$

Illustration 1.4.7

Obtain the dual of :

Minimise $Z = 40x_1 + 120\,x_2$

Subject to $x_1 - 2x_2 \leq 8$

$3x_1 + 5\,x_2 = 90$

$15x_1 + 44x_2 \geq -90$

$2x_1 + 3x_2 \geq 2$

and $x_1, x_2 \geq 0$

Solution

For each primal constraint, assign a dual variable:

- y_1 for constraint (1) $\leq$
- y_2 for constraint (2) $=$
- y_3 for constraint (3) $\geq$
- y_4 for constraint (4) $\geq$

Rule:

- $\leq \rightarrow$ dual variable ≥ 0
- $= \rightarrow$ dual variable (no sign restriction)
- $\geq \rightarrow$ dual variable ≤ 0

So,

- $y_1 \geq 0$
- y_2 free
- $y_3 \leq 0$
- $y_4 \leq 0$

▶ Since the primal is a minimization with constraints involving $\leq, =, \geq$, the dual will be a maximization.

- Dual objective function: formed from the RHS of primal constraints
- Dual constraints: formed from coefficients of x_1 and x_2 in the primal

Dual Problem

Maximize $Z^*=8y_1 +90y_2 -90y_3 +2y_4$

Subject to:

(formed from columns of primal constraints)

Coefficients of x_1 : $y_1 +3y_2+15y_3 +2y_4 \leq 40$

Coefficients of x_2 :

$-2y_1+5y_2 +44y_3 +3y_4 \leq 120$

And: $y_1 \geq 0$

y_2 free

$y_3 \leq 0$

$y_4 \leq 0$

Illustration 1.4.8

Primal (Minimization) Problem:

Minimize

$Z= 40x_1 +200\ x_2$

Subject to $4x_1+ 40x_2\geq 160$

$3x_1+ 10x_2\geq 60$

$8x_1+ 10x_2\geq 80$

and $x_1, x_2 \geq 0$

Solution

To solve the primal using the simplex method:

- Convert $\geq$ to $=$ by subtracting surplus variables and adding artificial variables.
- Use the Big M method to deal with artificial

Standard Form:

Minimize $Z=40x_1+200x_2+0S_1+0S_2+0S_3+MA_1+MA_2 +MA_3$

Subject to

$4x_1 + 40x_2 - S1 + 0S2 + 0S3 + A1 + 0A2 + 0A3 = 160$

$3x_1 + 10x_2 + 0S1\ S2 + 0S3 + 0A1 + A2 + 0A3 = 60$

$8x_1 + 10x_2 + 0S1 + 0S2 - S3 + 0A1 + 0A2 + A3 = 80$

And $x_1\ \ x_2\ S1\ S2\ S3\ A1\ A2\ A3 \geq 0$

Simplex Table I

	Cj		40	200	0	0	0	M	M	M	
XB	Basic Variable	Solution	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3	Min. Ratio
M	A_1	160	4	40*	-1	0	0	1	0	0	4→
M	A_2	60	3	10	0	-1	0	0	1	0	6
M	A_3	80	8	10	0	0	-1	0	0	1	8
	Zj	300M	15M	60M	-M	-M	-M	M	M	M	
	Cj - Zj		40-15M	200-60M	M	M	M	0	0	0	

Simplex Table 2

	Cj		40	200	0	0	0	M	M	
XB	Basic Variable	Solution	x_1	x_2	S_1	S_2	S_3	A_2	A_3	Min. Ratio
200	x_2	4	1/10	1	-1/40	0	0	0	0	40
M	A_2	20	2	0	1/4	-1	0	1	0	10
M	A_3	40	7*	0	1/4	0	-1	0	1	40/7→
	Zj	800+60M	20+9M	200	-5M/2	-M	-M	M	M	
	Cj - Zj		20-9M	0	5M/2	M	M	0	0	

Simplex Table 3

	Cj		40	200	0	0	0	M	
XB	Basic Variable	Solution Values	x_1	x_2	S_1	S_2	S_3	A_2	Min. Ratio
200	x_2	24/7	1/10	1	-1/40	0	0	0	240
M	A_2	60/7	2	0	1/4	-1	0	1	30 →
40	x_1	40/7	7*	0	1/4	0	-1	0	-160
	Zj	(6400+60M) /7	40	200	(5M-120) /28	-M	(2M-20) /7	M	
	Cj - Zj		0	0	(5M-120) /28	M	(2M-20) /7	0	0

Simplex Table 4

XB	Cj Basic Variable	Solution Values	200 x_1	40 x_2	0 S_1	0 S_2	0 S_3
200	x_2	3	0	1	-21/560	1/20	0
0	S3	30	0	0	5/8	-7/2	1
40	x_1	10	1	0	1/8	-1/2	0
	Zj	1000	40	200	-5/2	-10	0
	Cj - Zj		0	0	5/2	10	0

Since all Cj - Zj in the index row are ≥ 0, so the current solution is optimal.

x_1 =10, x_2 =3 and the minimum value of

Z= 4×10 +200× 3

=400+600

= 1000

The dual of the above primal can be written as:

Maximise $Z^* = -160y_1+60y_2+80y_3$

Subject to $4y_1+3y_2+8y_3 \leq 40$

$40y_1+10y_2+10y_3 \leq 200$

And $y_1\ y_2\ y_3 \geq 0$

Simplex Table 5

XB	Cj → Basic Variable	Solution Value	160 Y_1	60 Y_2	80 Y_3	0 S_1	0 S_2	Min. ratio
80	S_1	20	0	2	7*	1/7	-1/10	20/7 →
160	Y_1	5	1	1/4	1/4	-1/28	2/70	20
	Zj	800	160	40	40	0	4	
	Cj – Zj		0	20	40	0	-4	

Simplex Table 6

Cj Basic Variable	160 Solution Value	60 Y_1	80 Y_2	0 Y_3	0 S_1	0 S_2	Min. Ratio
S_1	40	4	3	8	1	0	10
S_2	200	40*	10	20	0	1	5 ← Key Row
Zj	0	0	0	0	0	0	
Cj – Zj		160	60	80	0	0	

Simplex Table 7

	Cj →		160	60	80	0	0	
XB	Basic Variable	Solution Value	Y_1	Y_2	Y_3	S_1	S_2	Min. Ratio
80	Y_3	20/7	0	2/7*	1	1/7	0	10→
160	Y_1	30/7	1	5/28	0	-1/28	7/20	2.4
	Zj	6400/7	160	360/7	80	40/7	40/7	
	Cj – Zj		0	60/7	0	-40/7	-24/7	

Simplex Table 8

	Cj		160	60	80	0	0
XB	Basic Variable	Solution Value	Y_1	Y_2	Y_3	S_1	S_2
60	y_2	10	0	1	7/2	1/2	-1/20
160	y_1	5/2	1	0	-5/8	-1/8	3/80
	Zj	1000	160	60	110	0	0
	Cj – Zj	0	0	0	-30	-10	-3

Since all Cj - Zj in the index row are ≥ 0, so the current solution is optimal.

$y_1=5/2$, $y_2=10$, $Z=160(5/2)+60(10)$

$=400+600=1000$

Both Primal and Dual have same optimal value:

$Z=1000$

Summarised Overview

This chapter explains the Simplex Method as an effective tool for solving LPPs with multiple variables. It covers the standard form, introduction of slack, surplus, and artificial variables, and outlines how to perform simplex iterations using tableau. The steps include identifying entering and leaving variables, calculating the pivot element, and updating the tableau to reach the optimal solution. It also introduces special cases like maximization, minimization (Big M Method), mixed constraints, and duality in LPP. Numerous illustrations are provided to help understand how to construct and solve simplex tables for both feasible and infeasible cases.

Self-Assessment Question

1. What is Simplex Method and why is it used?
2. What is meant by the 'standard form' of a Linear Programming Problem?
3. What are the key limitations of the graphical method that the Simplex Method overcomes?
4. Explain the role of slack, surplus, and artificial variables in the simplex method.
5. What is a pivot element in the simplex table
6. When do we use the Big M Method in solving LPPs?

Assignments

1. In which situations are artificial variables introduced in the simplex method?
2. How do you convert a minimization problem with '≥' constraints into a form suitable for the simplex method?
3. Use the simplex method to solve the following LP problem.

   Maximize $Z = 50\,x_1 + 70\,x_2 + 4x_3$

   subject to the constraints

   $x_1 + x_2 \leq 70$

   $x_1 + 2\,x_2 \leq 100$

   $2\,x_1 + x_2 \leq 120$

   where

   $x_1, x_2, \geq 0$
4. Minimise $Z = x_1 + x_2$

   subject to $2\,x_1 + x_2 \geq 4$

   $x_1 + 7\,x_2 \geq 7$

   Where $x_1, x_2, \geq 0$
5. Write the dual of the following LPP

   Maximize $Z = 50x_1 + 120x_2$

   subject to the constraints = $2\,x_1 + 4\,x_2 \leq 80$

   $3\,x_1 + x_2 \leq 60$

   Where $x_1, x_2, \geq 0$

Reference

1. Hillier, F. S., & Lieberman, G. J. (2020). Introduction to Operations Research (11th ed.). McGraw-Hill Education.
2. Sharma, J.K., Operations Research: Theory and Application (6th ed.).Trinity press
3. Dantzig, G. B. (1951). Maximization of a Linear Function of Variables Subject to Linear Inequalities. In Activity Analysis of Production and Allocation (pp. 339–347). Wiley.
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Suggested Reading

1. Hillier, F. S., & Lieberman, G. J. (2020). Introduction to Operations Research (11th ed.). McGraw-Hill Education.
2. Sharma, J.K., Operations Research: Theory and Application (6th ed.).Trinity press

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

02 BLOCK

Transportation And Assignment

Block Content

SREENARAYANAGURU OPEN UNIVERSITY

Unit 1

Transportation

Learning Outcomes

After completing this unit, the learner will be able to:

- comprehend the concept of transportation problem
- describe the assumptions of transportation problem
- solve the transportation problem

Background

The Transportation Problem is a specialized type of linear programming problem where the goal is to determine the most cost-effective way to distribute a product from several suppliers (sources) to several consumers (destinations), minimizing the total transportation cost. Unlike general LPPs, transportation problems have a structured format allowing the use of methods like NWCM, LCM, VAM, and MODI. These problems are fundamental in logistics and supply chain optimization.

Keywords

Transportation cost, Allocation, Feasible Solution, North-West Corner Method, Least Cost Method

Discussion

- Minimization of Cost

Imagine there are three factories in Kerala that produce cement. Each factory produces a certain number of cement bags. At the same time, there are four construction sites in different towns that need cement bags in different quantities. The challenge is:

How do we transport cement from factories to construction sites in such a way that the transportation cost is the least possible, while making sure that supply meets demand? This situation is what we call a Transportation Problem in Operations Research.

The transportation problem is a unique form of linear programming where the main objective is to minimize the cost involved in shipping a product from multiple sources (or origins) to various destinations. Due to the specific structure of this problem, the standard simplex method is not effective for finding a solution. Instead, specialized techniques are required. In this context, an origin refers to the point from which goods are sent, while a destination is the point where the goods are received. The transportation cost is calculated per unit of shipment between these locations.

2.1.1 Terminologies used in Transportation Problem

2.1.1.1 Transportation Cost:

The cost per unit of transporting goods from an origin to a destination. These costs form a cost matrix.

2.1.1.2 Allocation (Shipment)

The number of units assigned (transported) from a specific origin to a specific destination.

2.1.1.3 Balanced Transportation Problem

- Supply equals demand

A Balanced Transportation Problem refers to a scenario where the total supply from all origins exactly matches the total demand at all destinations. For instance, if four factories collectively produce 1000 units and the combined requirement of four warehouses is also 1000 units, then the transportation problem is considered balanced.

2.1.1.4 Unbalanced Transportation Problem

- Mismatch between supply & demand

An Unbalanced Transportation Problem occurs when the total supply from the origins does not equal the total demand at the destinations. For example, if the combined output of four factories is 1000 units, but the total requirement across four warehouses is either 900 or 1100 units, the problem is considered unbalanced. To convert such a problem into a balanced transportation problem, a dummy origin or dummy destination is added, depending on whether there is excess supply or demand. This dummy source or destination is assigned zero transportation cost per unit.

2.1.1.5 Feasible Solution

- Satisfies supply and demand
- No violations

A Feasible Solution to a transportation problem is a solution that:

- Satisfies all supply (availability) and demand (requirement) constraints, and
- Does not violate any transportation restrictions (i.e., shipments are non-negative).

2.1.1.6 Basic Feasible Solution (BFS)

- Starting points

A Basic Feasible Solution is a special kind of feasible solution that:

- Has exactly (m + n – 1) number of occupied (non-zero) cells or allocations, where m = number of origins and n = number of destinations, and
- These allocations must be independent (i.e., they do not form a closed loop).

It is called "basic" because it's the starting point for optimization techniques (like the MODI method or stepping stone method).

2.1.1.7 Optimal Solution

- Maximum cost reduction

An Optimal Solution is a feasible (or basic feasible) solution that:

- Minimizes the total transportation cost (or maximizes profit, in some cases),
- Still satisfies all supply and demand constraints, and
- Meets the condition that no further cost reduction is possible by shifting allocations.

An optimal solution provides the least possible total cost of transporting goods from sources to destinations.

2.1.2 Basic structure of transportation problem

		Destination				
		D1	D2	D3	D4	Supply (s_i)
Source	O1	C_{11}	C_{12}	C_{13}	C_{14}	S_1
	O2	C_{21}	C_{22}	C_{23}	C_{24}	S_2
	O3	C_{31}	C_{32}	C_{33}	C_{34}	S_3
	O4	C_{41}	C_{42}	C_{43}	C_{44}	S_4
Demand (d_j):		d_1	d_2	d_3	d_4	

In the above table D1, D2, D3 and D4 are the destinations where the products/goods are to be delivered from different sources S1, S2, S3 and S4. Si is the supply from the source Oi. dj is the demand of the destination Dj. Cij is the cost when the product is delivered from source Si to destination Dj.

2.1.3 Basic Assumptions of the Transportation Problem

1. Linearity of Cost

- Transportation cost constant

The cost of transportation per unit is constant, regardless of the quantity shipped. That means total transportation cost is a linear function of the number of units transported.

2. Homogeneous Products

- Identical goods

All goods being transported are identical and divisible. One unit from one source is the same as one unit from another.

3. Supply and Demand Are Known and Fixed

- Supply and demand are known

The availability at each origin and the requirement at each destination are known in advance and remain constant.

4. Objective Is to Minimize Total Cost

- Minimise transportation cost

The goal is to minimize total transportation cost (or maximize profit in the case of profit-based models).

5. Instantaneous and Unlimited Transportation Capacity

- Non restrictions on routes

There are no capacity restrictions on transportation routes unless specifically mentioned. The model assumes goods can be moved without delay or capacity limits.

2.1.4 Mathematical Formulation of the Problem

Let us formulate the problem consider

- m : Number of origins (sources)
- n: Number of destinations
- C_ij: Cost of transporting one unit from origin iii to destination j
- x_ij: Number of units transported from origin iii to destination j (decision variable)
- a_i: Supply available at origin i
- b_j: Demand required at destination j

Objective Function:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{i=1}^{m} xij\ cij$$

Constraints:

$$\sum_{i=1}^{m} xij = a_i \quad \text{for i= 1,2,3....m}$$

$$\sum_{i=1}^{m} xij = bj \quad \text{for j = 1,2,3.....n}$$

Non-negativity Constraints:

The number of units transported cannot be negative. xij≥0 for all i and j

2.1.5 Methods of Solving Transportation Problem

2.1.5.1 North-West Corner Method

This is the most systematic and easiest method for obtaining an initial feasible solution to a transportation problem.

Step 1:

Construct an empty m × n matrix, where m = number of sources (rows) and n = number of destinations (columns).

- Systematic and easy

Step 2:

Write the supply values at the end of each row and the demand values at the bottom of each column.

Step 3:

Start with the top-left cell (i.e., North-West corner) of the matrix.

Make the maximum possible allocation to that cell — this will be the minimum of supply and demand for that row and column.

Step 4:

Adjust the supply and demand by subtracting the allocated quantity.

Step 5:

- If supply becomes 0, move down to the next row in the same column.
- Go back to Step 4.

Step 6:

- If demand becomes 0, move right to the next column in the same row.
- Go back to Step 4.

Step 7:

If supply equals demand, make the allocation and cross out both the row and column (but move only in one direction to avoid degeneracy).

Step 8:

Continue this process until the entire supply and demand are exhausted and all allocations are completed.

Illustration 2.1.1

Solve the following by North West Corner Method (NWCM)

	W_1	W_2	W_3	Supply S_1
P1	7	6	9	20
P2	5	7	3	28
P3	4	5	8	17
Di	21	25	19	65

Solution:

- Start from the top-left (north-west) corner, i.e., P1 → W1.
- Allocate as much as possible (minimum of supply and demand).
- Adjust the supply and demand, and move right if demand becomes 0 or down if supply becomes 0.

Allocation Table with NWCM Steps (Allocations Marked in Cells)

W1 (21)	W2 (25)	W3 (19)	Supply
20 (7)			0
1 (5)	25 (7)	2 (3)	0
		17 (8)	0
0	0	0	

1. P1 → W1: min(20, 21) = 20 → allocate 20

   → New: P1 = 0, W1 = 1

   → Move to P2 → W1

2. P2 → W1: min(28, 1) = 1 → allocate 1

   → New: P2 = 27, W1 = 0

   → Move to P2 → W2

3. P2 → W2: min(27, 25) = 25 → allocate 25

   → New: P2 = 2, W2 = 0

   → Move to P2 → W3

4. P2 → W3: min(2, 19) = 2 → allocate 2

   → New: P2 = 0, W3 = 17

   → Move to P3 → W3

5. P3 → W3: min(17, 17) = 17 → allocate 17

   → All supplies and demands now fulfilled

Final NWCM Allocation Table:

W1	W2	W3	Supply	
P1	20	–	–	20
P2	1	25	2	28
P3	–	–	17	17
Demand	21	25	19	65

Cost Calculation

	W1	W2	We	Supply
P1	20 × 7 = 140	–	–	20
P2	1 × 5 = 5	25 × 7 = 175	2 × 3 = 6	28
P3	–	–	17 × 8 = 136	17

20×7)+(1×5)+(25×7)+(2×3)+(17×8)=140+5+175+6+136=462

Illustration 2.1.2

Find the feasible solution of the following transportation problem using NWCM

	W1 (4)	W2 (7)	W3 (6)	W4 (13)	Supply
F1 (6)	14	25	45	5	6
F2 (8)	65	25	35	55	8
F3 (16)	35	3	65	15	16
Demand	4	7	6	13	Total: 30

Solution

Step-by-Step Allocation (NWCM)

Start at the top-left (F1 → W1) and allocate as much as possible:

W1 (4)	W2 (7)	W3 (6)	W4 (13)	Supply	
F1 (6)	4 (14)	2 (25)			0
F2 (8)		5 (25)	3 (35)		0
F3 (16)			3 (65)	13 (15)	0
Demand	0	0	0	0	

Each cell shows: allocation (cost)

Total Transportation Cost Calculation

F1 → W1: 4×14=56

F1 → W2: 2×25=50

F2 → W2: 5×25=125

F2 → W3: 3×35=105

F3 → W3: 3×65=195

F3 → W4: 13×15=195

Total Cost = 56+50+125+105+195+195=726

2.1.5.2 Least Cost Method (LCM)

- Priortising routes with low cost

The Least Cost Method aims to minimize overall transportation costs by prioritizing routes with the lowest unit cost. It works by directing the largest possible shipments through the most economical paths first.

Step 1:

Begin by identifying the cell in the transportation table with the smallest unit cost. Allocate the maximum possible quantity to this cell (whichever is smaller between supply and demand). Once a row or column has fulfilled its requirement (supply or demand), eliminate it from further consideration. If both a row and a column are satisfied at the same time, eliminate only one (usually the row). If multiple cells share the same lowest cost, choose the one where the largest allocation is possible.

Step 2:

After adjusting the remaining supplies and demands, repeat the process by selecting the next lowest cost cell from the uncrossed portion of the table. Again, allocate as much as possible, and eliminate the row or column that gets fully satisfied.

Step 3:

Continue this process until all supply and demand requirements are met. The resulting allocation is a feasible solution, though it may not be optimal. Also, it's important to note that this initial solution might be degenerate in some cases.

Illustration 2.1.3

Solve the following transportation problem using Least Cost Method (LCM)

	W1 (4)	W2 (7)	W3 (6)	W4 (13)	Supply
F1 (6)	14	25	45	5	6
F2 (8)	65	25	35	55	8
F3 (16)	35	3	65	15	16
Demand	4	7	6	13	30

Solution

Find the lowest cost cell:

F3 → W2 = 3

- Supply: 16
- Demand: 7

Allocate 7 units

- Remaining supply at F3 = 16 - 7 = 9
- Remaining demand at W2 = 0 (fulfilled) → Cross out W2

Next lowest cost:

F1 → W4 = 5

- Supply: 6
- Demand: 13

Allocate 6 units

- Remaining supply at F1 = 0 (fulfilled) → Cross out F1
- Remaining demand at W4 = 13 - 6 = 7

Next lowest cost:

F3 → W4 = 15

- Supply: 9
- Demand: 7

Allocate 7 units

- Remaining supply at F3 = 9 - 7 = 2
- Remaining demand at W4 = 0 → Cross out W4

Next lowest cost:

F1 is crossed out; W2 and W4 are also done

Now, lowest cost is F3 → W1 = 35

- Supply: 2
- Demand: 4

Allocate 2 units

- Remaining supply at F3 = 0 → Cross out F3
- Remaining demand at W1 = 4 - 2 = 2

Only F2 left

Remaining cells:

- F2 → W1 = 65, demand = 2

Allocate 2 units

- Supply left at F2 = 8 - 2 = 6
- W1 demand = 0 → Cross out W1

Only F2 → W3 remains

- Supply: 6
- Demand: 6

Allocate 6 units

Final Allocation Table

	W1 (4)	W2 (7)	W3 (6)	W4 (13)
F1				6 (5) = 30
F2	2 (65) = 130		6 (35) = 210	
F3	2 (35) = 70	7 (3) = 21		7 (15) = 105

Total Transportation Cost

$(2\times65)+(6\times35)+(2\times35)+(7\times3)+(7\times15)+(6\times5)$

$=130+210+70+21+105+30=566$

Illustration 2.1.4

A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units (in 100s) per week of a product, respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 8, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34

Use Least Cost Method (LCM) to find an initial basic feasible solution to the transportation problem

Solution

Pick the cell with the lowest cost, then allocate the maximum possible quantity (min of supply and demand), and adjust the table accordingly.

Minimum cost = 8 → S3 → D2

- Supply = 18
- Demand = 8

Allocate 8

- Remaining S3 = 10
- D2 fulfilled → cross out D2

Next lowest cost = 10 → S1 → D4

- Supply = 7
- Demand = 14

Allocate 7

- S1 = 0 → cross out S1
- D4 = 7 remaining

Next lowest cost = 20 → S3 → D4

- Supply = 10
- Demand = 7

Allocate 7

- S3 = 3
- D4 fulfilled → cross out D4

Next lowest cost = 19 → S2 and S3 remain, D1 and D3 remain

→ 19 → S1 is done

→ 30 → S2 → D2 (crossed out)

→ 40 → S3 → D1

- Supply = 3
- Demand = 5

Allocate 3

- S3 = 0 → cross out S3

- D1 = 2 remaining

Only S2 left

- D1 remaining = 2
- D3 = 7

→ Next lowest = 40 → S2 → D3

Allocate 7

- S2 = 2
- D3 = 0 → cross out D3

→ Next = 70 → S2 → D1

Allocate 2

Final Allocation Table

	D1 (5)	D2 (8)	D3 (7)	D4 (14)	Supply
S1				7(10) =70	0
S2	2 (70) = 140		7(40) =280		0
S3	3 (40) = 120	8(8)=64		7(20) =140	0

Total Transportation Cost

= (7×10)+ (2×70)+(7×40)+(3×40)+(8×8)+(7×20)

= 70+140+280+120+64+140=814

2.1.5.3 Vogel's Approximation Method (VAM)

Imagine you are shopping and comparing two options. Instead of only looking at the cheapest item, you also consider the penalty of not choosing the next best option.

- Consider penalty cost for best allocation

VAM works the same way:

- For each row and column, we calculate a penalty (the difference between the two smallest costs).
- This penalty tells us how much we 'lose' if we don't pick the cheapest option there.
- Then, we give preference to the row or column with the highest penalty because that's where wrong choices would cost us the most.

Vogel's Approximation Method (VAM) is a technique used to find an initial feasible solution to a transportation problem. It considers the opportunity cost of not choosing the cheapest route (by using penalties).

- Initial feasible solution

Vogel's Approximation Method is often considered superior to the North-West Corner Rule (NWCR) and Least Cost Method (LCM) because it typically yields an initial solution that is closer to the optimal. The method is based on minimizing the penalty cost—the extra cost that would be incurred if we skip the best possible allocation in a row or column.

Step 1: Compute Penalties

For each row and column, calculate the penalty by finding the difference between the lowest and the second-lowest transportation costs in that row or column. This represents the additional cost incurred if the lowest-cost option is not selected.

Step 2: Choose the Highest Penalty

Identify the row or column with the highest penalty value. Within that row or column, choose the cell with the lowest transportation cost, and allocate as much as possible (based on supply and demand).

In case of a tie in penalty values, break the tie by choosing the row or column where the largest allocation can be made.

Step 3: Update Supply/Demand

Adjust the remaining supply and demand. Cross out the row or column where the supply or demand becomes zero. If both are satisfied at once, cross out only one and set the other to zero to avoid degeneracy.

Rows or columns with zero supply or demand should be excluded from future penalty calculations.

Step 4: Repeat

Continue repeating steps 1 through 3 until all the supply and demand constraints are fulfilled.

Illustration 2.1.5

A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units (in 100s) per week of a product, respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 8, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34

To solve this problem use Vogel's Approximation Method

Step 1: Compute penalties (first iteration)

- Rows:

S1 → 10, 19 → Penalty = 19 – 10 = 9

S2 → 30, 40 → Penalty = 40 – 30 = 10

S3 → 8, 20 → Penalty = 20 – 8 = 12

- Columns:

D1 → 19, 40 → Penalty = 40 – 19 = 21

D2 → 8, 30 → Penalty = 30 – 8 = 22

D3 → 40, 50 → Penalty = 50 – 40 = 10

D4 → 10, 20 → Penalty = 20 – 10 = 10

Highest penalty = 22 (Column D2)

Lowest cost in D2 = 8 (S3 → D2)

Allocate min(18, 8) = 8

→ S3 = 10 left, D2 = 0 → Cross out D2

Step 2: Recalculate penalties (exclude D2)

- Rows:

S1 → 10, 19, 50 → 10, 19 → Penalty = 19 – 10 = 9

S2 → 40, 60, 70 → 40, 60 → Penalty = 60 – 40 = 20

S3 → 20, 40, 70 → 20, 40 → Penalty = 40 – 20 = 20

- Columns:

D1 → 19, 40 → Penalty = 40 – 19 = 21

D3 → 40, 50, 70 → 40, 50 → Penalty = 50 – 40 = 10

D4 → 10, 20, 60 → 10, 20 → Penalty = 20 – 10 = 10

Highest penalty = 21 (D1)

Lowest cost in D1 = 19 (S1 → D1)

Allocate min(7, 5) = 5

→ S1 = 2 left, D1 = 0 → Cross out D1

Step 3: Recalculate penalties (exclude D1, D2)

- Rows:

S1 → 50, 10 → 10, 50 → Penalty = 40

S2 → 40, 60 → Penalty = 60 – 40 = 20

S3 → 20, 70 → Penalty = 70 – 20 = 50

- Columns:

  - D3 → 40, 50, 70 → 40, 50 → Penalty = 10
  - D4 → 10, 20, 60 → 10, 20 → Penalty = 10

Highest penalty = 50 (Row S3)

Lowest cost in S3 = 20 (S3 → D4)

Allocate min(10, 14) = 10

→ S3 = 0, D4 = 4 left → Cross out S3

Step 4: Recalculate penalties (exclude D1, D2, S3)

- Rows:

  - S1 → 50, 10 → Penalty = 40
  - S2 → 40, 60 → Penalty = 20

- Columns:

D3 → S1 = 50, S2 = 40 → Penalty = 10

D4 → S1 = 10, S2 = 60 → Penalty = 50

- Highest penalty = 50 (Column D4)

Lowest cost in D4 = 10 (S1 → D4)

Allocate min(2, 4) = 2

→ S1 = 0, D4 = 2 → Cross out S1

Step 5: Only S2 remains

D3 = 7

D4 = 2

S2 = 9

Allocate 7 to D3 → (S2 → D3 = 40)

Allocate 2 to D4 → (S2 → D4 = 60)

Final Allocation Table (VAM)

	D1 (5)	D2 (8)	D3 (7)	D4 (14)
S1	5 (19) = 95			2 (10) = 20
S2			7(40)= 280	2 (60) = 120
S3		8(8)=64		10 (20) = 200

Total Transportation Cost

=(5×19)+(2×10)+(7×40)+(2×60)+(8×8)+(10×20)

=95+20+280+120+64+200=779

Summarised Overview

The transportation problem ensures optimal allocation of goods from multiple origins to multiple destinations at minimum cost while satisfying supply and demand constraints. It assumes cost linearity, homogeneity of goods, and known supplies/demands. If the total supply and demand are unequal, dummy rows or columns are added to balance the problem. solutions begin with an initial basic feasible solution using methods like: North-West Corner Method (NWCM) – simple and systematic. Least Cost Method (LCM) – prioritizes lowest cost cells. Vogel's Approximation Method (VAM) – considers penalty cost for best allocations.

Self-Assessment Question

1. What are the assumptions underlying transportation models?
2. Explain how the Least Cost Method is applied for an initial feasible solution.
3. Describe the steps in Vogel's Approximation Method.
4. How do you detect and resolve degeneracy in a transportation problem?

Assignments

1. Find the Feasible solution of the following problem using NWCM

	W1	W2	W3	W4	Supply
F1	14	25	45	5	6
F2	65	25	35	55	8
F3	35	3	65	15	16
Demand	4	7	5	13	30

2. Solve the following by Vogel's method

	A	B	C	Supply
1	6	8	4	14
2	4	9	8	12
3	1	2	6	5
Demand	6	10	15	31

Reference

1. Sharma, R. R. K., & Sharma, K. D. (2003). A new dual based procedure for the transportation problem. *European Journal of Operational Research, 144*(3), 560-564.
2. Gass, S. I. (1990). On solving the transportation problem. *Journal of the Operational Research Society, 41*(12), 1113-1114.
3. Kapoor, V. K. (2000). *Operations research*. New Delhi: Sultan Chand & Sons.

Suggested Reading

1. Paneerselvam, R. (2017). *Operations Research*. PHI Learning Pvt. Ltd.
2. Sharma, S. D. (2014). *Operations Research*. Kedarnath Ramnath & Co
3. Kapoor, V. K. (2000). *Operations research*. New Delhi: Sultan Chand & Sons.

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

Unit 2

Optimality Test

Learning Outcomes

After completing this unit, the learner will be able to:

- understand methods of optimality test of transportation problem
- apply MODI Method to find optimal solution
- handle degeneracy in transportation problem

Background

The Transportation Problem is a key model in Operations Research used for determining the most cost-efficient way to distribute a product from multiple sources (factories, warehouses) to multiple destinations (customers, outlets) while meeting supply and demand constraints. This linear programming problem minimizes the total cost of transportation and assumes constant per-unit costs, known and fixed supply and demand, and identical units.

Keywords

MODI Method, Optimal Solution, Degeneracy

Discussion

- Low transportation cost

Once an initial basic feasible solution (IBFS) is obtained for a transportation problem using methods like North-West Corner Rule (NWCR), Least Cost Method (LCM), or Vogel's Approximation Method (VAM), the next important step is to check whether that solution is optimal, i.e., whether it results in the lowest possible transportation cost. This process is known as the Optimality Test.

2.2.1 Methods of Optimality Test

- Optimality testing

The most widely used technique for optimality testing in transportation problems is Modified Distribution Method (MODI Method) (also called the u–v Method) and Stepping stone method.

2.2.1.1 Modified Distribution Method (MODI)

We have learned how to set up a transportation problem and how to find an initial feasible solution using methods like the North-West Corner Rule, Least Cost Method, or Vogel's Approximation Method (VAM).

But remember, those are only starting solutions. They may not be the best (optimal) solution. After getting an initial solution, we need to test if it is optimal (minimum cost) or not.

- If it is optimal → we stop.
- If not → we improve the solution until it becomes optimal.

This is where the MODI Method comes in. It helps us to:

- Check optimality of the current solution.
- Improve the solution step by step until the least-cost allocation is reached

- Optimization

The Modified Distribution Method (MODI) is used to check whether a basic feasible solution of a transportation problem is optimal. If it's not, the method helps to improve the solution by adjusting allocations in a way that reduces total cost.

Step 1:

Once you have a basic feasible solution using methods like Vogel's Approximation, North-West Corner, or Least Cost Method, identify which cells have allocations (occupied) and which are empty (unoccupied).

Let the cost of each cell be denoted as Cij .

We now assign values to Ui (for rows) and Vj (for columns) using the condition:

Ui+ Vj= Cij (only for occupied cells)

Start by setting one variable (usually Ui =0) and solve for the others. There will be m+n−1 such equations.

Step 2:

Now calculate cell evaluations (also called improvement indices or opportunity costs), denoted as dij, for each unoccupied cell using:

$dij = Cij - (Ui + Vj)$

Step 3:

If all dij ≥ 0 , then the current solution is optimal.

If any dij < 0, the solution is not optimal, and there is room for cost improvement.

Step 4:

i. For the most negative dij make a reallocation using a loop (increase and decrease allocations alternately along the loop).

ii. Form a closed loop involving that unoccupied cell and some occupied cells (alternating horizontal and vertical moves).

iii. Adjust the allocations: increase in the selected cell and alternate +/− in the loop.

iv. Recalculate Ui, Vj and dij and repeat until all dij ≥ 0

Illustration 2.2.1

Solve the following Transportation Problem

I	II	III	Supply	
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	

STEP 1: Initial Feasible Solution using Vogel's Approximation Method (VAM)

- Compute penalties (row and column penalties)

Row	Lowest	2nd Lowest	Penalty
R1	2	4	2
R2	1	3	2
R3	4	5	1
R4	1	2	1

Column	Lowest	2nd Lowest	Penalty
C1	1	2	1
C2	3	4	1
C3	1	2	1

Highest Penalty = **2**, tie between R1 and R2

We choose R2 (smallest cost = **1** at (2,3))

- Allocate min(8,18) = 8 to (2,3)
- Update: Supply R2 = 0; Demand C3 = 18 − 8 = 10

  ▶ Remove R2 (supply exhausted)

Recompute penalties. Now:

- R1: 2, 4 → Penalty = 2
- R3: 4, 5 → Penalty = 1
- R4: 1, 2 → Penalty = 1
- C1, C2, C3 = same

Pick **R1** (penalty = 2), smallest cost = 2 at (1,1)

- Allocate min(5, 7) = **5** to (1,1)
- Supply R1 = 0, Demand C1 = 7 – 5 = 2

  ▶ Remove R1 (supply exhausted)

Recompute:

- R3: 4, 5 → P = 1
- R4: 1, 2 → P = 1
- C1: 1, 3 → P = 2

Pick **C1**, min cost = 1 at (4,1)

- Allocate min(14,2) = **2** to (4,1)
- Supply R4 = 12, Demand C1 = 0

  ▶ Remove C1

Remaining: C2 (9), C3 (10)

Pick **R4**, smallest cost = 2 at (4,3)

Allocate min(12,10) = **10** to (4,3)

Supply R4 = 2, Demand C3 = 0

Remove C3

Remaining: C2 (9)

Pick **R4**, only cell left = (4,2)

- Allocate min(2,9) = **2** to (4,2)
- Supply R4 = 0, Demand C2 = 7

Only one row left (R3), allocate remaining demand:

(3,2): Allocate 7 to fulfil C2

Initial VAM Allocation Table

	I	II	III	Supply
From 1	5			0
From 2			8	0
From 3		7		0
From 4	2	2	10	0
Demand	0	0	0	

Step 2: Optimality Test using MODI Method

Assign U_I and V_j

Occupied cells:

(1,1) = 5 → c = 2

(2,3) = 8 → c = 1

(3,2) = 7 → c = 4

(4,1) = 2 → c = 1

(4,2) = 2 → c = 6

(4,3) = 10 → c = 2

Let $U_1 = 0$

From (1,1): $U_1 + V_1 = 2 \Rightarrow \ = 2$

From (4,1): $U_4 + V_1 = 1 \Rightarrow U_4 + 2 = 1 \Rightarrow U_4 = -1$

From (4,2): $U_4 + V_2 = 6 \Rightarrow -1 + V_2 = 6 \Rightarrow V_2 = 7$

From (3,2): $U_3 + V_2 = 4 \Rightarrow V_3 + 7 = 4 \Rightarrow V_3 = -3$

From (2,3): $U_3 + V_2 = 1$

From (4,3): $-1 + V_3 = 2 \Rightarrow V_3 = 3$

Then from (2,3): $U_2+3= 1 \Rightarrow U_2=-2$

Final values:

U_i	V_j
$u_1 = 0$	$v_1 = 2$
$u_2 = -2$	$v_2 = 7$
$u_3 = -3$	$v_3 = 3$
$u_4 = -1$	

- Find the minimum allocation on the negative sign cells:

Negative cells = (4,2) = 2, (2,3) = 8 → Minimum = **2**

- Adjust allocations:

Apply +2 and −2 alternately:

Cell	Adjustment	**New Allocation**
(2,2)	+2	2
(4,2)	−2	0 (becomes unoccupied)
(4,3)	+2	12
(2,3)	−2	6

New Allocation Table

	I	II	III	Supply
From 1	5			0
From 2		2	6	0
From 3		7		0
From 4	2		12	0
Demand	0	0	0	

Occupied cells = 6

Run MODI again:

Step 1: Solve U_i, V_j

Use occupied cells:

From (1,1): $U_1+ V_1= 2 \Rightarrow \quad = 2$

From (4,1): $U_4 + V_1= 1 \rightarrow U_4=-1$

From (4,3): $U_4+ V_3 = 2 \rightarrow -1+V_3 =2 \; V_3 = 3$

From (2,3): $U_2+ V_3 =1 \rightarrow U_2 = -2$

From (2,2): $-2+ U_2 = 3 \rightarrow U_2 = 5$

From (3,2): $U_3+5= 4 \;\; U_3=-1$

Step 2: Compute new $d_{ij} = C_{ij} - (U_i + V_j)$

Cell (i,j)	Cij	Ui + Vj	d_{ij}
(1,2)	7	0 + 5 = 5	2
(1,3)	4	0 + 3 = 3	1
(2,1)	3	-2 + 2 = 0	3
(3,1)	5	-1 + 2 = 1	4
(3,3)	7	-1 + 3 = 2	5

All $d_{ij} \geq 0$ → **Optimal solution reached**

Final Optimal Allocation

	I	II	III	Supply
From 1	5			0
From 2		2	6	0
From 3		7		0
From 4	2		12	0
Demand	0	0	0	

Total Transportation Cost:

Use only allocated cells:

Z = (1,1) 5×2 + (2,2) 2×3 + (2,3) 6×1 + (3,2) 7×4 + (4,1) 2×1 + (4,3) 12×2

Z=5×2+2×3+6×1+7×4+2×1+12×2=10+6+6+28+2+24= 76

Illustration 2.2.2

Find Solution using LCM, also find optimal solution using modi method,

Distribution centre						
		D1	**D2**	**D3**	**D4**	**Supply**
Plant	**P1**	19	30	50	12	7
	P2	70	30	40	60	10
	P3	40	10	60	20	18
Requirement		5	8	7	15	

Solution.

An initial basic feasible solution is obtained by Least Cost Method and is shown in table 1.

Table 1

Distribution centre						
		D1	D2	D3	D4	Supply
Plant	P1	19	30	50	12 (7)	7
	P2	70 (3)	30	40 (7)	60	10
	P3	40 (2)	10 (8)	60	20 (8)	18
Requirement		5	8	7	15	

Initial basic feasible solution

$12 \times 7 + 70 \times 3 + 40 \times 7 + 40 \times 2 + 10 \times 8 + 20 \times 8 =$ Rs. 894.

Calculating u_i and v_j using $u_i + v_j = c_{ij}$

Substituting $u_1 = 0$, we get

$u_1 + v_4 = c_{14} \Rightarrow 0 + v_4 = 12$ or $v_4 = 12$

$u_3 + v_4 = c_{34} \Rightarrow u_3 + 12 = 20$ or $u_3 = 8$

$u_3 + v_2 = c_{32} \Rightarrow 8 + v_2 = 10$ or $v_2 = 2$

$u_3 + v_1 = c_{31} \Rightarrow 8 + v_1 = 40$ or $v_1 = 32$

$u_2 + v_1 = c_{21} \Rightarrow u_2 + 32 = 70$ or $u_2 = 38$

$u_2 + v_3 = c_{23} \Rightarrow 38 + v_3 = 40$ or $v_3 = 2$

Table 2: Distribution centre

		D1	D2	D3	D4	Supply	u_i
Plant	P1	19	30	50	12 (7)	7	0
	P2	70 (3)	30	40 (7)	60	10	38
	P3	40 (2)	10 (8)	60	20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Calculating opportunity cost d_{ij} using $c_{ij} - (u_i + v_j)$

Unoccupied cells	Opportunity cost d_{ij}
(P_1, D_1)	$c_{11} - (u_1 + v_1) = 19 - (0 + 32) = -13$
(P_1, D_2)	$c_{12} - (u_1 + v_2) = 30 - (0 + 2) = 28$
(P_1, D_3)	$c_{13} - (u_1 + v_3) = 50 - (0 + 2) = 48$

(P_2, D_2)	$c_{22} - (u_2 + v_{2)} = 30 - (38 + 2) = -10$
(P_2, D_4)	$c_{14} - (u_2 + v_{4)} = 60 - (38 + 12) = 10$
(P_3, D_3)	$c_{33} - (u_3 + v_{3)} = 60 - (8 + 2) = 50$

Table 3: Distribution centre

		D1	D2	D3	D4	Supply	u_i
Plant	P1	-13 19	28 30	48 50	12 (7)	7	0
	P2	70 (3)	-10 30	40 (7)	10 60	10	38
	P3	40 (2)	10 (8)	50 60	20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Now choose the smallest (most) negative value from opportunity cost dij (i.e., –13) and draw a closed path from P1D1. The following table shows the closed path.

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	-13 19 +	28 30	48 50	– 12 (7)	7	0
	P2	70 (3)	-10 30	40 (7)	10 60	10	38
	P3	40 (2) –	10 (8)	50 60	+ 20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Choose the smallest value with a negative position on the closed path (i.e., 2), it indicates the number of units that can be shipped to the entering cell. Now add this quantity to all the cells on the corner points of the closed path marked with plus signs and subtract it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

Now again calculate the values for u_i & v_j and opportunity cost. The resulting matrix is shown below.

Table 5

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19 (2)	[28] 30	[61] 50	12 (5)	7	0
	P2	70 (3)	[-23] 30	40 (7)	[-3] 60	10	51
	P3	[13] 40	10 (8)	[63] 60	20 (10)	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Choose the smallest (most) negative value from opportunity cost (i.e., –23). Now draw a closed path from P2D2.

Table 6

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	+ 19 (2)	[28] 30	[61] 50	– 12 (5)	7	0
	P2	– 70 (3)	[-23] 30 +	40 (7)	[-3] 60	10	51
	P3	[13] 40	– 10 (8)	[63] 60	+ 20 (10)	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Table 7: MODI Method

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19 (5)	[28] 30	[38] 50	12 (2)	7	0
	P2	[23] 70	30 (3)	40 (7)	[20] 60	10	28
	P3	[13] 40	10 (5)	[40] 60	20 (13)	18	8
Requirement		5	8	7	15		
v_j		19	2	12	12		

Since all the current opportunity costs are non–negative, this is the optimal solution. The minimum transportation cost is:

$$19 \times 5 + 12 \times 2 + 30 \times 3 + 40 \times 7 + 10 \times 5 + 20 \times 13 = \text{Rs. } 799$$

2.2.2 Unbalanced Transportation Problem

- Supply not matches demand

A transportation problem is considered balanced when the total supply from all sources equals the total demand at all destinations. If these values are not equal, the problem is unbalanced and requires adjustments to be solved as a balanced problem. This imbalance can be addressed by introducing a dummy source or destination with a cost of zero to balance the problem before applying standard transportation problem solution methods.

2.2.2.1 Supply Exceeds Demand

There are two situations:

When demand is less than supply (total supply exceeds total demand)

If total supply exceeds total demand, a "dummy" destination is added to absorb the excess supply. The transportation cost to this dummy destination is set to zero.

Illustration 2.2.3

we have three factories (A, B, C) supplying goods to three warehouses (X, Y, Z), with the following supply and demand:

	X	Y	Z	Supply
A	4	6	8	50
B	2	5	7	60
C	3	4	6	40
Demand	**30**	**70**	**20**	

Solve this Transportation Problem

Solution

Here, the total supply is 50 + 60 + 40 = 150, and the total demand is 30 + 70 + 20 = 120. Since supply exceeds demand, this is an unbalanced problem.

Step 1: Balancing the Problem

To balance the problem, we need to add a dummy destination to absorb the excess supply. The excess supply is calculated as:

150 (total supply) -- 120 (total demand)

We add a dummy column (let's call it **D**) with a demand of 30. The transportation cost to this dummy destination is set to zero.

The modified transportation table becomes:

	X	Y	Z	D (Dummy)	Supply
A	4	6	8	0	50
B	2	5	7	0	60
C	3	4	6	0	40
Demand	**30**	**70**	**20**	**30**	

Now, the total supply (150) equals the total demand (150), making the problem balanced.

Step 2: Solving the Balanced Problem

Now that the problem is balanced, we can solve it using any of the standard methods, such as the North-West Corner Rule, Least Cost Method, or Vogel's Approximation Method.

Here we solve it by using the Least Cost Method

1. Select the cell with the lowest cost: B-X (cost = 2).
2. Allocate 30 units to B-X (min of 60 supply at B and 30 demand at X).
3. Update the remaining supply at B to 30 and cross out column X (demand met).
4. Select the next lowest cost: B-Y (cost = 5).
5. Allocate 30 units to B-Y (min of 30 supply at B and 70 demand at Y).
6. Update the remaining supply at B to 0 and cross out row B (supply exhausted).
7. Select the next lowest cost: C-Y (cost = 4).
8. Allocate 40 units to C-Y (min of 40 supply at C and 40 remaining demand at Y).
9. Update the remaining supply at C to 0 and cross out row C (supply exhausted).
10. Allocate the remaining units to the dummy destination: A-D (cost = 0).
11. Allocate 30 units to A-D (remaining supply at A).
12. Update the supply at A to 0 and cross out row A (supply exhausted).

Final Allocation Table

	X	Y	Z	D (Dummy)
A	0	0	20	30
B	30	30	0	0
C	0	40	0	0

2.2.2.2 Demand Exceeds Supply

When demand is more than supply (total demand exceeds total supply)

If total demand exceeds total supply, a "dummy" source is added to cover the shortfall in supply. The transportation cost from this dummy source is set to zero.

Illustration 2.2.4

A company has three warehouses (A, B, and C) that supply goods to three retail outlets (X, Y, and Z). The supply capacities of the warehouses and the demands of the outlets are as follows:

	X	Y	Z	Supply
A	4	6	8	30
B	5	4	3	40
C	6	8	5	20
Demand	35	50	25	

Identify whether the transportation problem is balanced.

1. If unbalanced, balance the problem appropriately.
2. Find the initial basic feasible solution using the Least Cost Method (LCM).
3. Compute the total transportation cost for your solution.

Solution

- Total Supply = 90
- Total Demand = 110

→ Demand exceeds Supply → Unbalanced Problem

Make It Balanced

Add a dummy source (D) to compensate for the shortage of 20 units.

Assume the cost from the dummy source to all destinations is **0** (since no real shipment):

	X	Y	Z	Supply
A	4	6	8	30
B	5	4	3	40
C	6	8	5	20
Dummy D	0	0	0	20
Demand	35	50	25	

Least Cost Method (LCM) Allocation

Allocation is made in the lowest cost cell available, then adjust supply/demand and strike out fulfilled rows/columns.

1. Cell (D,Z) → Cost = 0 → Min

- Supply = 20, Demand = 25

→ Allocate 20

→ Supply of D exhausted, Demand at Z becomes 5

2. ◇ Cell (B,Z) → Cost = 3

- Supply = 40, Demand = 5

→ Allocate 5

→ Z fully satisfied, Demand at Z = 0, strike column Z

3. ◇ Cell (B,Y) → Cost = 4

- Supply = 35, Demand = 50
- Allocate 35

→ B exhausted, Demand at Y becomes 15

4. ◇ Cell (A,X) → Cost = 4

- Supply = 30, Demand = 35

→ Allocate 30

→ A exhausted, Demand at X = 5

5. ◇ Cell (C,Y) → Cost = 8

- Supply = 20, Demand = 15

→ Allocate 15

→ Demand at Y = 0, Supply at C = 5

6. ◇ Cell (C,X) → Cost = 6

- Supply = 5, Demand = 5

→ Allocate 5

→ Both supply and demand exhausted

Final Allocation Table

	X	Y	Z	Supply
A	30			30
B		35	5	40
C	5	15		20
D			20	20
Demand	35	50	25	

Step 4: Total Transportation Cost

Only real sources (A, B, C) contribute to cost. Dummy allocations (cost 0) are ignored.

$Z = (30\times4) + (35\times4) + (5\times3) + (5\times6) + (15\times8)$

$= 120+140+15+30+120 = 425$

2.2.3 Degeneracy in Transportation Problem

- Basic variables is less than the number of rows plus number of columns minus one

In the Transportation Problem, degeneracy occurs when the number of basic allocations in a feasible solution is less than (m + n − 1), where:

- m = number of rows (sources)
- n = number of columns (destinations)

A non-degenerate basic feasible solution must have exactly (m + n − 1) positive allocations.

Epsilon

To handle degeneracy, we introduce a very small amount close to zero into one or more unoccupied cells (if necessary) so that the total number of occupied cells becomes m + n − 1. This tiny value, represented by the Greek letter **ε** (epsilon), is treated as a valid allocation. Once introduced, **ε** remains in the solution until the degeneracy is resolved or the final optimal solution is reached—whichever happens first.

It may occur at

1. At the initial solution
2. At subsequent iterations of testing of the optimal solution

Illustration 2.2.5

A company has two warehouses (W1 and W2) and three retail outlets (R1, R2, R3). The supply and demand are given below:

Supply:

- W1: 20 units
- W2: 30 units

Demand:

- R1: 10 units
- R2: 10 units
- R3: 30 units

The cost per unit of transporting goods from each warehouse to each retail outlet is as follows:

	R1	R2	R3	Supply
W1	2	3	1	20
W2	5	4	8	30
Demand	10	10	30	

Use the Least Cost Method to find an initial basic feasible solution.

Check whether the initial solution is degenerate.

If degeneracy exists, explain how to resolve it.

Present the modified allocation table after resolving degeneracy.

Solution

Total supply = 50, and total demand = 50 → Balanced Transportation Problem

Step 1: Find Initial Basic Feasible Solution

Cell (W1, R3): Cost = 1 → allocate min(20, 30) = 20

Remaining: W1 = 0, R3 = 10

W1 is exhausted

Cell (W2, R3): Cost = 8 → allocate min(30, 10) = 10

Remaining: W2 = 20, R3 = 0

Cell (W2, R1): Cost = 5 → allocate min(20, 10) = 10

Remaining: W2 = 10, R1 = 0

Cell (W2, R2): Cost = 4 → allocate min(10, 10) = 10

Remaining: W2 = 0, R2 = 0

Allocation Table

R1	R2	R3	Supply	
W1	-	-	20	20
W2	10	10	10	30
Demand	10	10	30	

Step 2: Check for Degeneracy

To be non-degenerate, the number of allocations = m + n – 1 = 2 + 3 – 1 = 4

But here, we have only 3 allocations

So, this is a degenerate solution

Step 3: Resolve Degeneracy

Add a very small quantity ε (epsilon) to any unoccupied cell, ensuring no loop is formed.

Let's choose cell (W1, R1), and allocate ε units.

Modified Allocation Table (with ε)

	R1	R2	R3	Supply
W1	ε	-	20	20
W2	10	10	10	30
Demand	10	10	30	

Now, we have 4 occupied cells, and can proceed to test for optimality.

Summarised Overview

The transportation problem is a structured linear programming model designed to minimize the cost of distributing goods from multiple sources to multiple destinations, subject to supply and demand constraints. An initial basic feasible solution is first obtained using methods like the North-West Corner Rule, Least Cost Method, or Vogel's Approximation Method. This solution is then tested for optimality using techniques such as the Modified Distribution Method (MODI), which calculates opportunity costs for unoccupied cells to determine if cost improvements are possible. If any opportunity cost is negative, reallocations are made using a closed loop until all costs are non-negative, indicating optimality. In cases where the number of allocations is less than (rows + columns − 1), degeneracy occurs and is resolved by assigning a small value ε to unoccupied cells. When total supply and demand are unequal, dummy rows or columns with zero cost are added to balance the problem. These steps ensure an efficient, cost-effective transportation plan.

Self-Assessment Question

1. When is a dummy row or column introduced in transportation problems?
2. How do you detect and resolve degeneracy in a transportation problem?
3. Explain how ε (epsilon) is used to address degeneracy.
4. Explain treatment of unbalanced transporation
5. Briefly explain MODI Method.
6. Short note on optimality test process.

Assignments

1. A company has factories at F1, F2, and F3 that supply products to warehouses at W1, W2 and W3. The weekly capacities of the factories are 200, 160 and 90 units, respectively. The weekly warehouse requirements are 180, 120 and 150 units, respectively. The unit shipping costs (in rupees) are as follows:

	W1	W2	W3	Supply
F1	16	20	12	200
F2	14	8	18	160
F3	26	24	16	90
Demand	180	120	150	450

Determine the optimal distribution for this company in order to minimize its total shipping cost.

2. A manufacturer wants to ship 22 loads of his product as shown below. The matrix gives the kilometres from sources of supply to the destinations.

	D1	D2	D3	D4	D5	Supply
S1	5	8	6	6	3	8
S2	4	7	7	6	5	5
S3	8	4	6	6	4	9
Demand	4	4	5	4	8	

3. The shipping cost is Rs 10 per load per km. What shipping schedule should be used in order to minimize the total transportation cost?

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Suggested Reading

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Space for Learner Engagement for Objective Questions

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Unit 3

Assignment Problem

Learning Outcomes

After completing this unit, the learner will be able to:

- Comprehend the concept of assignment problem
- Solve assignment problem by using Hungarian method
- Handle maximisation case of assignment problem

Background

The Assignment Problem is a fundamental concept within Operations Research, falling under the umbrella of linear programming. It addresses a specific type of resource allocation challenge where the primary objective is to optimize the assignment of a set of resources (e.g., individuals, machines) to a set of tasks or jobs. This optimization typically aims to either minimize the total cost or time incurred, or to maximize the total profit generated from these assignments. The most widely recognized and efficient method for solving assignment problems is the Hungarian Method. Developed by Hungarian mathematician Dénes Kőnig and later improved by Harold W. Kuhn, this algorithm provides a systematic approach to finding the optimal assignment pattern. It works by transforming the original cost/profit matrix into an equivalent matrix where the optimal assignment can be identified by finding a unique set of zeros. For maximization problems, the Hungarian method is adapted by converting the profit matrix into an opportunity loss (cost) matrix.

Keywords

Hungarian Method, Assignment Problem, Optimisation

Discussion

- Assigning tasks to agents in an optimal way

The Assignment Problem is a specific kind of linear programming problem aimed at minimizing the cost or time required to complete a set of tasks by assigning them to a group of individuals. In real-life situations, we often encounter the challenge of assigning various workers to different tasks. Since individuals possess varying skill levels, their efficiency in performing the same job differs. These differences are usually measured in terms of cost, profit, or time required to complete a task.

Imagine you are the manager of a company with 4 workers and 4 jobs. Each worker has a different efficiency for each job, and the time (or cost) they take is different.

The question is: How do we assign each worker to exactly one job so that the total cost (or time) is minimum?

This real-life situation is what we call an Assignment Problem.

2.3.1 Assumptions of Assignment Problem

1. Equal Number of Jobs and Persons:

The number of jobs is equal to the number of persons (i.e., it is a square matrix). If not, dummy rows or columns are added to make it square.

2. One-to-One Assignment:

Each person is assigned to exactly one job, and each job is assigned to exactly one person.

3. Indivisibility:

Jobs are indivisible, meaning a job cannot be divided among multiple persons, and a person cannot be assigned to more than one job.

4. Known Cost/Time/Profit:

The cost, time, or profit of assigning each person to each job is known in advance and is fixed.

5. **Objective of Optimization:**

The objective is to either minimize the total cost or time, or maximize the total profit of the assignment.

6. **Independent Assignments:**

The cost or effectiveness of assigning one person to a job is independent of the assignments of other persons to other jobs.

7. **No Constraints on Jobs/Persons:**

It is assumed that every person is capable of doing every job, although with different efficiencies (unless certain assignments are prohibited, in which case very high or low values are used to reflect infeasibility)

2.3.2 Hungarian Method of Solving an Assignment Problem

Following are teh method of solving an assignment problem using Hungarian Method.

- Steps of Hungarian methods

Step 1: Check whether the given matrix is square. If not, make it square by adding a suitable number of dummy rows (or columns) with 0 cost/time elements.

Step 2: Locate the smallest cost element in each row of the cost matrix. Subtract the smallest element of each row from every element of that row.

Step 3: In the resulting cost matrix, locate the smallest element in each column and subtract the smallest element of each column from every element of that column.

Step 4: In the resulting matrix, search for an optimum assignment as follows:

i. Examine the rows successively until a row with exactly one zero is found. Draw a rectangle around this zero (assigned 0) and cross out all other zeroes in the corresponding column.

Proceed in this manner until all the rows have been examined. If there is more than one zero in any row, do not touch that row; pass on to the next row.

ii. Repeat step (i) above for the columns of the resulting cost matrix.

iii. If a row or column of the reduced matrix contains more than one zero, arbitrarily choose a row or column having the minimum number of zeroes.

Arbitrarily select any zero in the row or column so chosen. Draw a rectangle around it and cross out all the zeroes in the corresponding row and column.

Repeat steps (i), (ii), and (iii) until all the zeroes have either been assigned or crossed.

iv. If each row and each column of the resulting matrix has one and only one assigned 0, the optimum assignment is made in the cells corresponding to assigned 0. The optimum solution is attained. Otherwise, go to the next step.

Step 5: Draw the minimum number of horizontal and/or vertical lines through all the zeroes as follows:

i. Tick mark (✓) the rows in which assignment has not been made.

ii. Tick mark (✓) columns, which have zeroes in the marked rows.

iii. Tick mark (✓) rows (not already marked) which have assignments in marked columns. Then tick mark (✓) columns, which have zeroes in newly marked rows, if any. Continue this process.

iv. Draw straight lines through all unmarked rows and marked columns.

Step 6: Revise the cost matrix as follows:

i. Find the smallest element not covered by any of the lines.

ii. Subtract this from all the uncovered elements and add it to the elements at the intersection of the two lines.

iii. Other elements covered by the lines remain unchanged.

Step 7: Repeat the procedure until an optimum solution is attained.

Illustration 2.3.1

A computer centre has four expert programmers and needs to develop four application programmes. The head of the computer centre, estimates the computer time (in minutes) required by the respective experts to develop the application programmes as follows:

Programmer/ Programme	A	B	C	D
Programmer 1	120	100	80	90
Programmer 2	80	90	110	70
Programmer 3	110	140	120	100
Programmer 4	90	90	80	90

Find the assignment pattern that minimises the time required to develop the application programmes.

Solution

Subtract the minimum element of each row from every element of that row. Note that the minimum element in the first row is 80. So 80 is to be subtracted from every element of the first row, i.e., from 120, 100, 80 and 90, respectively. As a result, the elements of the first row of the resulting matrix would be 40, 20, 0, 10, respectively. Similarly, we obtain the elements of the other rows of the resulting matrix. Thus, the resulting matrix is:

Programmer/ Programme	A	B	C	D
Programmer 1	40	20	0	10
Programmer 2	10	20	40	0
Programmer 3	10	40	20	0
Programmer 4	10	10	0	10

Now subtract the minimum element of each column from every element of that column in the resulting matrix. The minimum element in the first column is 10. So 10 is to be subtracted from every element of the first column, i.e., from 40, 10, 10, and 10, respectively. As a result, the elements of the first column of the resulting matrix are 30, 0, 0, 0, respectively. Similarly, we obtain the elements of the other columns of the resulting matrix. Thus, the resulting matrix is:

Programmer/ Programme	A	B	C	D
Programmer 1	30	10	0	10
Programmer 2	0	10	40	0
Programmer 3	0	30	20	0
Programmer 4	0	0	0	10

Now, starting from first row onward, we draw a rectangle around the 0 in each row having a single zero and cross all other zeroes in the corresponding column. Here, in the very first row we find a single zero. So, we draw a rectangle around it and cross all other zeroes in the corresponding column. We get:

Programmer/ Programme	A	B	C	D
Programmer 1	30	10	0	10
Programmer 2	0	10	40	0
Programmer 3	0	30	20	0
Programmer 4	0	0	0	10

In the second, third and fourth row, there is no single zero. Hence, we move column-wise. In the second column, we have a single zero. Hence, we draw a rectangle around it and cross all other zeroes in the corresponding row. We get:

Programmer/ Programme	A	B	C	D
Programmer 1	30	10	0	10
Programmer 2	0	10	40	0
Programmer 3	0	30	20	0
Programmer 4	0	0	0	10

In the matrix above, there is no row or column, which has a single zero. Therefore, we first move row-wise to locate the row having more than one zero. The second row has two zeroes. So, we draw a rectangle arbitrarily around one of these zeroes and cross the other one. Let us draw a rectangle around the zero in the cell (2, A) and cross the zero in the cell (2, D). We cross out the other zeroes in the first column. Note that we could just as well have selected the zero in the cell (2, D), drawn a rectangle around it and crossed all other zeroes. This would have led to an alternative solution.

In this way, we are left with only one zero in every row and column around which a rectangle has been drawn. This means that we have assigned only one operation to one operator. Thus, we get the optimum solution as follows:

Programmer/ Programme	A	B	C	D
Programmer 1	30	10	0	10
Programmer 2	0	10	40	0
Programmer 3	0	30	20	0
Programmer 4	0	0	0	10

Note that the assignment of jobs should be made on the basis of the cells corresponding to the zeroes around which rectangles have been drawn. Therefore, the optimum solution for this problem is:

1 → C, 2 → A, 3 → D, 4 → B

This means that programmer 1 is assigned programme C, programmer 2 is assigned programme A, and so on.

The minimum time taken in developing the programmes is = 80 + 80 + 100 + 90 = 350 min.

Illustration 2.3.2

A company is producing a single product and selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in five more cities that do not have any agency of the company.

The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to the additional cities in such a way that the travelling distance is minimised. The distances (in km) between the surplus and deficit cities are given in the following distance matrix.

Deficit City

Surplus city	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	70	185
D	50	50	80	80	110
E	55	35	70	80	105

Determine the optimum assignment schedule

Solution

Subtracting the minimum element of each row from every element of that row, we have:

	I	II	III	IV	V
A	30	0	45	60	70
B	15	0	10	40	55
C	30	0	45	60	75
D	0	0	30	30	60
E	20	0	35	45	70

Subtracting the minimum element of each column from every element of that column, we have:

	I	II	III	IV	V
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

We now assign zeroes by drawing rectangles around them as explained in Example 1. Thus, we get:

	I	II	III	IV	V
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

Since the number of assignments is less than the number of rows (or columns), we proceed from Step 5 onwards of the Hungarian method as follows:

i. we tick mark (✓) the rows in which the assignment has not been made. These are the 3rd and 5th rows.

ii. we tick mark (✓) the columns which have zeroes in the marked rows. This is the 2nd column.

iii. we tick mark (✓) the rows which have assignments in marked columns. This is the 1 st row.

Again we tick mark (✓) the column(s) which have zeroes in the newly marked row. This is the 2nd column, which has already been marked. There is no other such column. So, we have:

	I	II	III	IV	V	
A	30	0	35	30	15	✓
B	15	0	0	10	0	
C	30	0	35	30	20	✓
D	0	0	20	0	5	
E	20	0	25	15	15	✓
		✓				

We draw straight lines through unmarked rows and marked columns as follows:

	I	II	III	IV	V	
A	30	0	35	30	15	✓
B	15	0	0	10	0	
C	30	0	35	30	20	✓
D	0	0	20	0	5	
E	20	0	25	15	15	✓
		✓				

We proceed as follows, as explained in step 6 of the Hungarian method:

i. We find the smallest element in the matrix not covered by any of the lines. It is 15 in this case.

ii. We subtract the number '15' from all the uncovered elements and add it to the elements at the intersection of the two lines.

iii. Other elements covered by the lines remain unchanged.

	I	II	III	IV	V
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	5
D	0	15	20	0	0
E	5	0	10	0	0

We repeat Steps 1 to 4 of the Hungarian method and obtain the following matrix:

	I	II	III	IV	V
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	5
D	0	15	20	0	0
E	5	0	10	0	0

Since each row and each column of this matrix has one and only one assigned 0, we obtain the optimum assignment schedule as follows:

A → V, B → III, C → II, D → I, E → IV

Thus, the minimum distance is

200 + 130 + 110 + 50 + 80 = 570 km

2.3.3 Maximization In Assignment Problem

- Maximised performance

There are problems where certain facilities have to be assigned to a number of jobs, so as to maximize the overall performance of the assignment. The Hungarian Method can also solve such assignment problem, as it is easy to obtain an equivalent minimization problem by converting every number in the matrix to an opportunity loss. The problem of maximization can be converted into minimization case by selecting the largest element among all the elements of the profit matrix and then subtracting it from all other elements in the matrix including itself.

Illustration 2.3.3

Five jobs are to be processed and 5 machines are available. Any machine can process any job until resulting profit in Rupees as follows.

Jobs \ Machines	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment pattern that maximises sales

Convert to Cost Matrix (Max Element = 41)

Subtract each element from 41:

Jobs \ Machines	**A**	**B**	**C**	**D**	**E**
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

Solve using Hungarian Method

We now apply the Hungarian Method (Row and Column Reduction, Assign Zeroes, etc.)

Row Reduction (Subtract row minimum from each row):

Jobs \ Machines	A	B	C	D	E
1 (min=1)	8	2	0	12	0
2 (min=1)	0	16	12	19	4
3 (min=0)	0	14	8	11	4
4 (min=0)	19	3	0	5	5
5 (min=1)	11	7	0	5	1

Column Reduction (Subtract column minimum from each column):

Column minimums:

- A: 0
- B: 2
- C: 0
- D: 5
- E: 0

Jobs \ Machines	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

Assignment (Crossing zeros and assigning)

Assign one '0' per row/column such that no row or column gets more than one assignment.

Let's try assigning (only one per row & column):

- Row 1: assign to column C (0)
- Row 2: assign to column A (0)
- Row 3: skip for now
- Row 4: assign to column D (0)
- Row 5: assign to column E (1), better to take C or D (we try C)

But now we go step-by-step and apply minimal covering lines and adjust matrix, but to save time, here's the optimal assignment (after full Hungarian method):

Final Assignment and Maximum Profit

Job	Assigned Machine	Profit
1	C	40
2	A	40
3	B	27
4	D	36
5	E	39

Total Maximum Profit = 40 + 40 + 27 + 36 + 39 = ₹182

Summarised Overview

The assignment problem involves assigning tasks to agents in a way that minimizes cost or maximizes profit, with each agent assigned to only one task. The Hungarian Method is a systematic and efficient approach used to solve this problem, particularly for square matrices. In minimization cases, it reduces the matrix through row and column operations to create zeros and identifies an optimal assignment. For maximization problems, the matrix is first converted to a minimization form by subtracting each element from the highest value in the matrix. The method then proceeds as usual, ensuring an optimal one-to-one assignment with maximum total benefit.

Self-Assessment Question

1. Define the Assignment Problem in your own words, highlighting its primary objective
2. List and briefly explain three core assumptions of the Assignment Problem.

k \ Job	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

3. Explain how a maximization assignment problem is converted into a minimization problem for solving with the Hungarian Method.
4. Explain steps in Hungarian method of solving assignment problem.

Assignments

1. 1 In the context of the Hungarian Method, what is the purpose of drawing lines to cover all zeros? What does the "smallest uncovered element" signify, and how is it used in the next step?
2. Find Solution of Assignment problem using Hungarian method (MIN case)
3. Find Solution of Assignment problem using Hungarian method (MAX case)

ork\Job	I	II	III	IV
A	42	35	28	21
B	30	25	20	15
C	30	25	20	15
D	24	20	16	12

Reference

1. Kuhn, H. W. (1955). The Hungarian method for the assignment problem. *Naval Research Logistics Quarterly*, *2*(1-2), 83–97.
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Unit 4

Balanced and Unbalanced Assignment

Learning Outcomes

After completing this unit, the learner will be able to:

- explain the concept of prohibited assignment
- differentiate between Transportation and Assignment problems

Background

The Assignment Problem, a specialized area within linear programming in Operations Research, deals with allocating a set of resources (like workers or machines) to an equal number of tasks or jobs. The core objective is to find an optimal one-to-one assignment that either minimizes total cost or time or maximizes total profit. A crucial aspect of this problem is its structure, particularly whether it is "balanced" or "unbalanced." A balanced assignment problem is characterized by an equal number of resources and tasks, forming a square matrix. This ideal scenario allows for direct application of solution methodologies like the Hungarian Method. However, real-world situations often present unbalanced assignment problems, where the number of resources and tasks do not match. To address this, a preparatory step is required where "dummy" rows or columns (representing imaginary resources or tasks with zero cost) are introduced. This artificial balancing transforms the unbalanced problem into a balanced one, enabling the use of standard algorithms. Furthermore, the concept of prohibited assignments, where certain resource-task pairings are impossible or undesirable, is handled by assigning an infinitely high cost to those specific cells, ensuring they are excluded from the optimal solution. Understanding these distinctions and the techniques to manage them is fundamental for accurate and practical application of assignment problem solutions.

Keywords

Balanced problem, Unbalanced problem, Prohibited assignment

Discussion

- No. of raw = no. of columns

A balanced transportation problem occurs when the number of rows equals the number of columns. For example, if there are 4 workers and 4 jobs, the problem is considered balanced.

In contrast, an unbalanced transportation problem arises when the number of rows and columns are not equal. For instance, if there are 4 workers and only 3 jobs, the problem is unbalanced. To handle this, a dummy job (an imaginary job) is added to balance the matrix before proceeding with the solution.

2.4.1 Balanced Assignment Problem

Illustration 2.4.1

A company has 4 employees (E1, E2, E3, E4) and 4 tasks (T1, T2, T3, T4). The cost (in ₹) of assigning each employee to a task is given below:

	T1	T2	T3	T4
E1	9	2	7	8
E2	6	4	3	7
E3	5	8	1	8
E4	7	6	9	4

The objective is to assign one task to each employee such that the total cost is minimized.

Solution

Row Reduction

Subtract the minimum element of each row from all elements in that row.

	T1	T2	T3	T4	Min
E1	7	0	5	6	2
E2	3	1	0	4	3
E3	4	7	0	7	1
E4	3	2	5	0	4

Column Reduction

Subtract the minimum element of each column from all elements in that column.

	T1	T2	T3	T4
E1	4	0	5	6
E2	0	1	0	4
E3	1	7	0	7
E4	0	2	5	0

Assignment (Cover zeros using minimum lines)

Let's try assigning:

- E1 → T2 (0)
- E2 → T1 (0)
- E3 → T3 (0)
- E4 → T4 (0)

Each employee gets a unique task. No line adjustment needed — optimal assignment found.

Final Assignment and Total Minimum Cost

Employee	Assigned Task	Cost
E1	**T2**	**2**
E2	**T1**	**6**
E3	**T3**	**1**
E4	**T4**	**4**

Total Minimum Cost = ₹2 + ₹6 + ₹1 + ₹4 = ₹13

2.4.2 Unbalanced Assignment Problem

- No. of raw ≠ no. of columns

An assignment problem is considered unbalanced when the number of rows does not match the number of columns. To convert it into a balanced problem, we add dummy row(s) if there are fewer rows than columns, or dummy column(s) if there are fewer columns than rows. The values in these dummy rows or columns are set to zero. Once the problem is balanced in this way, it can be solved using the Hungarian method as previously described.

Illustration 2.4.2

	J1	J2	J3	J4	J5
M1	6	2	5	2	6
M2	7	6	8	2	5
M3	6	8	7	8	7

M4	3	4	9	4	9
M5	7	4	6	5	8
M6	4	8	4	6	7

A company is faced with the problem of assigning six different machines to five different jobs. Estimated cost is given above.

Solution

Add Dummy Job (J6) with all costs = 0

	J1	J2	J3	J4	J5	J6
M1	6	2	5	2	6	0
M2	7	6	8	2	5	0
M3	6	8	7	8	7	0
M4	3	4	9	4	9	0
M5	7	4	6	5	8	0
M6	4	8	4	6	7	0

Row Reduction

Subtract the minimum value in each row from all elements of that row.

Row Minima:

- M1: 0
- M2: 0
- M3: 0
- M4: 0
- M5: 0
- M6: 0

All rows already contain 0s, so no change in this step.

Column Reduction

Subtract the minimum value in each column from all elements of that column.

Column Minima:

- J1: min = 3
- J2: min = 2
- J3: min = 4

- J4: min = 2
- J5: min = 5
- J6: min = 0

Now subtract each column's minimum:

	J1	J2	J3	J4	J5	J6
M1	3	0	1	0	1	0
M2	4	4	4	0	0	0
M3	3	6	3	6	2	0
M4	0	2	5	2	4	0
M5	4	2	2	3	3	0
M6	1	6	0	4	2	0

Cover all zeros with the minimum number of lines

We now attempt to cover all zeros using the minimum number of horizontal and vertical lines.

Covered zeros:

- Column J6 has 6 zeros → cover this column
- Row M1 has 2 uncovered zeros → cover M1
- Row M4 has one zero → already covered via J6
- Row M2 has one zero → already covered via J6
- Row M3 has one zero → already covered via J6
- Row M5 has one zero → already covered via J6
- Row M6 has one zero → already covered via J6

We can cover all zeros with 4 lines (1 column and 3 rows). Since this is less than 6, continue to adjust.

Modify Matrix

Find the smallest uncovered value = 1

Subtract 1 from all uncovered elements.

Add 1 to elements at the intersection of two lines.

Leave the rest unchanged.

Apply the adjustments. Updated matrix:

	J1	J2	J3	J4	J5	J6
M1	3	0	1	0	1	0
M2	3	3	3	0	0	1
M3	2	5	2	5	1	1
M4	0	2	5	2	4	0
M5	3	1	1	2	2	0
M6	0	5	0	3	1	0

Repeat Covering Zeros

Now check how many lines are needed to cover all zeros.

It can now be done with 6 lines, so proceed to final assignment.

Assignment

Based on the Hungarian method and the analysis of the reduced cost matrix, the following optimal assignments are determined:

M1	J2
M2	J4
M3	J6 (Dummy)
M4	J1
M5	J5
M6	J3

Total Cost Calculation

Using the original cost matrix, the total minimum cost is calculated as follows:

- M1 → J2 = 2
- M2 → J4 = 2
- M3 → J6 = 0 (dummy)
- M4 → J1 = 3
- M5 → J5 = 8
- M6 → J3 = 1

Total Minimum Cost = 2 + 2 + 0 + 3 + 8 + 1 = ₹16

2.4.3 Prohibited or Restricted problem

- One or mre restrictions

A Prohibited problem is the one in which there are one or more restrictions. E. g. say there are 4 contractors – C1, C2, C3 & C4. And there are 4 roads to be repaired – R1, R2, R3 & R4. But contractor C2 cannot or is not allowed to work on R3. This is a prohibited problem

Then we assign a very high or infinite value ∞ (represented by M) to C2-R3 and proceed with solution. Throughout the solution steps, M does not change. Since M is infinity, no assignment is possible in M.This is done to restrict the entry of this pair of resource-activity in the final solution.

Illustration 2.4.3

A company has taken the third floor of a multi-storied building for rent to locate one of its zonal offices. There are five main rooms in this floor to be assigned to five managers. Each room has its own advantages and disadvantages. Some have two cupboards, some are closer to the washrooms or to the canteens, some are of big sizes and are on different floors, etc. Each of the five managers were asked to rank their room preferences amongst the rooms 201, 302, 103, 304 and 205. Their preferences were recorded in a table as indicated below:

Managers

M1	M2	M3	M4	M5
302	302	103	302	201
103	304	201	205	302
304	205	304	304	104
	201	205	103	
		302		

Most of the managers did not include all five rooms in the list since they were not satisfied with some of them. Assuming that their preferences can be quantified by numbers, find out which manager should be assigned which room so that their total preference ranking is minimum.

Solution:

Let us give the ranks 1, 2, 3, 4 and 5 to the first, second, third, fourth and fifth preferences. We assign ∞ to the cells for which no preference is given. Thus, the given problem can be represented by the following assignment table:

	201	302	103	304	205
M1	∞	1	2	3	∞
M2	4	1	∞	2	3
M3	2	5	1	3	4
M4	∞	1	4	3	2
M5	1	2	∞	3	∞

Let us now solve this assignment problem using the Hungarian method. Following Steps 1 to 4, we get:

	201	302	103	304	205
M1	∞	0	1	1	∞
M2	3	0	∞	0	1
M3	1	4	0	1	2
M4	∞	0	3	1	0
M5	0	1	∞	1	∞

Now, since each row and each column has one and only one assigned zero, the optimum assignment, i.e., the assignment with maximum satisfaction is made and is given by:

M1 → 302, M2 → 304, M3 → 103, M4 → 205, M5 → 201.

Table 2.4.1 : Difference between Transportation and Assignment problems

Basis of Comparison	Transportation Problem	Assignment Problem
Definition	A problem of transporting goods from sources to destinations at minimum cost.	A special case of transportation problem where tasks are assigned to agents at minimum cost.
Objective	To determine the most cost-effective distribution of products from multiple sources to multiple destinations.	To assign tasks/jobs to agents/ resources so that the total cost or time is minimized.
Structure	Generally involves m sources and n destinations.	Typically involves n agents and n tasks (square matrix).
Matrix Size	Can be rectangular (m × n).	Always a square matrix (n × n); if not, it is balanced by dummy rows/columns.
Values in Matrix	Represents unit transportation cost.	Represents cost or time of assigning one agent to one task.
Total Supply = Total Demand	Required for balance. If not equal, a dummy source/ destination is added.	Always assumed equal; if not, dummy rows or columns are added to make it square.
Solution Method	Methods include: Northwest Corner, Least Cost, Vogel's Approximation, MODI method.	Solved using Hungarian Method or Linear Programming.

Applications	Supply chain, logistics, production-distribution networks.	Job assignment, task scheduling, resource allocation.
Number of Allocations in Solution	Up to m + n – 1 basic variables.	Exactly n allocations (one per row and column).
Complexity	More complex, especially with large supply/demand nodes.	Less complex; specialized for one-to-one matching problems.

Summarised Overview

This chapter comprehensively covers the concepts of balanced and unbalanced assignment problems, detailing their characteristics and solution methodologies. It begins by defining a balanced assignment problem as one where the number of assignees equals the number of tasks, leading to a square cost matrix. The chapter then demonstrates the step-by-step application of the Hungarian Method for solving such a balanced minimization problem, involving row reduction, column reduction, and iterative zero assignment until an optimal solution is found, as exemplified by assigning employees to tasks for minimum cost. Subsequently, it introduces unbalanced assignment problems, which occur when the number of rows and columns differ, and explains the crucial step of adding dummy rows or columns with zero costs to balance the matrix, making it solvable by the Hungarian Method, illustrated through a machine-to-job assignment example requiring matrix modification after initial attempts to cover zeros. Finally, the chapter addresses prohibited assignments, explaining how to handle them by assigning a very high (infinite) cost, and concludes by providing a clear tabular comparison highlighting the fundamental differences in definition, objective, structure, and solution methods between transportation and assignment problems.

Self-Assessment Question

1. Distinguish between a "balanced" and an "unbalanced" assignment problem
2. Briefly explain the Minimization vs. Maximization conversion in assignment problems. If a profit matrix has values ranging from 10 to 50, and the highest profit is 50, how would a cell with an original profit of 20 be transformed for a minimization problem?
3. In the context of "Prohibited or Restricted problems," why is a very high or infinite value (M or ∞) assigned to the restricted cells.
4. List out the difference between Transportastion probblems and Assignment problems.

Assignments

1. What are the differences between Transportation and Assignment problems
2. Find Solution of Assignment problem using Hungarian method (MAX case)

Work \ Job	A	B	C	D	E
A	8	2	x	5	4
B	10	9	2	8	4
C	5	4	9	6	x
D	3	6	2	8	7
E	5	6	10	4	3

3. In an unbalanced assignment problem, why are dummy rows or columns added, and what cost values are typically assigned to them?

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Space for Learner Engagement for Objective Questions

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03 BLOCK

Decision Theory

Block Content

Unit 1

Introduction

Learning Outcomes

After completing this unit, the learner will be able to:

- get an idea about the components of decision problem
- prepare payoff table
- prepare regret table

Background

Decision-making is a fundamental function of management and a critical aspect of organizational success. In the context of business and management, effective decision-making enables managers to choose the best course of action from among several alternatives to achieve predetermined goals. Whether in routine operations or strategic planning, decision-making is required at every level of an organization.

Keywords

Decision making, Courses of Action, Events, Payoff, Regret table

Discussion

Decision-making is essentially about selecting among alternatives. However, it is not always straightforward, especially within organizations where situations can be quite challenging. Both new and experienced managers need strong decision-making abilities. Being able to tackle complex problems and devise effective strategies is crucial—not just for managing teams efficiently, but also for advancing important organizational changes and achieving strategic goals.

- Selection from alternative

Decision-making may be defined as “the process of deciding to adopt a certain course of action from various alternatives, to achieve a set of pre- determined goals.”

According to George R. Terry, "Decision-making is the selection based on certain criteria from two or more alternatives”

Steps in Decision Making

▶ Defining of the Problem:

The decision-making process begins with either addressing an issue or working toward a specific objective. Thus, the first crucial step is to clearly identify the problem or define the goal. During this stage, the decision-maker must thoroughly outline the various possible alternatives to the problem in a well-organized manner.

▶ Data Collection:

- Gathering necessary information

Once the issue is identified, the next step involves gathering the necessary information to assess it properly. Reliable and precise data form the foundation of effective decision-making. Accurate information provides a comprehensive understanding of the situation and helps in evaluating options accurately. Data can come from within the organization or external sources such as reports, publications, and market trends. It is essential to ensure that the data is current, relevant, and dependable. If sufficient secondary data is unavailable, primary data should be collected for thorough analysis.

▶ Model Development:

To assess the available alternatives, decision-makers often use models. These models, built on collected data and different scenarios, help simulate real-life situations to choose the best option. By examining the relationship between various choices, decision-makers can better visualize possible outcomes. In many cases, using models proves to be a valuable tool in making informed decisions.

▶ Evaluation:

While models are initially assessed using assumed variables, this step involves testing them against real-world elements. The decision-maker must now examine the model using practical and realistic parameters to determine its effectiveness and practicality. This stage verifies whether the assumptions and outcomes of the model hold true in actual situations.

- **Making the Decision:**

- Choose appropriate solution

After thoroughly evaluating the models and reviewing the relevant data, the decision-maker is ready to choose the most appropriate solution. The soundness of the decision largely depends on the accuracy of the data and the credibility of the models developed. Without dependable information and adaptable models, reaching a conclusive decision may not be feasible.

- **Implementing Follow-up Measures:**

- Follow-up plans

An essential part of decision-making is predicting its consequences and preparing to manage them. The decision-maker should design follow-up plans to address potential reactions and outcomes—both short-term and long-term—of the chosen course of action. Since one decision often leads to new circumstances requiring further choices, decision-making is seen as an ongoing, dynamic process.

Components of a Decision Problem

1. Acts (Courses of Action)

The Alternative Courses of Action, or Strategies, are the acts that are available to decision-makers. The decision analysis involves a selection among two or more Alternative Courses of Action, and the problem is to choose the best of these alternatives in order to achieve the objective.

e.g., A_1, A_2, A_3.

A = {A_1: Advertise heavily, A_2: Moderate advertising, A_3: No advertising}

2. Events (States of Nature)

These represent the external environment, such as demand level or market conditions. The events identify the occurrences which are outside of the decision-maker's control and which determine the level of success for the given act. These events are often called states of nature or outcomes.

Example:

S = {S_1: High Demand, S_2: Medium Demand, S_3: Low Demand}

3. Payoff

Each combination of a course of action and a state of nature is associated with payoff, which means the net benefit to the decision-maker that accrues from a given combination of decision alternatives and events. They are also known as Conditional Profit Values or Conditional Economic Consequences.

Payoff Table

For a given problem, a payoff table lists the states of nature, i.e., outcomes or events, which are mutually exclusive as well as collectively exhaustive and a set of given courses of action. For each combination of the state of nature and the courses of action, the payoff is calculated.

It helps visualize all possible outcomes (payoffs) for different decisions given various states of nature (uncontrollable future events).

Example:

A street food vendor, "Samosa King," sells samosas. He has to decide how many samosas to prepare each morning. Unsold samosas at the end of the day have to be discarded (no scrap value). He earns a profit of ₹5 per samosa sold. Each samosa costs him ₹2 to make.

He has analyzed past sales data and believes there are three possible demand scenarios for the day, each with a certain probability:

- **Low Demand:** 100 samosas (Probability: 0.30)
- **Medium Demand:** 150 samosas (Probability: 0.50)
- **High Demand:** 200 samosas (Probability: 0.20)

Samosa King can choose to prepare one of three quantities: 100, 150, or 200 samosas.

Constructing the Payoff Table:

The payoff table will show the profit for each combination of "decision alternative" (samosas prepared) and "state of nature" (demand).

- Profit per sold samosa: ₹5 (Selling Price) - ₹2 (Cost) = ₹3
- Loss per unsold samosa: ₹2 (Cost) - ₹0 (Scrap Value) = ₹2

Let's calculate the payoff for each cell:

Decision Alternatives	Low Demand (100)	Medium Demand (150)	High Demand (200)
100 samosa	300	300	300
150 samosa	200	450	450
200 samosa	100	350	600

Explanation of calculations:

- **If 100 samosas prepared:**

Low Demand: All 100 are sold. Profit = 100×₹3=₹300.

Medium Demand: All 100 are sold (demand is 150, but only 100 available). Profit = 100×₹3=₹300.

High Demand: All 100 are sold (demand is 200, but only 100 available). Profit = 100×₹3=₹300.

- **If 150 samosas prepared:**

Low Demand: 100 sold, 50 unsold. Profit = (100×₹3)−(50×₹2)=₹300−₹100=₹200.

Medium Demand: All 150 are sold. Profit = 150×₹3=₹450.

High Demand: All 150 are sold (demand is 200, but only 150 available).

Profit = 150×₹3=₹450.

If 200 samosas prepared:

Low Demand: 100 sold, 100 unsold.

Profit = (100×₹3)−(100×₹2)=₹300−₹200=₹100.

Medium Demand: 150 sold, 50 unsold.

Profit = (150×₹3)−(50×₹2)=₹450−₹100=₹350.

High Demand: All 200 are sold. Profit = 200×₹3=₹600.

Regret Table (Opportunity Loss Table)

The Opportunity Loss has been defined as to be the difference between highest possible profit for a state of nature and the actual profit obtained for the particular action taken, i.e., an opportunity loss is the loss incurred due to failure of not adopting the best possible course of action or strategy. Opportunity loss are calculated separately for each state of nature that might occur. For a given state of nature, the Opportunity loss of possible course of action is the difference between the payoff value for that course of action and the payoff for the best possible course of action that could have been selected.

Example:

Let's use the same "Samosa King" example to illustrate how to construct a regret table.

Constructing the Regret Table:

To create the regret table, for each state of nature (each column), we identify the maximum payoff in that column. Then, for every cell in that column, we subtract its payoff from the maximum payoff in that column.

Step 1: Identify the maximum payoff for each state of nature.

- **Low Demand (100 samosas):** The maximum payoff in this column is ₹300 (from preparing 100 samosas).
- **Medium Demand (150 samosas):** The maximum payoff in this column is ₹450 (from preparing 150 samosas).
- **High Demand (200 samosas):** The maximum payoff in this column is ₹600 (from preparing 200 samosas).

Step 2: Calculate the Regret for each cell.

Regret = (Maximum Payoff for that State of Nature) - (Payoff of the specific decision in that State of Nature)

Regret Table calculations:

Decision	Low Demand (100) (Max Payoff = ₹300)	Medium Demand (150) (Max Payoff = ₹450)	High Demand(200) (Max Payoff = ₹600)
100 samosa	₹300 - ₹300 = ₹0	₹450 - ₹300 = ₹150	₹600 - ₹300 = ₹300
150 samosa	₹300 - ₹200 = ₹100	₹450 - ₹450 = ₹0	₹600 - ₹450 = ₹150
200 samosa	₹300 - ₹100 = ₹200	₹450 - ₹350 = ₹100	₹600 - ₹600 = ₹0

Regret Table

Decision	Low Demand (100)	Medium Demand (150)	High Demand (200)
100	0	150	300
150	100	0	150
200	200	100	0

Decision Making Situations

- Decision making under certainty
- Decision making under uncertainty- Maximin criterion, Maximax criterion, Minimax criterion, Hurwicz alpha criterion and Laplace criterion.
- Decision making under risk – EMV, EOL and EVPI
- Decision making under competition- Decision tree, game theory, etc.

Decision Making Under Certainty

- Taking decisions

In this type of environment, the decision-maker is fully aware of the outcomes associated with each available option. Such decision problems are based on the assumption that only one particular state of nature is relevant, and the decision-maker treats it as a given, assuming complete certainty about its occurrence. For instance, imagine someone has ₹20,000 to invest for a year. They can either deposit it in a savings account offering 6% interest or invest in a government treasury note yielding 10%. If both options are risk-free and guaranteed, it is certain that the treasury note would be the more profitable choice.

Summarised Overview

Decision-making is the structured process of selecting the best course of action from among multiple alternatives to achieve specific objectives. It begins with identifying a problem or goal, followed by collecting relevant data, developing models to evaluate alternatives, and analyzing these models using realistic assumptions. Based on this evaluation, a suitable decision is made, and follow-up actions are planned to manage its impact. In managerial contexts, decision-making is essential for solving problems, managing teams effectively, and implementing strategic changes. Tools like payoff and regret tables help visualize outcomes under different scenarios, especially when dealing with uncertainty or risk. Thus, decision-making is both a logical and dynamic process central to organizational success.

Self-Assessment Question

1. What are the steps involved in decision making ?
2. Define decision-making. How is it significant in managerial functions?
3. Why is decision-making considered a continuous process?
4. Describe the components of a decission problem.
5. What is a day of table?
6. Describe the regret table.

Assignments

1. Prepare a case study of a real or fictional business scenario and apply the decision-making process to solve a specific problem.
2. What is a payoff table? How does it assist in decision-making?
3. What is a Regret Table?
4. A toy manufacturer is considering a project of manufacturing a dancing doll with three different movement designs. The doll will be sold at an average of Rs 10. The first movement design using 'gears and levels' will provide the lowest tooling and set up cost of Rs 1,00,000 and Rs 5 per unit of variable cost. A second design with spring action will have a fixed cost of Rs 1,60,000 and variable cost of Rs 4 per unit. Yet another design with weights and pulleys will have a fixed cost of Rs 3,00,000 and variable cost Rs 3 per unit. The demand events that can occur for the doll and the probability of their occurrence is given below.

	Demand (units)	Probability
Light demand	25,000	0.10
Moderate demand	1,00,000	0.70
Heavy demand	1,50,000	0.20

(a)Construct a payoff table for the above project. (b) Which is the optimum design?

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Unit 2

Decision Making Under Risk

Learning Outcomes

After completing this unit, the learner will be able to:

- describe decision making under risk
- calculate Expected Monetary Value
- calculate Expected Value of Perfect Information (EVPI)

Background

In real-world decision-making, especially in business and management, uncertainty is a constant factor. However, in many cases, while the exact outcome is unknown, the probabilities of different outcomes can be estimated using historical data, expert judgment, or statistical methods. This forms the basis of decision-making under risk. Unlike decision-making under pure uncertainty, this approach assumes that the decision-maker knows all possible outcomes and their likelihoods. This knowledge allows the use of structured, quantitative tools to guide decisions.

Keywords

Risk, Decision making, Expected Monetary Value, Expected Opportunity Loss, Expected Value of Perfect Information

Discussion

- Decision making under uncertainity

Risk implies a degree of uncertainty and an inability to fully control the outcomes or consequences of such an action. Decision making under Risk refers to situations where the decision-maker knows all the possible outcomes of an action and the probability of each outcome. Imagine you are a farmer deciding which crop to plant. The outcome depends on the weather. If there's good rainfall, paddy gives high profit. If rainfall is low, maize might give better returns. You don't know the future for sure—but you do know the probabilities of each weather condition. This situation is called Decision Making under Risk—because you don't know exactly what will happen, but you know the likelihood of each outcome

Since the chances of each event happening are known, decisions can be made using logical methods like Expected Monetary Value (EMV), Expected Opportunity Loss (EOL), and Expected Value of Perfect Information (EVPI). These methods help choose the best option by comparing the risks and benefits of each alternative.

Expected Monetary Value (EMV)

Under Decision Making Under Risk, the decision maker is supposed to have believable evidential information, knowledge, experience, or judgment to enable him to assign probability values to the likelihood of occurrence of each state of nature. Probabilities could be assigned to future events by reference to similar previous experiences and information. Sometimes, past experience or past records often enable the decision maker to assign probability values to the likely possible occurrences of each state of nature. Knowing the probability distribution of the states of nature, the best decision is to select that course of action which has the largest expected payoff value.

- Optimise the expected pay off

The most widely used criterion for evaluating the alternative course of action is the expected monetary value or expected utility. The objective of decision making here is to optimize the expected payoff which may mean either maximization of expected profit or minimization of expected regret.

Example:

Acts	E_1 (.3)	E_2 (.4)	E_3 (.3)
A_1	2.5	2	-1
A_2	4	2.6	0
A_3	3	1.8	1

Formula:

EMV=P1 O1+P2 O2+P3 O3

EMV Calculation:

- A_1: 0.3×2.5 + 0.4×2 + 0.3×(-1) = 0.75 + 0.8 - 0.3 = 1.25
- A_2: 0.3×4 + 0.4×2.6 + 0.3×0 = 1.2 + 1.04 + 0 = 2.24
- A_3: 0.3×3 + 0.4×1.8 + 0.3×1 = 0.9 + 0.72 + 0.3 = 1.92

Best Choice: A_2 (highest EMV)

Expected Opportunity Loss

Expected Opportunity Loss (EOL) is a decision-making method used under conditions of risk. It measures the cost of not choosing the best possible alternative for each state of nature. Suppose you had two choices:

- Option A gives you ₹500 profit.
- Option B gives you ₹800 profit.

If you chose A, you didn't actually lose money, but you 'lost the chance' to make an extra ₹300.That ₹300 is called opportunity loss or regret.

In decision making under risk, we don't just look at profits—we also think about how much we might regret not choosing the best option once the outcome is known.

- If we always knew the future, we would choose the best option every time.
- But since we don't, we calculate the average regret (using probabilities).

This average regret is called the Expected Opportunity Loss (EOL).

EOL is calculated by creating a regret table, which shows the difference between the best payoff and the payoff from each action. The decision-maker then selects the option with the lowest EOL, as it represents the least expected regret.

Steps:

- Make a regret table (subtract best value in each column from others).
- Multiply each regret by the probability.
- Add these for each act.

Example: Regret Table & EOL Calculation

A company is deciding between 3 projects (**A_1, A_2, A_3**) under 3 possible market conditions (**S_1, S_2, S_3**). The profits (in ₹ thousands) are as follows:

Acts	**S_1**	**S_2**	**S_3**
A_1	30	50	40
A_2	40	30	20
A_3	50	60	10

Step 1: Create the Regret Table

To create the regret table, subtract each value in a column from the maximum value of that column.

Acts	S_1 (max = 50)	S_2 (max = 60)	S_3 (max = 40)
A_1	50 - 30 = 20	60 - 50 = 10	40 - 40 = 0
A_2	50 - 40 = 10	60 - 30 = 30	40 - 20 = 20
A_3	50 - 50 = 0	60 - 60 = 0	40 - 10 = 30

Regret table

Acts	S_1	S_2	S_3
A_1	20	10	0
A_2	10	30	20
A_3	0	0	30

Step 2: Assume Probabilities

Let the probabilities of S_1, S_2, S_3 be:

$P(S_1) = 0.3$, $P(S_2) = 0.4$, $P(S_3) = 0.3$

Step 3: Calculate EOL for each Act

- $EOL(A_1) = (20 \times 0.3) + (10 \times 0.4) + (0 \times 0.3) = 6 + 4 + 0 = 10$
- $EOL(A_2) = (10 \times 0.3) + (30 \times 0.4) + (20 \times 0.3) = 3 + 12 + 6 = 21$
- $EOL(A_3) = (0 \times 0.3) + (0 \times 0.4) + (30 \times 0.3) = 0 + 0 + 9 = 9$

Since **A_3 has the lowest EOL (9)**, the best decision is to choose **Act A_3**.

Expected Value of Perfect Information

EVPI is the maximum amount a decision maker would pay for perfect information about future events. EVPI is a decision-making tool used under risk. It tells us the maximum amount a decision-maker should be willing to pay to get perfect information about which state of nature will occur in the future. In other words, it measures the value of removing uncertainty completely.

Formula:

EVPI=Expected value with perfect information (EPPI)−Expected Monetary Value (EMV)

A higher EVPI means that getting perfect information could really improve the decision, while a lower EVPI means it's not worth spending much to reduce uncertainty.

Example:

A company is planning to launch a new product. The market demand could be High, Medium, or Low, with the following probabilities and expected profits (in ₹):

Alternatives	High Demand (P = 0.5)	Medium Demand (P = 0.3)	Low Demand (P = 0.2)
Aggressive Launch	₹1,20,000	₹60,000	₹-30,000
Moderate Launch	₹80,000	₹70,000	₹40,000
Conservative Launch	₹50,000	₹40,000	₹30,000

Step 1: Compute Expected Monetary Value (EMV) for each alternative

EMV (Aggressive Launch):

= (0.5 × ₹1,20,000) + (0.3 × ₹60,000) + (0.2 × ₹-30,000)

= ₹60,000 + ₹18,000 – ₹6,000 = ₹72,000

EMV (Moderate Launch):

= (0.5 × ₹80,000) + (0.3 × ₹70,000) + (0.2 × ₹40,000)

= ₹40,000 + ₹21,000 + ₹8,000 = ₹69,000

EMV (Conservative Launch):

= (0.5 × ₹50,000) + (0.3 × ₹40,000) + (0.2 × ₹30,000)

= ₹25,000 + ₹12,000 + ₹6,000 = ₹43,000

Step 2: Determine the Best EMV

The highest EMV is for Aggressive Launch = ₹72,000

Best Decision under Uncertainty (based on EMV): Aggressive Launch

Step 3: Compute Expected Payoff with Perfect Information (EPPI)

Now, assume perfect knowledge of demand. For each state of nature, choose the best alternative: (column wise or based on market demand)

- High Demand: Best payoff = ₹1,20,000 (Aggressive)
- Medium Demand: Best payoff = ₹70,000 (Moderate)
- Low Demand: Best payoff = ₹40,000 (Moderate)

Now calculate EPPI:

= (0.5 × ₹1,20,000) + (0.3 × ₹70,000) + (0.2 × ₹40,000)

= ₹60,000 + ₹21,000 + ₹8,000 = ₹89,000

Step 4: Calculate EVPI

EVPI=Expected Payoff with Perfect Information (EPPI)−Best EMV

EVPI=₹89,000–₹72,000=₹17,000

Interpretation:

The Expected Value of Perfect Information (EVPI) is ₹17,000.

This means that the company would be willing to pay up to ₹17,000 to obtain perfect market demand information before making a decision.

Summarised Overview

The topic of decision-making under risk includes key analytical tools such as Expected Monetary Value (EMV), Expected Opportunity Loss (EOL), and Expected Value of Perfect Information (EVPI). EMV helps identify the option with the highest average payoff based on probabilities. EOL provides an alternative approach by focusing on minimizing regret—the difference between the chosen option and the best possible outcome in hindsight. EVPI quantifies the value of having perfect information, helping assess whether it's worth investing in more accurate forecasts. Together, these tools help decision-makers make rational, risk-aware, and data-driven choices.

Self-Assessment Question

1. What is decision-making under risk, and how is it different from decision-making under uncertainty?
2. Define Expected Monetary Value (EMV). What is its significance in risk-based decisions?
3. What does Expected Opportunity Loss (EOL) measure in decision-making?
4. What is expected opportunity loss?

Assignments

1. Write short notes on the following:

   a) EMV

   b) EOL

   c) EVPI

   d) Regret Table

2. A manufacturing company is considering three different investment strategies:

- Strategy A
- Strategy B
- Strategy C

The profits (in ₹ thousands) under different market conditions are:

Strategy	Boom (P = 0.4)	Steady (P = 0.3)	Recession (P = 0.3)
A	100	80	-20
B	80	90	60
C	60	70	70

a) Compute the EMV for each strategy and recommend the best option.

b) Prepare a regret table and compute the EOL for each strategy.

c) Calculate EVPI and interpret its meaning.

3. Explain the step by step process of computation of expected monetary value for each alternative

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1. Taha, H. A. (2017). *Operations Research: An Introduction* (10th ed.). Pearson Education.
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Unit 3

Decision Making under Uncertainty

Learning Outcomes

After completing this unit, the learner will be able to:

- elucidate different decision criteria available under the condition of uncertainty
- get an awareness on maximin criteria
- get an ideal about Maximax, Mini-max regret, Hurwitz criterion and Laplace criterion
- construct Decision Tree

Background

In many real-world business and economic situations, decisions must be made without having complete knowledge about future outcomes. This is known as decision-making under uncertainty. In such scenarios, probabilities of future events (states of nature) are unknown due to lack of historical data, new market conditions, or unprecedented circumstances (e.g., launching a new product or entering a new market).

The challenge lies in choosing a strategy without any statistical support or forecast probability. Hence, various decision-making criteria are employed based on the attitude of the decision-maker (pessimistic, optimistic, balanced, or rational).

Keywords

Maximin Criterion, Maximax Criterion, Hurwicz Criterion, Laplace criterion, Decision tree

Discussion

In decision making under uncertainty, the decision maker has no knowledge regarding any of the states of nature outcomes. Under this condition, the probabilities associated with occurrence of different states of nature are not known, i.e., there is no historical data available or no relative frequency which could indicate the probability of the occurrence of a particular state of nature. In other words, the decision maker has no way of calculating the expected payoff for his course of action or strategies. Such situations arise when a new product is introduced in the market or a new plan is set up.

In business situations, there are many problems of this nature, and here the choice of a decision or strategy is very largely depended on the policy of an organization. The number of different decision criteria available under the condition of uncertainty are:

1. Maximin Criterion (Pessimistic Approach)

Imagine you are deciding where to invest your savings.

- Option A could give you very high returns, but if things go wrong, you might lose a lot.
- Option B doesn't give very high returns, but it guarantees at least a decent profit.

If you are the type of person who always thinks: 'What's the worst that can happen?' and then chooses the option where the worst outcome is the best among all, then you are using the Maximin Criterion

The maximin criterion is based on a pessimistic rule assuming that the worst situation is likely to occur in the future and we could like to maximize the profits. It focuses on the worst-case scenario. Suitable for risk-averse decision-makers.

- Identify the minimum payoff in each decision alternative (row).
- Choose the maximum among those minimum values.

Example:

Alternatives	Outcome 1	Outcome 2	Outcome 3	Min payoff
A	20	30	10	10
B	25	5	15	5
C	10	15	20	10

Maximin=maximum of minimum

So identify raw minimum = 10,5 and 10

And then find maximum from these minimum

→ Maximin = Max (10, 5, 10) = 10 → Choose A or C

2. Maximax Criterion (Optimistic Approach)

The Maximax criterion is based on an optimistic rule assuming that the best situation will occur in the future. It chooses the decision with the best of the best payoffs. It is used by the decision-maker who is very optimistic and willing to take risks.

- Identify the maximum payoff in each row.
- Choose the maximum among them.

Example:

Maximax = maximum of maximum

From the above problem identify raw maximum

Alternatives	Max Payoff
A	30
B	25
C	20

→ Maximax = Max (30, 25, 20) = 30 → Choose A

3. Minimax Regret Criterion (Regret Theory)

This technique calculates the opportunity loss or regrets for not taking the best decision under each state of nature.

Regret = Opportunity loss = Best outcome in a state − Actual outcome.

- Create a regret table by subtracting each value from the best outcome in its column.
- Identify the maximum regret for each alternative.
- Choose the alternative with the minimum of these regrets.

Example:

Alternatives	O1	O2	O3
A	20	30	10
B	25	5	15
C	10	15	20

Step 1: Find column max:

O1 = 25, O2 = 30, O3 = 20

Step 2: Regret table:

Alternatives	O1 (25-)	O2 (30-)	O3 (20-)	Max Regret
A	5	0	10	10
B	0	25	5	25
C	15	15	0	15

Minimax= minimum of maximum

Identify raw maximum = 10, 25 and 15

→ Minimax regret = Min (10, 25, 15) = 10 → Choose A

4. Hurwicz Criterion (Weighted Average Approach)

The Hurwicz Criterion, also referred to as the criterion of realism, is a decision-making method that blends optimistic and pessimistic viewpoints. It assigns weights to the best and worst possible outcomes of each alternative using a realism coefficient (alpha). The option with the highest weighted average of these outcomes is then selected.

Uses a coefficient of realism (α), where:

α ($0 \leq \alpha \leq 1$) reflects the decision-maker's optimism.

$\alpha = 1$ → fully optimistic

$\alpha = 0$ → fully pessimistic

Formula:

Hurwicz Value = $\alpha \times$(Best Payoff)+$(1-\alpha)\times$(Worst Payoff)

Decision Rule:

- Calculate the Hurwicz value for each alternative.
- Choose the alternative with the highest Hurwicz value.

Example:

Alternative	Max Payoff	Min Payoff	α = 0.6	Hurwicz Value
A	30	10	0.6	0.6×30 + 0.4×10 = 22
B	25	5	0.6	0.6×25 + 0.4×5 = 17
C	20	10	0.6	.06×20+ 0.4×10=16

→ Choose Alternative A, which have maximum value

5. Criterion of Rationality (Laplace Criterion)

The criterion of rationality assumes that, in the absence of known probabilities, all possible outcomes are equally likely. This approach is based on the principle of insufficient reason. The decision-maker calculates the average payoff for each alternative and chooses the one with the highest average value, considering it the most logical or rational choice when no further information is available.

Example – Criterion of Rationality (Laplace Criterion)

A business must choose one of three marketing strategies (A, B, or C), but has no idea which market condition (Good, Average, Poor) will occur. The payoffs (in ₹ lakhs) under each condition are:

Strategy	Good	Average	Poor
A	60	40	20
B	80	30	10
C	50	50	50

Since probabilities are unknown, we assume equal probability for all states:

P(Good) = P(Average) = P(Poor) = 1/3 (here 3 states of nature)

The total probability = 1/3+1/3+1/3=1

Calculate Average (Expected) Payoff for Each Strategy

A: (60 + 40 + 20) 1/3 = 40

B: (80 + 30 + 10) 1/3 = 40

C: (50 + 50 + 50) 1/3 = 50

Decision:

Since Strategy **C** has the **highest average payoff (₹50 lakhs)**, it is selected under the **Criterion of Rationality**.

Decision Tree

Any problem that can be represented in a decision table can also be visually illustrated using a decision tree. A decision tree is a diagrammatic approach that outlines decisions, chance events, probabilities, and payoffs.

Decision tree analysis involves creating a visual diagram that outlines the sequence of decisions to be made, the possible outcomes of those decisions, and the associated payoffs. It provides a clear, step-by-step representation of the decision-making process. A decision tree is made up of nodes, branches, probabilities, and payoffs.

There are two main types of nodes:

- **Decision node:** Depicted as a square, this node marks a point where the decision-maker must choose between different available actions. The various options are shown as branches extending from the decision node.
- **Chance node:** Represented by a circle, this node follows each decision and indicates the point where the outcome, which depends on uncertainty, will be revealed. These outcomes are governed by probability and lead to different possible consequences.

Branches that reflect choices or natural states arise from and link different nodes. Two categories of branches exist:

- **Decision branch**: This branch denotes a path of action and extends from a decision node.
- **Chance branch**: It is the branch leading away from a chance node and represents the state of nature of a set of chance events. The assumed probabilities of the states of nature are written alongside their respective chance branch
- **Terminal branch**: Any branch that makes the end of the decision tree (not followed by either a decision or chance node), is called a terminal branch. A terminal branch can represent either a course of action. The terminal points of a decision tree are supposed to be mutually exclusive points so that exactly one course of action will be chosen.

Example

An executive has to make a decision. Has been made events may lead such that any of the four research Mein Oscar the results are R1 R2 R3 and R4. Probabilities of occurrence of these results are as follows. R1= 0.5, R2 = 0.2, R3 = 0.2 and R4 = 0.1

Decision	R1	R2	R3	R4
D1	14	10	9	8
D2	10	10	10	10
D3	11	10	8	7
D4	13	11	7	5

Show this decision situation in the form of a decision tree and indicate the most prefer decision and corresponding expected value

Solution

EMV Calculations:

EMV(D1) = 14×0.5 + 10×0.2 + 9×0.2 + 8×0.1 = 11.6

EMV(D2) = 10×0.5 + 10×0.2 + 10×0.2 + 10×0.1 = 10.0

EMV(D3) = 11×0.5 + 10×0.2 + 8×0.2 + 7×0.1 = 9.8

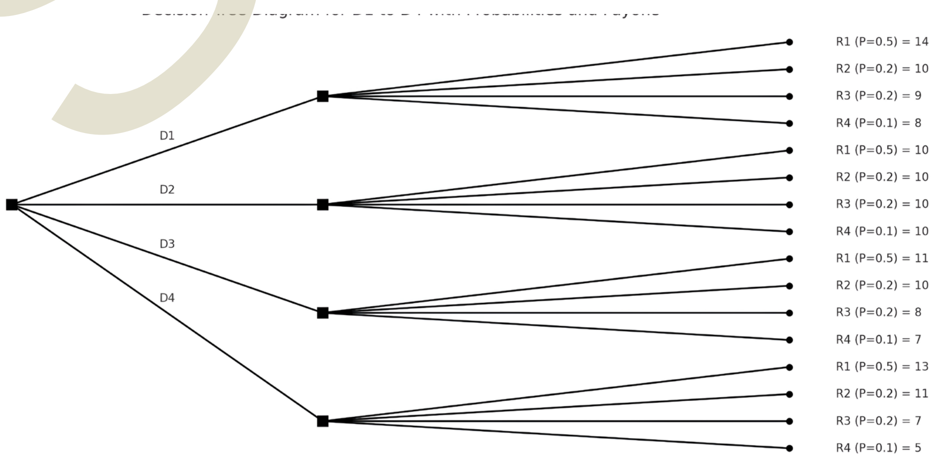
EMV(D4) = 13×0.5 + 11×0.2 + 7×0.2 + 5×0.1 = 10.6

Best Decision:

Since D1 has the highest EMV (11.6), the most preferred decision is D1.

Decision Tree Diagram

The following diagram shows the decision tree for alternatives D1 to D4 with associated outcomes and their probabilities and payoffs:



Decision Tree Diagram representing the four decision alternatives (D1 to D4) and their outcomes based on the different states of nature (R1 to R4), along with their associated probabilities and payoffs.

- Square nodes represent decision points.
- Circle nodes represent chance events.
- The values next to chance nodes show the state of nature, its probability, and the payoff.

Summarised Overview

Decision making under uncertainty arises when the probabilities of future outcomes are unknown, often due to lack of data or new, unpredictable scenarios. In such cases, decision-makers rely on specific criteria to guide choices: the Maximin criterion focuses on the best of the worst outcomes (pessimistic), while Maximax considers the best possible outcomes (optimistic). The Minimax Regret approach minimizes potential regret from not choosing the best option. The Hurwicz criterion blends optimism and pessimism using a weighted factor (α), and the Laplace criterion assumes all outcomes are equally likely, selecting the option with the highest average payoff. A decision tree can visually map these options, outcomes, probabilities, and payoffs to support better judgment.

Self-Assessment Question

1. What is the key difference between decision-making under risk and under uncertainty?
2. Define the Hurwicz criterion. What role does the coefficient of realism (α) play?
3. How does a decision tree help in solving un Differentiate between decision nodes and chance nodes in a decision tree.
4. What are the main component of decision tree analysis?

Assignments

1. Application of Decision Criteria

Given the following payoff table, apply all five criteria (Maximin, Maximax, Minimax Regret, Hurwicz with α = 0.7, Laplace) and recommend the best alternative:

Alternatives	Outcome 1	Outcome 2	Outcome 3
A	25	15	10
B	20	30	5
C	15	25	20

2. A firm is considering three strategies (S1, S2, S3). Future demand can be low, medium, or high with respective probabilities of 0.2, 0.5, and 0.3. The payoffs (in ₹ lakhs) are given below.

Draw a decision tree and compute the Expected Monetary Value (EMV) for each strategy. Recommend the best strategy.

Strategy	Low Demand	Medium Demand	High Demand
S1	20	40	50
S2	10	50	60
S3	30	30	30

3. Describe the advantages and limitations of Decision tree analysis.

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1. Kapoor, V. K. (2000). *Operations research*. New Delhi: Sultan Chand & Sons.
2. Hillier, F. S., & Lieberman, G. J. (2021). *Introduction to operations research* (11th ed.). New York, NY: McGraw-Hill Education.
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04 BLOCK

Network Analysis

Block Content

Unit 1

Network Analysis

Learning Outcomes

After completing this unit, the learner will be able to:

- Understand the concept of Network Analysis
- Describe a network diagram
- Elucidate stages of Project Management
- Identify the concept of critical path

Background

Imagine you are planning a wedding, a house construction, or even a college event. There are many activities to be done, some can go together, but some must wait until others are finished. For example, you cannot paint a house before the walls are built. Network Analysis is simply a method to plan and manage such projects. It shows the order of activities, how they are connected, and how much time they may take. Network analysis is a project planning and scheduling technique commonly used in large-scale activities such as construction, maintenance, fabrication, procurement, computer implementation, and research and development. It is widely applied in operations research to model and solve various problems. This unit discusses about the concept of Network Analysis in details.

Keywords

Network analysis, Activity, Dummy activity, Critical path

Discussion

4.1.1 Meaning

- Project planning and scheduling technique

Network analysis is a project planning and scheduling technique commonly used in large-scale activities such as construction, maintenance, fabrication, procurement, computer implementation, and research and development. It is widely applied in operations research to model and solve various problems. This method involves outlining a project by identifying and analyzing all the related tasks or activities. Complex projects are divided into smaller, manageable activities, which are then arranged in a logical sequence based on their dependencies. This helps in efficiently organizing and managing the overall workflow of the project. Let us consider an example of

Organizing a College Annual Day

Activities involved:

A. Booking Auditorium
B. Arranging Sound System
C. Arranging Seating (chairs, decorations)
D. Inviting Chief Guest
E. Preparing Stage program
F. Final Rehearsal
G. Conducting the event

So in this example:

First, Booking auditorium (A) must be done before anything else.

Once the auditorium is booked → you can Arranging sound system (B) and Arranging seating (C)

Inviting Chief Guest (D) can also be done independently after booking the auditorium.

Preparing Stage Program (E) depends on having the chief guest confirmed.

When sound, seating, and stage program are ready → you can hold the Final Rehearsal (F).

Finally, the Event (G) is conducted after the rehearsal.

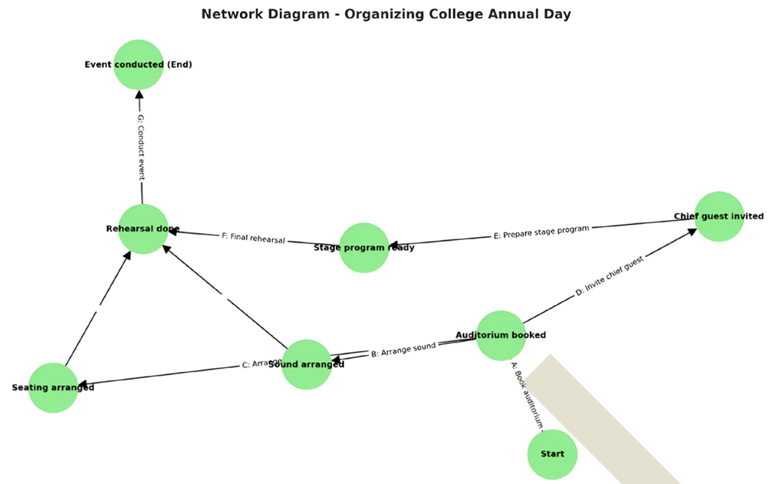


Figure 4.1.1

This network diagram shows the sequence of activities required to organize a College Annual Day. The project starts with booking the auditorium, after which parallel activities such as arranging sound, seating, and inviting the chief guest can take place. Once the chief guest is invited, the stage program is prepared. After sound, seating, and stage program are ready, the final rehearsal is conducted, leading to the successful organization of the event

4.1.2 Objectives of Network Analysis

- Technique for breaking large projects into smaller manageable activities

Network analysis is a valuable technique for breaking down large projects into smaller, manageable activities, which can be effectively analyzed using tools like PERT. The key goals of network analysis include:

1. **Aids in Planning** : It serves as a crucial tool for organizing, scheduling, and monitoring various project activities, helping in overall project planning.
2. **Understanding Activity Relationships**: Network analysis identifies how different tasks are linked and depend on one another, allowing for a better grasp of technical and functional connections across project elements.
3. **Managing Costs:** It enables cost estimation related to project delays and resource usage. By analyzing activity speeds and resource needs, total costs can be minimized.
4. **Reducing Maintenance Time:** When maintenance or production overhead is high, network analysis supports decisions to reduce downtime efficiently, even if resources are costly.

5. **Shortening Project Duration:** It helps reschedule and reallocate resources in a way that reduces the total completion time of a project.
6. **Managing Idle Resources**: Network analysis ensures optimal utilization of labor and materials by minimizing fluctuation and waste. For instance, avoiding the unnecessary hiring of excess workers for a limited timeframe reduces idle time and cost.
7. **Preventing Delays and Interruptions:** It promotes a disciplined and structured scheduling process, unlike traditional methods. This helps in reducing production delays and controlling large-scale projects effectively.

4.1.3 Basic Concepts of Network Analysis

1. Events (Nodes) ○

- Specific point in time that marks either the beginning or end of one or more activities

An event (also called a node) is a specific point in time that marks either the beginning or end of one or more activities. Events don't consume any time or resources. They are represented by circles in a network diagram. Think of an event as a milestone as it signifies that something has been completed, but it's not the work itself.

Example: In a project to build a house, 'Foundation complete' would be an event. It's a specific moment in time when the task of laying the foundation is finished.

2. Activities (Arrows) →

- Task that consumes time and/or resources

An activity is a task that consumes time and/or resources (like labour or materials). It's the actual work being done. Activities are shown as arrows connecting two events: a starting event and an ending event.

Types of Activities

- Different types of activities

a. **Predecessor Activity**: This is a task that must be completed before another activity can begin. It's the prerequisite.

Example: Before you can paint the walls of a room, you must first apply primer. Priming is the predecessor activity to painting.

b. **Successor Activity:** This is a task that can only start after a predecessor activity is completed. It's the follow-up task.

Example: After completing the foundation of a house (predecessor), you can begin framing the walls. Framing is the successor activity.

c. **Parallel (Concurrent) Activities**: These are tasks that can be done at the same time because they don't depend on each other.

Example: During a kitchen remodel, the electrician can install new wiring while the plumber installs new pipes. Electrical work and plumbing are parallel activities.

3. Critical Activity ⚠

A critical activity is a task on the critical path, which is the longest sequence of activities in a project network. If a critical activity is delayed, the entire project will be delayed. A key characteristic is that a critical activity has zero float (or slack), meaning there's no extra time available for it.

Example: Imagine you're driving from point A to point B. The critical path is the fastest route, and any delay on this route will make you late. The critical activities are the stretches of road on that fastest route.

4. Dummy Activity (Dashed Arrows) ━ ➡

A dummy activity is a special kind of activity that doesn't consume any time or resources. It is used solely to show a logical relationship or dependency between activities when a simple arrow isn't enough. It's represented by a dashed arrow.

Why are Dummy Activities needed?

- **To Maintain Correct Precedence**: Sometimes, a project's logic requires one activity to depend on the completion of multiple other activities, but not all of them. A dummy activity can be used to show this specific, one-to-one relationship.
- **To Differentiate Between Activities**: If two or more activities share the same start and end events, this creates an ambiguous diagram. A dummy activity is used to give each activity a unique set of start and end events.

Example:

Consider two tasks:

Activity A: Laying the foundation.

Activity B: Installing plumbing.

Activity C: Framing the walls.

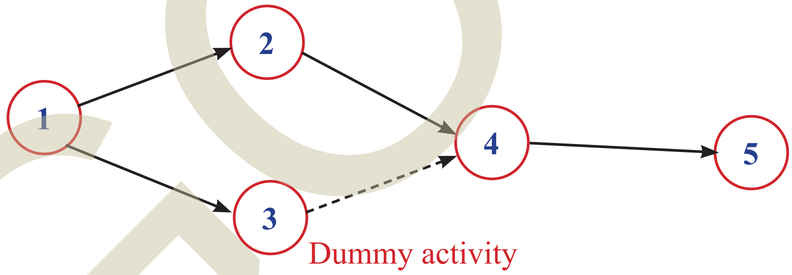
Let's say Activity C (Framing) can only start after both Activity A (Foundation) and Activity B (Plumbing) are complete.

In a network diagram, we would draw an arrow for Activity A and an arrow for Activity B, both ending at the same node. Then, an arrow for Activity C would start from that node. This works.

However, what if another activity, say Activity D (Electrical wiring), depends only on Activity A (Foundation), but not on Activity B?

If we draw an arrow for Activity D starting from the same node where A and B end, the diagram would incorrectly imply that D also depends on the completion of both A and B.

To fix this, we use a dummy activity. We draw a dummy activity (dashed arrow) from the end of Activity A to the end of Activity B's node. This way, Activity D can start from the end of Activity A's node, correctly showing its dependency, while Activity C still correctly depends on both A and B.



Dummy End Activity

- Represents the end of the project when no real successor activity follows.
- Used to define the final completion point of all activities.

5. Network Diagram:

A graphical representation of the project that shows the sequence and dependencies between different activities using nodes and arrows.

6. Path:

A path is a sequence of activities connecting the start node to the end node. Each path has a total duration based on the time required to complete its activities.

Critical Path:

The longest path through the network with the least slack or float, determining the minimum time required to complete the entire project. Delays on this path will delay the whole project.

7. Float

The amount of flexible time that a non-critical activity can be delayed without affecting the project's overall duration.

- Total Float: Delay allowed without affecting the project deadline. It is the difference between earliest start time and latest start time or earliest finish time and latest finish time.

Total float = LS – ES Or LF-EF

- Free Float: Delay allowed without affecting the start of subsequent activities.

Free float = Earliest start time of successor activity – Earliest Finish time of present activity

- Independent float

It is that portion of the total float within which an activity can be delayed for start without affecting float of the preceding activity.

Independent float= ES of next activity – LF of previous activity - Duration

8. Slack

The term slack is associated with events. It is the difference between the latest occurrence time and its earliest occurrence time.

4.1.4 Rules to frame a network diagram

1. The network diagram arrows represent activities and circles, the events. The length of the arrow is of no significance.
2. Each activity must start and end in a node. The tail of an activity represents the start and the end, the completion of work.
3. The event number 1 denotes the start of a project and is called an initial node or event. All activities emerging from event 1 should not be preceded by any other activity. Events carrying the highest number denotes the completion events.

4. Events should be numbered in an ascending order

5. only one activity can span across a pair of events. An event number should not be repeated or duplicated. Two activities should not be identified by the same completion events. Activities must be represented either by their symbol or by the corresponding order of starting and completion events.

6. No dangling is allowed unless specially decided in the questions.

7. The logical sequence between activities must follow the following rules.

An event can't occur until all the incoming activities into it have been completed. Though a dummy activity does not consume either resources or time, even then it has to follow the rules stated above

4.1.5 Fulkerson's Rule for Numbering of Events in Network Analysis

- Systematic method to assign unique and logical numbers to each event

Fulkerson's Rule is a systematic method used in network diagrams (PERT/CPM) to assign unique and logical numbers to each event (node) in a way that maintains direction and dependency of activities. It ensures no looping or confusion in the sequencing of tasks.

Rules of Fulkerson's Method:

1. **Start with the initial event (node):**

Assign it the number 1 (or the lowest number).

This node should have no predecessor.

2. **Proceed in the logical order of activities:**

An event is numbered only after all its preceding activities are numbered.

3. **Numbers increase along the direction of flow:**

Always assign higher numbers to successor events.

No two events should have the same number.

4. **No backward numbering:**

The event numbers should strictly progress forward; there should be no loop or cycle.

5. Dummy activities may be used

When multiple activities begin or end at the same node and unique identification is needed, use dummy activities to maintain correct precedence.

Example

Suppose a project has the following activities:

Activity	From Event	To Event
A	Start	…
B	Start	…
C	After A	…
D	After A	…
E	After B	…
F	After C & D	…
G	After E & F	End

Step 1: Draw a rough network

- Start node → A → C/D → F
- Start node → B → E → G
- F + E → G → End

Step 2: Numbering with Fulkerson's Rule

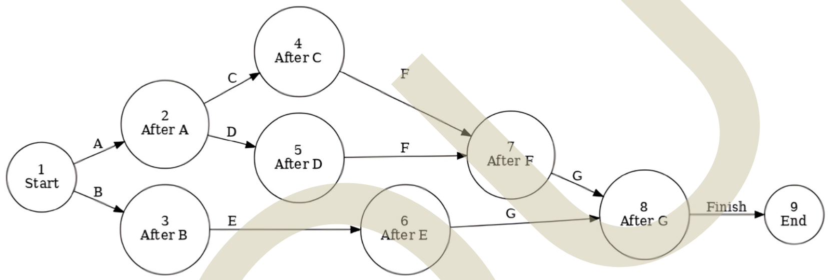
1. Start event = 1.
2. From event 1: Activities A and B go out → create events 2 and 3.
3. Activity C and D depend on A → so from event 2, create events 4 and 5.
4. Activity E depends on B → from event 3, create event 6.
5. Activity F depends on both C and D → merge events 4 and 5 into event 7.
6. Activity G depends on E (event 6) and F (event 7) → merge into event 8.
7. End event gets the highest number = 9.

Final Numbering

- Event 1: Start
- Event 2: After A
- Event 3: After B

- Event 4: After C
- Event 5: After D
- Event 6: After E
- Event 7: After F
- Event 8: After G
- Event 9: End

This ensures that all activities flow from lower-numbered to higher-numbered events, satisfying Fulkerson's Rule.



4.1.6 Stages of Project Management

Network analysis is a vital tool in project management, and it is important for every project manager to apply this technique to guide projects effectively toward successful completion.

Project Management consists of three stages

1. Project Planning Stage

In order to visualize the sequencing or precedence requirements of the activities in a project, it is helpful to draw a network diagram. For this, following tips are adopted. Identify various activities to be performed in the project.

2. Determine requirements of resources such as men, material, machines, money, etc., for carrying out activities listed above.
3. Assign responsibility for each work package.
4. Allocate resources to work packages. Estimate cost and time at various levels of project completion.
5. Develop work performance criteria. Establish control channels for project personnel.

2. Scheduling Stage

- Determining when each activity is to be performed

Once all work packages have been identified and given unique names or identities, scheduling of the project i.e., when each of the activities required to be performed is taken up. The various steps involved during this phase are listed below.

1. Identify all people who will be responsible for each task.
2. Estimate the expected durations of each activity, taking into consideration the resources required for their execution in most economic manner.
3. Specify the interrelationship among various activities.
4. Develop a network diagram showing the sequential interrelationship between various activities.
5. Based on time estimates, calculate the total project duration.
6. Identify path, calculate floats, carry out resources, levelling exercise for critical or scarce resources, taking into account resource constraints if any.

3. Project Control Stage

- Evaluating actual progress or status against the plan

Project control refers to evaluating actual progress or status against the plan. If significant differences are observed, then the scheduling and resources allocation decisions are changed to update and revise the uncompleted part of the project. In other words, remedial or reallocation of resources measures are adopted in such cases.

Illustration 4.1.1

A project has the following activities labelled B,C………,Q. The precedence constraints are:

B < E

C < G

G < L

E < G

E < H

G < H

H < J

L < M

H < M

M < N

H < I

J < P

P < Q

X < Y ⇒ X must finish before Y starts

Draw the activity-on-node network diagram that represents these precedence constraints.

Solution

Step-by-Step Precedence Construction:

1. **Start node (1):**

   Activities B (1→2) and C (1→3)

2. **From node (2):**

   E (2→4)

   F (2→11) (not constrained but shown in the diagram)

3. **From node (3):**

   G (3→4)

4. **From node (4):**

   H (4→5)

5. **From node (5):**

   J (5→8)

   L (5→6)

6. **From node (6):**

   M (6→7)

7. **From node (7):**

   N (7→9)

8. Also from node (5):

I (5→10)

9. From node (8):

P (8→10)

10. From node (9) and (10):

Q (9→11 and 10→11 merged)

Network Diagram

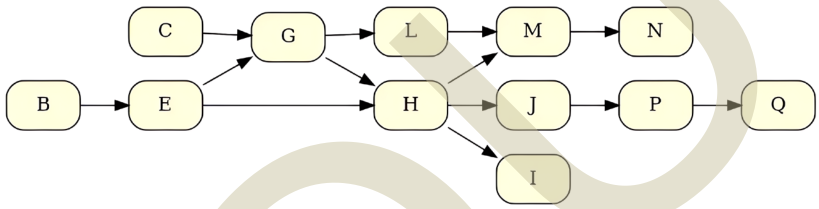


Illustration 4.1.2

Draw a network diagram comprising activities A, B, C……… ,G and that the following constrains are satisfied:

A, B < D

A, B < E

C, D, E < F

C, D, E < G

X < Y ⇒ X must finish before Y starts

Solution

1. Start node (1):

Activities A (1→2), B (1-3) and C (1→4) Can start simultaneously.

2. From node (3):

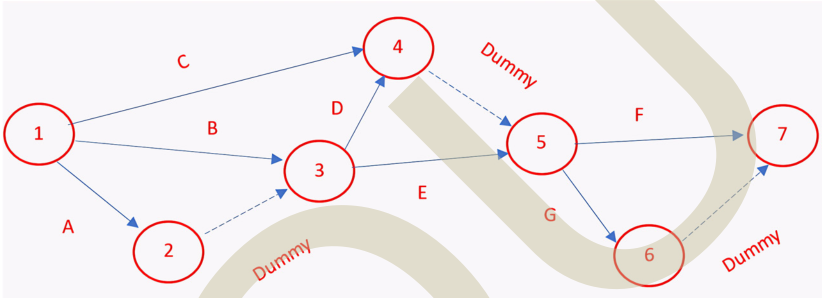
- D (3→4)
- E (3→5) and after completing activity A (not constrained but shown in the diagram as dummy activity (2-3))

3. **From node (5):**

- F (5→7) and after completing activity C and D (not constrained but shown in the diagram as dummy activity (4-5)
- G (5→6) and after completing activity C and D (not constrained but shown in the diagram as dummy activity (4-5)

4. **From node (6)**

- A dummy activity concluding the network



4.1.7 Activity Time and Critical Path

- Sequence of connected activities

A critical path is a sequence of connected activities that leads from the beginning of the project to the end of the project. The longest path in the network is called the critical path. The critical path is of great importance as it determines the duration of the entire project. If any activity on the critical path is delayed, then the entire project will be delayed. Every network has a critical path. It is possible to have multiple critical path.

The minimum time required for a project completion is equal to the longest time path or sequences of connected activities, through the network which describe the precedence relationship of project activities. The longest time path is also referred to as the critical path

4.1.8 Critical Activities or Bottleneck Activities:

- All activities in the critical path

All the activities on the critical path are called bottleneck activities or critical activities. Critical means that delay in the completion of one or more of these activities will cause delay in the completion date in the project. So, such activities require special attention of the project manager. Critical activities require special attention of the project manager because of the following reason.

1. Critical path is important as its length determines the minimum time required for the completion of the project
2. Critical path is shown by ⟹ (Double lined arrow) in network. Those activities must be performed more rapidly for the purpose of attaining stipulated project completion time. Any delay in activity on the critical path cause delay in the completion of the project.

4.1.9 Computation of Critical path

When network is larger, it is very difficult or maybe impossible to find the critical path by listing all the paths and selecting the longest one. Therefore, we need to develop an algorithm. That means a systematic approach to determine the critical path. For this critical path calculation we proceed in two phases:

4.1.9.1 Forward Pass Through the Network [Earliest Start (ES) Time Rule]

In project management, a forward pass is a technique used when analysing a project's timeline on a network diagram. By systematically moving from the beginning to the end of the project, this calculation determines the earliest possible start date and earliest possible finish date for every individual task. This process is crucial for understanding the shortest overall duration required to complete the entire project, as it identifies how soon the project can be realistically finished.

- Forward pass

The Forward Pass is the first half of the Critical Path Method (CPM) calculation. It is a systematic left-to-right calculation across your project network diagram to determine the earliest possible times each activity can start and finish. The "Earliest Start (ES) Time Rule" is a crucial part of this calculation.

Purpose of the Forward Pass and ES/EF:

- **Earliest Start (ES)**: The earliest time an activity can begin, given that all its preceding activities have been completed.
- **Earliest Finish (EF)**: The earliest time an activity can be completed, calculated as ES + Duration.
- **Project Duration**: The EF of the very last activity (or activities) in the project network represents the earliest possible completion time for the entire project.

The Earliest Start (ES) Time Rule

The rule for determining the Earliest Start (ES) of any activity is as follows:

1. **For the First Activity in the Project:**

- If an activity has no predecessors (it's a starting activity), its ES is typically 0 (or the project start date).
- Example: If Task A is the first task, ES(A) = 0.

2. **For Subsequent Activities (with Predecessors):**

The ES of an activity is the maximum (latest) of the Early Finish (EF) times of all its immediate predecessors.

This is the core of the "ES time rule" for activities that depend on others. An activity cannot begin until all the activities it depends on are fully completed. Therefore, you must wait for the one that finishes latest.

Steps of the Forward Pass with the ES Rule:

1. Start at the beginning: Identify all activities with no predecessors. Set their ES to 0 (or project start).
2. Calculate EF for these activities: For each, EF = ES + Duration.
3. Move to the next set of activities: For activities that have predecessors, apply the ES rule:

- Look at all the direct predecessors for the current activity.
- Find the EF for each of those predecessors.
- The largest (latest) of these EF values becomes the ES for the current activity.

4. Calculate EF for the current activities: Again, EF = ES + Duration.
5. Continue this process through the entire network diagram, moving from left to right (start to finish), until you have calculated the ES and EF for every single activity in the project.

Illustration 4.1.3

The following table shows the list of project activities, their immediate predecessors, and duration in days

Activity	Predecessor	Duration (days)
A	– (Start)	4
B	– (Start)	3
C	A	5

D	B	6
E	C, D	2

Calculate project duration. Determine the Earliest Start (ES), Earliest Finish (EF), for each activity.

Solution

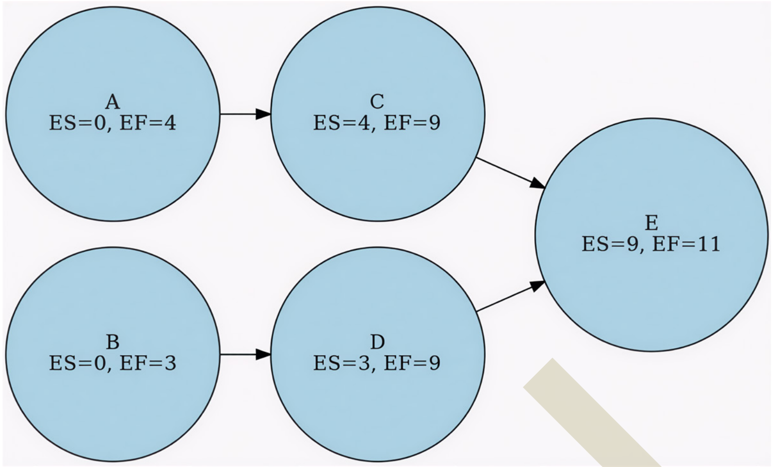
Step 1: Rules of Forward Pass

1. ES (Earliest Start) of the first activity = 0.
2. EF (Earliest Finish) = ES + Duration.
3. For activities with multiple predecessors, take the maximum EF of predecessors as the ES.

Step 2: Calculations

- Activity A
  - ES = 0
  - EF = 0 + 4 = 4
- Activity B
  - ES = 0
  - EF = 0 + 3 = 3
- Activity C (after A)
  - ES = EF of A = 4
  - EF = 4 + 5 = 9
- Activity D (after B)
  - ES = EF of B = 3
  - EF = 3 + 6 = 9
- Activity E (after C and D)
  - ES = max(EF of C, EF of D) = max(9, 9) = 9
  - EF = 9 + 2 = 11

This diagram shows the Forward Pass calculation in Network Analysis. Each node contains the activity name along with its Earliest Start (ES) and Earliest Finish (EF) times.



Activity	ES	EF
A	0	4
B	0	3
C	4	9
D	3	9
E	9	11

Project Duration = 11 days (final EF).

4.1.9.2 Backward Pass Through the Network [Latest Finish (LF) Time Rule]

After completing the Forward Pass to determine the earliest start (ES) and earliest finish (EF) times for all activities and the overall project duration, the backward pass begins at the project's end and works backward through the network diagram. Its primary purpose is to calculate:

- **Latest Finish (LF)**: The latest time an activity can be completed without delaying the overall project completion date.
- **Latest Start (LS)**: The latest time an activity can begin without delaying the overall project completion date, calculated as LF - Duration.

By performing the backward pass, one can identify how much flexibility (or "slack"/"float") each activity has, ultimately leading to the identification of the critical path.

The "Latest Finish (LF) Time Rule":

The rule for determining the Latest Finish (LF) of any activity is as follows:

1. For the Last Activity(ies) in the Project:

   - If an activity has no successors (it's an ending activity). It is LF is typically set equal to the project's Earliest Finish (EF) as determined by the forward pass. This assumes you want to finish the project as soon as possible.
   - Alternatively, if there is a fixed, externally imposed project deadline, that deadline can be used as the LF for the last activity(ies).

Example: If Task F is the last task, and the project's earliest completion time (from forward pass) is 21 days, then LF(F) = 21.

2. **For Preceding Activities (with Successors):**

   - The LF of an activity is the minimum (earliest) of the Latest Start (LS) times of all its immediate successors.
   - This is the core of the "LF time rule" for activities that are predecessors to others. An activity's latest finish time is constrained by when its earliest-starting successor needs to begin. If the current activity finishes any later than that, it will delay its successor, and potentially the entire project.

Steps of the Backward Pass with the LF Rule:

1. Start at the end: Identify all activities with no successors (the final activities). Set their LF equal to the project's overall Earliest Finish (determined in the forward pass).
2. Calculate LS for these activities: For each, LS = LF - Duration.
3. Move backward to the previous set of activities: For activities that have successors, apply the LF rule:

   - Look at all the direct successors for the current activity.
   - Find the LS for each of those successors.
   - The smallest (earliest) of these LS values becomes the LF for the current activity.

4. Calculate LS for the current activities: Again, LS = LF - Duration.
5. Continue this process through the entire network diagram, moving from right to left (finish to start), until you have calculated the LS and LF for every single activity in the project.

Illustration 4.1.4

Project Activities

Activity	Description	Predecessor(s)	Duration (days)
A	Start project	—	3
B	Site preparation	A	2
C	Foundation	A	4
D	Wall construction	B, C	5
E	Roofing	D	3
F	Painting	E	2
G	Inspection	F	1

Build Project Network Diagram and Critical Path

Solution

Forward Pass (Calculating ES and EF)

Activity	Predecessor	Duration	ES	EF (ES + Duration)
A	—	3	0	3
B	A	2	3	5
C	A	4	3	7
D	B, C	5	7	12
E	D	3	12	15
F	E	2	15	17
G	F	1	17	18

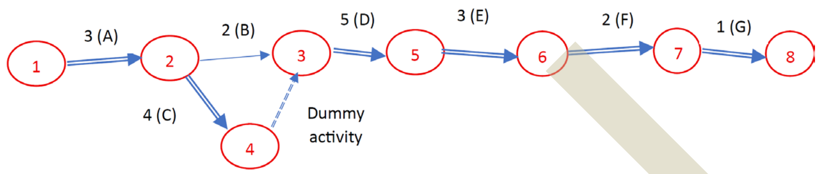
Backward Pass (Calculating LF and LS)

Activity	Successor	Duration	LF	LS (LF – Duration)
G	—	1	18	17
F	G	2	17	15
E	F	3	15	12
D	E	5	12	7
B	D	2	7	5
C	D	4	7	3
A	B, C	3	3	0

Activity Details

Activity	Predecessor(s)	Duration (days)	ES - EF - LS - LF
A	— (Start)	3	0 - 3 - 0 - 3

B	A	2	3 - 5 - 5 - 7
C	A	4	3 - 7 - 3 - 7
D	B, C	5	7 - 12 - 7 - 12
E	D	3	12 - 15 - 12 - 15
F	E	2	15 - 17 - 15 - 17
G	F	1	17 - 18 - 17 - 18



Critical activity = 1-2-4-3-5-6-7-8

Project duration =18 days

Illustration 4.1.5

The following table shows the activities and their durations (in days) for a project.

Activity	Duration (days)
1-2	20
1-3	25
2-3	10
2-4	12
3-4	6
4-5	10

Identify the Critical Path and Project Duration

Solution

Forward pass calculation

E1 =0,

E2= E1+20 (duration of 1-2) = 0+20 = 20

E3 = the maximum of E1+25 and E2+10

That is E1+25= 25, E2+10 =20+10 =30

That is E3 = 30

E4 = maximum of E3+6 and E2+12

E3 + 6 =30+6= 36, E2+12 = 20+12 = 32

That is E4 = 36

E5 = E4+10 = 36+10 =46

Backward pass calculation

L5 = 46

L4 = l5 – 10 = 46- 10 =36

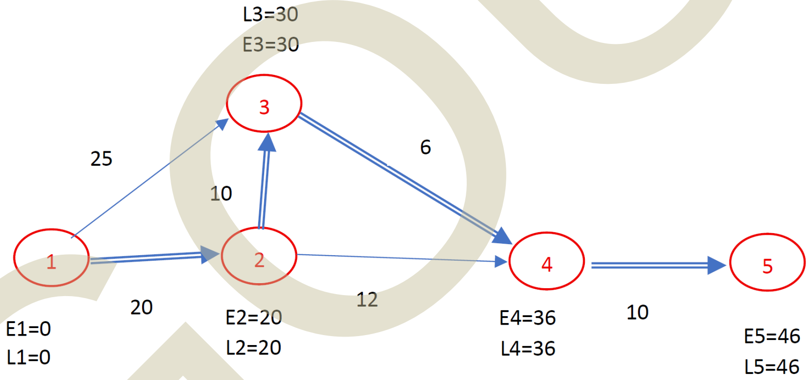
L3 = L4 – 6 = 36-6 = 30

L2 = The minimum of L3 – 10 and L4-12 (30-10=20 and 36-12=24)

That is L2 = 20

L1 = The minimum of L3 - 25 and L2 – 20 (30-25 = 5 and 20-20 =0)

That is L1 = 0



Critical path – 1-2-3-4-5

Project duration = 46 days

Summarised Overview

Network analysis is a vital technique in project management that facilitates the efficient planning, scheduling, and control of large-scale projects by breaking them into smaller, interrelated activities. Represented through network diagrams using nodes and arrows, it helps visualize task sequences, dependencies, and durations. Core components such as the Critical Path Method (CPM), Forward Pass, and Backward Pass calculations assist in identifying the longest path (critical path) and calculating the earliest and latest start and finish times of activities. This enables managers to pinpoint slack (or float) time, delays permissible without affecting the overall project schedule and allocate resources optimally. Fulkerson's rule ensures logical numbering of events within the diagram. The technique ultimately supports project managers in reducing costs, managing idle resources, preventing delays, and achieving timely project completion.

Self-Assessment Question

1. What do you mean by Network Analysis?
2. What is the significance of the critical path in project scheduling?
3. Describe the role of dummy activity in network diagrams.
4. Explain the objectives of Network Analysis.
5. Discuss the different types of activities.

Assignments

1. Explain the different stages of project management.
2. Explain the rules to frame a network diagram.
3. Explain Pulkerson's Rule for Numbering of Events.

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Suggested Reading

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4. Rajagopal, K. (2012). Operations Research. PH Learning Private Limited

Space for Learner Engagement for Objective Questions

Learners are encouraged to develop objective questions based on the content in the paragraph as a sign of their comprehension of the content. The Learners may reflect on the recap bullets and relate their understanding with the narrative in order to frame objective questions from the given text. The University expects that 1 - 2 questions are developed for each paragraph. The space given below can be used for listing the questions.

Unit 2

Slack

Learning Outcomes

After completing this unit, the learner will be able to:

- understand the concept of slack, float
- explain PERT
- differentiate between PERT and CPM

Background

Project management is a dynamic field that requires careful planning, execution, and control to achieve desired outcomes within defined constraints. Two fundamental aspects of project management are understanding scheduling flexibility and managing uncertainty, which are addressed by concepts like float (or slack) and PERT (Program Evaluation and Review Technique). Furthermore, real-world projects often face pressure to accelerate completion, leading to the critical concept of time-cost trade-off. This document provides a detailed overview of these essential project management tools and techniques, highlighting their definitions, applications, and interrelationships.

Keywords

Slack, Float, CPM, PERT, Crashing

Discussion

4.2.1 Slack

- Amount of time an activity or event can be delayed

Slack is the amount of time an activity or event can be delayed without delaying the overall project completion time.

- It indicates scheduling flexibility.
- If an activity has zero slack, it is on the critical path.

4.2.2 Float

- Buffer time available for an activity

Float refers to the buffer time available for an activity without affecting the project's overall timeline. It is the additional time obtained. It is calculated during network analysis (PERT/CPM).

In most cases, Slack and Float are used interchangeably, but technically:

- Slack is used for events (nodes).
- Float is used for activities (arrows).

- Shows how much extra time is available for an activity

Float is very useful for a manager because it shows how much extra time is available for an activity beyond its expected duration. It tells us how long an activity can be delayed without affecting the overall project timeline. If an activity has no float, it cannot be delayed at all without delaying the whole project. Such activities are called critical activities, and they lie on the critical path. So, any event with zero float or slack is part of the critical path, meaning it must be completed exactly on time.

i. Total Float (TF):

▶ **Definition:**

The maximum time an activity can be delayed without delaying the entire project.

▶ **Use:**

Shows how much flexibility is available for scheduling.

▶ **Formula:**

Total Float = Latest Finish – Earliest Finish

Or

TF = Latest Start – Earliest Start

Example:

If an activity is scheduled to start at day 5 but can be delayed up to day 8 without affecting the project, the total float is 3 days.

ii. **Free Float (FF):**

- **Definition:**

The amount of time an activity can be delayed without affecting the start time of its immediate next activity.

- **Use:**

Helps managers know whether a delay in one task will delay its following task.

- **Formula:**

Free Float = Earliest Start of next activity – Earliest Finish of current activity

- **Note:**

Free float is always less than or equal to total float.

iii. **Independent Float (IF):**

- **Definition:**

The time an activity can be delayed without affecting either the previous or the next activity, assuming both start and finish at their earliest and latest times.

- **Use:**

Shows the amount of flexibility that is fully within the activity itself, not depending on others.

- **Formula:**

Independent Float = Earliest Start of next activity – Latest Finish of previous activity – Duration

- **Note:**

Independent float can be zero, positive, or even negative (if the activity is tightly scheduled).

Table 4.2.1

Total Float, free float and independent float

Float Type	Delay Without Affecting	Can Be Negative?
Total Float (TF)	Entire Project Completion	Rarely

Free Float (FF)	Next (successor) Activity	No
Independent Float (IF)	Previous and Next Activities	Yes

4.2.3 CPM (Critical Path Method)

- Used for repetitive type project

It is generally used for repetitive type project or for those projects for which fairly accurate estimate of time for completion of activity can be made and for which cost estimates can be made with fair degree of accuracy. The critical path method can be used effectively in production planning, road systems and traffic schedule, communication network, etc. It is activity-oriented diagram.

4.2.3.1 Steps in CPM

1. List all project activities and estimate their durations.
2. Determine dependencies (which tasks must precede others).
3. Construct a network diagram (using nodes and arrows).
4. Perform Forward Pass to calculate:
    - Earliest Start (ES) – the earliest time an activity can start.
    - Earliest Finish (EF) – the earliest time an activity can finish.
5. Perform Backward Pass to calculate:
    - Latest Start (LS) – the latest time an activity can start without delaying the project.
    - Latest Finish (LF) – the latest time an activity can finish without delaying the project.
6. Identify the critical path (activities with zero slack).
7. Monitor and control project progress by paying close attention to critical activities.

Illustration 4.2.1

Calculate project duration and critical path

Activity	Duration (days)
1-2	20
1-3	25
2-3	10
2-4	12

3-4	6
4-5	10

Solution

Forward pass calculation

E1 =0,

E2= E1+20 (duration of 1-2) = 0+20 = 20

E3 = the maximum of E1+25 and E2+10

That is E1+25= 25, E2+10 =20+10 =30

That is E3 = 30

E4 = maximum of E3+6 and E2+12

E3 + 6 =30+6= 36, E2+12 = 20+12 = 32

That is E4 = 36

E5 = E4+10 = 36+10 =46

Backward pass calculation

L5 = 46

L4 = l5 – 10 = 46- 10 =36

L3 = L4 – 6 = 36-6 = 30

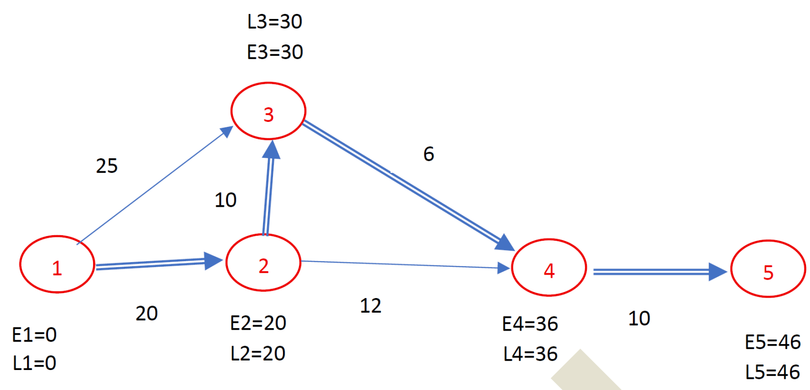
L2 = The minimum of L3 – 10 and L4-12 (30-10=20 and 36-12=24)

That is L2 = 20

L1 = The minimum of L3 - 25 and L2 – 20 (30-25 = 5 and 20-20 =0)

That is L1 = 0

Activity	Duration	Earliest time		Latest time		Float		
		ES	EF	LS	LF	Total float	Free float	Independent float
2-Jan	20	0	20	0	20	0	0	0
3-Jan	25	0	25	5	30	5	5	5
3-Feb	10	20	30	20	30	0	0	0
4-Feb	12	20	32	24	36	4	4	4
4-Mar	6	30	36	30	36	0	0	0
5-Apr	10	36	46	36	46	0	0	0



Critical path – 1-2-3-4-5

Project duration = 46 days

4.2.4 Program Evaluation and Review Technique – PERT

- Project management tool used to plan, schedule, and control complex projects

PERT is a project management tool used to plan, schedule, and control complex projects, especially when activity durations are uncertain. It is widely applied in R&D, construction, product development, and other areas involving unpredictable tasks. PERT is a method used to represent project activities in the correct sequence and timing. It is a project scheduling tool that helps plan, organize, and coordinate various tasks. PERT serves as a planning and control system for management by offering a clear structure or roadmap for executing a project. It ensures that all key events and activities are properly identified and arranged.

In PERT, activity times are not fixed. Instead, they are estimated using three different time values to account for uncertainty in project scheduling. These estimates help calculate the expected duration of each activity and assess the probability of project completion.

4.2.4.1 Steps in PERT

1. List activities and identify dependencies.
2. Draw the network diagram.
3. Estimate Optimistic Time, Most Likely Time, Pessimistic Time for each activity.
4. Calculate Expected Time (TE) and Variance.
5. Find the Critical Path (the longest expected time path).
6. Estimate Project Duration (sum of TE along the critical path).

7. Find Probability of Completion

4.2.4.2 Three Time Estimates in PERT:

1. Optimistic Time (O):

The shortest possible time in which an activity can be completed.

Assumes everything goes better than expected (ideal conditions).

2. Most Likely Time (M):

- Time Estimates in PERT

The best estimate of the time required to complete the activity under normal conditions.

Based on past experience or expert judgment.

3. Pessimistic Time (P):

The longest time the activity might take, assuming unfavourable conditions or unexpected delays.

Formula for Expected Time (TE):

$$TE = \frac{O + 4M + P}{6}$$

This formula gives more weight to the most likely time (M), providing a realistic average duration.

Example:

Suppose for an activity:

- Optimistic time (O) = 4 days
- Most likely time (M) = 6 days
- Pessimistic time (P) = 10 days

$$TE = \frac{4 + 4(6) + 10}{6} = 6.33 \text{ days}$$

Variance = (Tp – To /6)2

Illustration 4.2.2

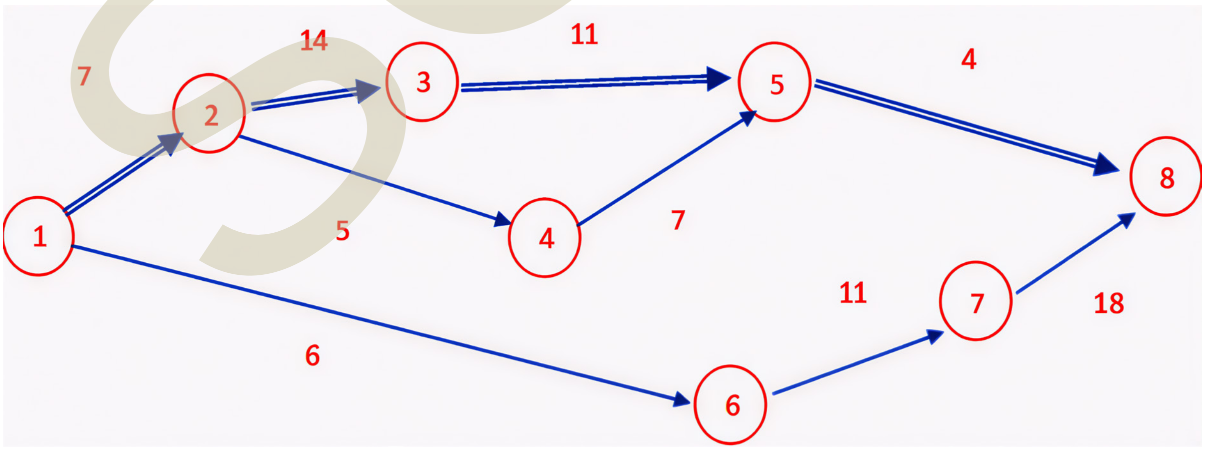
The following table lists the jobs of a network along with their time estimates

Jobs	Optimistic	Most likely	Pessimistic
1-2	3	6	15
1-6	2	5	14
2-3	6	12	30
2-4	2	5	8
3-5	5	11	17
4-5	3	6	15
6-7	1	4	7
5-8	1	4	7
7-8	4	19	28

Draw the project network, length and variance of the critical path, what is the probability of the project completed within 41 days

Solution

Jobs	Optimistic	Most likely	Pessimistic	$T_e = T_o + T_m + T_p / 6$	Variance = $(T_p - T_o /6)^2$
1-2	3	6	15	7	Square of (15-3)/6=4
1-6	2	5	14	6	4
2-3	6	12	30	14	16
2-4	2	5	8	5	1
3-5	5	11	17	11	4
4-5	3	6	15	7	4
6-7	3	9	27	11	16
5-8	1	4	7	4	1
7-8	4	19	28	18	16



Forward pass calculation of earliest time	Backward pass calculation of latest time
E1 = 0	L8 = E8 = 36
E2 = 0+7 =7	L7 = 36-18 = 18
E3 = 7+14 =21	L6 = 18-11 = 7
E4 = 7+5 = 12	L5 = 36-4 = 32
E5 = MAX (21+11, 12+7) = 32	L4 = 32-7 =25
E6 = 0+6 =6	L3 = 32-11 =21
E7 = 6+11 = 17	L2 = MIN (21-14, 25-5) =7
E8 = MAX (32+4, 17+18) = 36	L1 = MIN (7-7, 7-6) = 0

The optimum length of the critical path = 7+14+11+4 = 36

Critical path = 1-2-3-5-8

Variance of the critical path = 4+16+4+1 = 25

Standard deviation (σ) $\sqrt{\text{variance}}$

$= \sqrt{25} = 5$

The probability of the project completed within 41 days

t = 41 te = 36 $\sigma = 5$

Z = t – te/σ(Normal distribution)

Z = 41-36/5 = 5/5 = 1

Z = 1, area under standard normal curve = .34 (table value of normal distribution)

Since Z is positive, probability = .5+ .34 = .84

Therefor the probability of the project completed within 41 days = 0.84

Illustration 4.2.3

Suppose a project has the following activities:

Activity	Duration (days)	Predecessor
A	3	-
B	5	A
C	2	A
D	4	B, C
E	6	D

- Draw the network diagram for the given project.
- Identify all the paths from the start to the end of the project.

- Calculate the Earliest Start (ES), Earliest Finish (EF), Latest Start (LS), and Latest Finish (LF) for each activity.

Solution

Forward Pass (Earliest Times)

- Activity A
  - ES = 0
  - EF = ES + duration = 0 + 3 = 3
- Activity B (depends on A)
  - ES = EF of A = 3
  - EF = 3 + 5 = 8
- Activity C (depends on A)
  - ES = EF of A = 3
  - EF = 3 + 2 = 5
- Activity D (depends on B and C)
  - ES = max(EF of B, EF of C) = max(8, 5) = 8
  - EF = 8 + 4 = 12
- Activity E (depends on D)
  - ES = EF of D = 12
  - EF = 12 + 6 = 18

Project Duration = 18 days

2. Backward Pass (Latest Times)

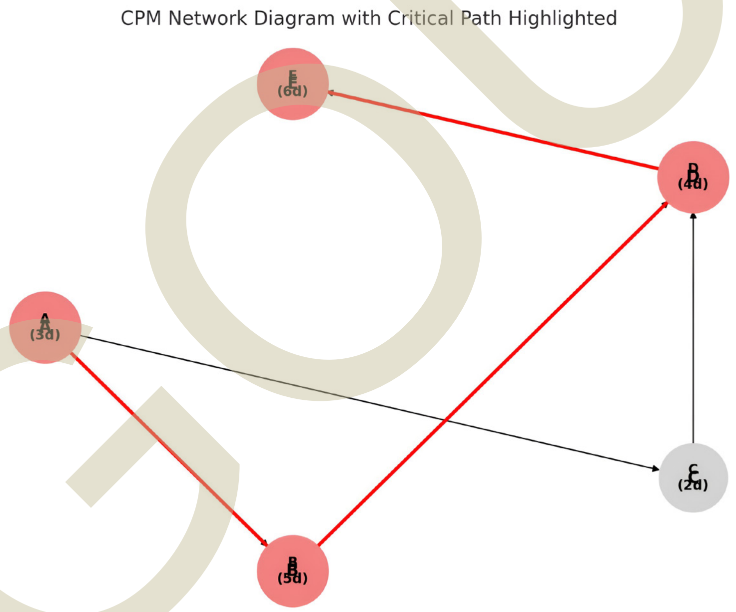
- Activity E
  - LF = Project Duration = 18
  - LS = LF – duration = 18 – 6 = 12
- Activity D (predecessor of E)
  - LF = LS of E = 12
  - LS = 12 – 4 = 8
- Activity B (predecessor of D)
  - LF = LS of D = 8
  - LS = 8 – 5 = 3
- Activity C (predecessor of D)
  - LF = LS of D = 8
  - LS = 8 – 2 = 6

- Activity A (predecessor of B and C)
  - LF = min(LS of B, LS of C) = min(3, 6) = 3
  - LS = 3 – 3 = 0

Activity	Duration (d)	ES	EF	LS	LF	Slack (d)
A	3	0	3	0	3	0
B	5	3	8	3	8	0
C	2	3	5	6	8	3
D	4	8	12	8	12	0
E	6	12	18	12	18	0

Critical Path: A → B → D → E (activities with Slack = 0).

Non-critical activity: C (Slack = 3 days).



The critical activities are shown in light coral, making it clear which tasks must be monitored closely.

Table 4.2.1 Difference Between CPM and PERT

Basis	PERT (Program Evaluation and Review Technique)	CPM (Critical Path Method)
1. Nature	Probabilistic (uncertain time estimates)	Deterministic (fixed time estimates)
2. Focus	Time-oriented planning	Time and cost-oriented planning

3. Activity Duration	Estimated using three times: Optimistic, Most Likely, Pessimistic	Fixed, known time durations
4. Application	Suitable for R&D projects, product development, etc.	Used in construction, manufacturing, maintenance
5. Time Estimates	Uses expected time (TE) based on probability	Uses single, fixed durations
6. Cost Control	Cost is not emphasized	Cost control is important and included
7. Type of Project	Ideal for non-repetitive and unique projects	Ideal for repetitive, well-defined projects
8. Critical Path	Can change as estimates vary	Usually remains stable once defined
9. Float Calculation	Not used extensively	Widely used for scheduling flexibility
10. Chart Used	Event-oriented (nodes represent events)	Activity-oriented (nodes represent activities)

4.2.5 Project Time- Cost Trade-Off

The objective of the time-cost trade-off analysis is to reduce the original project duration, determined form the critical path analysis, to meet a specific deadline, with the least cost. In addition to that it might be necessary to finish the project in a specific time to:

- Finish the project in a predefined deadline date.
- Recover early delays. - Avoid liquidated damages.
- Free key resources early for other projects
- Avoid adverse weather conditions that might affect productivity.
- Receive an early completion-bonus.
- Improve project cash flow

- Process of balancing the total project duration with the total cost

The Time-Cost Trade-Off in project management refers to the process of balancing the total project duration with the total cost.

This concept recognizes that reducing the time to complete a project (crashing) may lead to higher direct costs, while taking more time may increase indirect costs like overhead, labour, and supervision.

Key Concepts:

1. Normal Time and Normal Cost:

The planned duration and cost for completing an activity under normal working conditions.

2 Crash Time and Crash Cost:

The shortest possible time in which an activity can be completed, usually by applying extra resources, leading to higher cost.

3 Crashing:

The process of shortening the duration of activities on the critical path by spending more money, such as adding extra labour, using faster equipment, or working overtime.

4.2.6 Cost Slope:

The rate of increase in cost per unit of time reduction in an activity.

$$\text{Cost Slope} = \frac{\text{Crash Cost} - \text{Normal Cost}}{\text{Normal time} - \text{crash time}}\backslash$$

Summarised Overview

This unit explores essential project management concepts, beginning with float (or slack), which quantifies the flexibility in a project schedule by indicating how long an activity or event can be delayed without impacting the overall project completion. It differentiates between Total Float (impact on project completion), Free Float (impact on successor activity start), and Independent Float (no impact on previous or next activities), clarifying that "slack" is typically for events and "float" for activities. Program Evaluation and Review Technique (PERT) is a probabilistic tool for uncertain activity durations that uses optimistic, most likely, and pessimistic time estimates to calculate an expected duration, contrasting it with the deterministic Critical Path Method (CPM). The project time-cost trade-off, explains how "crashing" activities on the critical path to reduce project duration often increases direct costs. "Cost Slope" helps in making informed decisions about accelerating projects while managing additional expenditure.

Self-Assessment Question

1. What do you mean by slack?
2. What is crashing?
3. Explain the difference between Total Float and Free Float.
4. Explain project time cost trade off.
5. Explain PERT and its steps.

Assignments

1. An activity has an Optimistic Time (O) of 3 days, a Most Likely Time (M) of 5 days, and a Pessimistic Time (P) of 9 days. Calculate the Expected Time (TE) for this activity using the PERT formula.
2. What are the difference Between CPM and PERT
3. What is Independent Float?

Reference

1. Kerzner, H. (2017). *Project Management: A Systems Approach to Planning, Scheduling, and Controlling* (12th ed.). John Wiley & Sons.
2. Kapoor, V. K. (2000). *Operations research*. New Delhi: Sultan Chand & Sons.
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2. Prem, K & Hira, D. S (2017). *Operations Research.* S.Chand and Company Limited.
3. Kalavathy, S.(2013). *Operations Research* (4th Ed.), Vikas Publishing House Private Limited.
4. Rajagopal, K. (2012). *Operations Research*, PHI Learning

Space for Learner Engagement for Objective Questions

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Unit 3

Game Theory

Learning Outcomes

After completing this unit, the learner will be able to:

- Define Game Theory
- Understand the assumptions and features of Game Theory
- Understand Two Person Zero Sum Game

Background

Decision-making in various aspects of life, from economics and business to politics and everyday social interactions, rarely occurs in isolation. Often, the outcome of an individual's choice is not solely dependent on their own action but is profoundly influenced by the choices made by others. Game theory provides a powerful mathematical framework to analyse these situations of interdependent decision-making. Developed by mathematicians John von Neumann and Oskar Morgenstern in the mid-20th century, and later significantly expanded by John Nash and others, game theory offers insights into competitive and cooperative behaviours, helping to predict outcomes and design optimal strategies in interactive settings. This document will introduce the fundamental concepts of game theory, its key features, underlying assumptions, and a specific type of game known as the two-person zero-sum game.

Keywords

Game Theory, Players, Strategy, Payoffs

Discussion

4.3.1 Meaning of Game Theory

- Activity between two opposing parties or persons based a set of rules

A game is an activity between two opposing parties or persons based a set of rules, in which each side initiate a series of steps based on their strategies to convert the situation favourable to itself, winning or loss involve it.

Game theory is a theoretical framework used to analyse how individuals or entities, known as players, make decisions in situations where their outcomes depend on the choices of others. It helps to explain and predict behaviour in competitive or interactive social settings.

Game theory is the study of mathematical models of conflict and co-operation between intelligent rational decision makers. It deals with competitive situations of decision making under uncertainty.

4.3.2 Definition of Game Theory

Game theory is defined as "a theoretical framework for analysing situations in which players make interdependent decisions." The outcome for each participant depends not only on their own decisions but also on the decisions made by others. It is widely used in economics, political science, psychology, and business to understand competitive and cooperative behaviour.

4.3.3 Features of Game theory

Players:

Game theory involves two or more decision-makers known as players, each with their own interests and strategies.

Strategies:

Every player has a set of possible strategies or actions they can choose from to achieve their objectives.

Payoffs:

Each combination of strategies chosen by the players results in specific outcomes or payoffs, which can be in the form of profit, utility, or benefit.

Rationality:

Players are assumed to be rational, meaning they aim to maximize their own benefit or utility based on the available information.

Interdependence:

The outcome for one player depends not only on their own strategy but also on the strategies chosen by other players.

Conflict and Cooperation:

Game theory studies both competitive (conflict) and cooperative interactions between players.

Information:

Games may be played with complete or incomplete information. Complete information means all players know the structure of the game, while incomplete information implies some elements are unknown to one or more players.

Equilibrium Concept:

The theory seeks equilibrium solutions like Nash Equilibrium, where no player can benefit by changing their strategy while others keep theirs unchanged.

Repeated and One-shot Games:

Some games are played once (one-shot), while others are repeated over time, affecting the strategies and outcomes.

4.3.4 Assumptions of Game Theory

1. There are finite numbers of competitors (players).
2. The players act reasonably.
3. Every player strives to maximize gains and minimize losses.
4. Each player has finite number of possible courses of action.
5. The choices are assumed to be made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action.
6. The pay-off is fixed and predetermined.
7. The pay-offs must represent utilities.

4.3.5 Two-Person Zero-Sum Game

- A two-person zero-sum game is a type of game in game theory where:
- There are two players (also called competitors or decision-makers).
- One player's gain is exactly equal to the loss of the other.

The sum of outcomes (payoffs) for both players is always zero—hence the name "zero-sum."

Features:

1. **Two Players Only:**

The game involves exactly two participants.

2. **Opposing Interests:**

The game is purely competitive; one player's advantage is the other's disadvantage.

3. **Zero Total Payoff:**

The total gain and loss in the game adds up to zero.

For example, if Player A wins ₹100, Player B loses ₹100.

4. **Payoff Matrix:**

The game is often represented using a matrix showing payoffs for each combination of strategies.

5. **Saddle Point:**

An optimal strategy pair (if it exists) is called a saddle point, where the minimum of the row is equal to the maximum of the column.

6. **No Cooperation:**

Players do not cooperate; it is a fully competitive game.

4.3.6 Prisoner's Dilemma

The Prisoner's Dilemma is a fundamental concept in game theory that illustrates why two rational individuals might not cooperate, even if it appears that it is in their best interest to do so. The Prisoner's Dilemma was originally formulated in 1950 by Merrill Flood and Melvin Dresher, who were working at the RAND Corporation (research initiative and public sector consulting firm), a U.S. think tank.

The game was later formalized and given its now-famous name by Albert W. Tucker, a Princeton mathematician. Tucker created the "prison story" to make the abstract mathematical concept more relatable and easier to understand.

Summarised Overview

This unit introduces Game Theory as a theoretical framework for analyzing situations where multiple "players" make interdependent decisions, meaning each player's outcome is influenced by the choices of others. It highlights core features of game theory, including the presence of players with strategies, resulting payoffs for various strategy combinations, the assumption of player rationality (aiming to maximize their own benefit), and the interdependence of outcomes. Other important features include the study of conflict and cooperation, the distinction between complete and incomplete information, the pursuit of equilibrium concepts like Nash Equilibrium (where no player can improve their outcome by unilaterally changing their strategy, the Two-Person Zero-Sum Game, a specific type of game involving exactly two players where one player's gain precisely equals the other's loss, resulting in a total payoff of zero. Key features of these games include pure competition, the use of a payoff matrix, and the potential existence of a "saddle point" indicating an optimal strategy where the minimum of a row equals the maximum of a column.

Self-Assessment Question

1. What is Game Theory?
2. What is the primary purpose of game theory, and in what types of situations is it most applicable?
3. List and briefly explain five key features of game theory.
4. Define "Nash Equilibrium." Why is it an important concept in game theory?
5. What do you mean by Two Prison Zero sum game?
6. What is Prisoner's Dilemma?

Assignments

1. Describe the defining characteristics of a "Two-Person Zero-Sum Game", Provide a simple real-world example of such a game.

Reference

1. Dixit, A. K., Skeath, S., & Reiley, D. (2015). *Games of Strategy* (4th ed.). W. W. Norton & Company.
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Unit 4

Payoff Matrix

Learning Outcomes

After completing this unit, the learner will be able to:

- Understand payoff matrix
- Describe the strategies of game
- Find the saddle point
- Apply Maximin–Minmax Principle
- Apply Dominance Principle

Background

Building upon the foundational concepts of game theory, this unit delves into the practical tools and principles used to analyse and solve game theory problems. The payoff matrix serves as the primary visual representation of a game, systematically displaying the outcomes for each player based on their chosen strategies. The objective of players within this framework is to make optimal decisions by selecting the most advantageous strategies. We will explore various types of strategies, including pure and mixed, as well as decision-making criteria like maximin and minimax. A crucial concept for finding stable solutions in certain games is the saddle point, which represents an equilibrium. Finally, the Principle of Dominance offers a powerful method for simplifying complex games, making their analysis more manageable.

Keywords

Payoff matrix, Strategy, Saddle Point, Dominance

Discussion

4.4.1 Payoff Matrix

- Outcomes or returns that players receive at the end of a game

Payoff refer to the outcomes or returns that players receive at the end of a game. For example, in the case of profit-maximizing firms, payoffs are measured in terms of profits. A two-person zero-sum game is conveniently represented by a matrix. The matrix which shows the outcome of the game as the players select their particular strategies, is known as the pay-off matrix. It is important to assume that each player knows not only his own list of possible courses of action but also that of his opponent.

4.4.2 Decision of a Game

- Choosing the best possible course of action

The decision of a game refers to choosing the best possible course of action (strategy) for a player in a game situation, where outcomes depend not only on their own actions but also on the actions of others. It is aimed at maximizing gain (or minimizing loss) in a competitive setting.

The decision is based on:

- Payoff matrix (outcomes of each strategy combination),
- Assumptions about the opponent's behaviour,
- And the rationality of players.

4.4.3 Strategy in Game Theory

- complete plan of action a player will follow throughout the game

A strategy is a complete plan of action a player will follow throughout the game. It defines the player's action for every possible situation in the game.

4.4.4 Types of strategy

Type of Strategy	Description
Pure Strategy	A strategy where a player consistently chooses the same action. It is deterministic and does not involve any randomness. *Example: Always choosing Strategy A.*

Mixed Strategy	A strategy where a player chooses among available strategies randomly with assigned probabilities. It introduces chance into decision-making. *Example: Choosing Strategy A with 60% probability and Strategy B with 40%.*
Dominant Strategy	A strategy that results in a better payoff regardless of what the opponent does. If a player has a dominant strategy, they will always prefer it.
Dominated Strategy	A strategy that is always worse than another strategy, no matter what the opponent does. Rational players discard dominated strategies.
Maximin Strategy	A strategy in which a player maximizes the minimum gain, assuming the worst-case scenario. Used under conditions of uncertainty.
Minimax Strategy	A strategy aimed at minimizing the maximum loss. Mostly used in zero-sum games.
Nash Equilibrium Strategy	A set of strategies where no player can benefit by unilaterally changing their strategy. It reflects mutual best responses.

Maximin and Minimax are decision rules or criteria used to select a strategy, not strategies themselves. However, in the context of two-player zero-sum games, they are often referred to as strategies because they guide the player's choice of action.

4.4.5 Saddle Point

- Equilibrium point

The Saddle Point in a Payoff matrix is one which is the smallest value in its row and largest value in its column. The saddle point is also known as equilibrium point in the theory of games. Equilibrium means neither player can benefit by changing their strategy unilaterally. An element of a matrix that is simultaneously minimum of the row in which it occurs and the maximum of the column in which it occurs is a saddle point of the matrix game. Both players can choose one fixed strategy, and the outcome will be stable. The value at the saddle point is called the value of the game.

How to Find a Saddle Point?

1. Find the minimum value in each row → list of row minimums

2. Find the maximum value in each column → list of column maximums
3. If the maximum of the row minimums = minimum of the column maximums,

→ That common value is the saddle point.

4.4.6 Maximin–Minmax Principle

- Fundamental decision making rule used in two person-zero sum game

The Maximin–Minimax Principle is a fundamental decision-making rule used in two-person zero-sum games, especially when players are risk-averse or when there's no trust or cooperation between players.

Meaning:

- The Maximin–Minimax Principle helps in selecting optimal strategies for players by assuming that the opponent will act to minimize your gain.
- It is based on pessimistic (conservative) assumptions about the opponent's behaviour.

Maximin (used by Row Player / Maximizer):

1. For each strategy, find the minimum payoff the row player might get (worst-case scenario).
2. Then choose the strategy that gives the maximum among those minimum payoffs. "Maximize the minimum gain."

Minimax (used by Column Player / Minimizer):

For each strategy, find the maximum payoff (loss to the column player).

Then choose the strategy that gives the minimum among those maximum payoffs. "Minimize the maximum loss."

Principle:

In equilibrium,

Maximin value = Minimax value

If this equality holds, the game has a saddle point, and both players have pure strategy solutions.

Illustration 4.4.1

Consider the following payoff matrix for a two-person zero-sum game between Player A and Player B.

	B1	B2
A1	4	2
A2	3	5

Calculate the Maximin value for Player A and the Minimax value for Player B.

Determine whether the game has a saddle point

Solution

- Row player (A):
  - A1: min = 2
  - A2: min = 3

  → Maximin = 3 (A2)
- Column player (B):
  - B1: max = 4
  - B2: max = 5

  → Minimax = 4 (B1)

Since Maximin (3) ≠ Minimax (4), there is no saddle point, and the game requires mixed strategies.

Illustration 4.4.2

Consider the following payoff matrix

	B1	B2	B3
A1	5	6	7
A2	2	4	9
A3	1	3	6

Determine whether the game has a saddle point

Solution

Row Minimums:

A1 → 5

A2 → 2

A3 → 1

→ Maximin = 5

Column Maximums:

B1 → max = 5

B2 → max = 6

B3 → max = 9

→ Minimax = 5

Since Maximin = Minimax = 5,

→ Saddle point = 5,

→ At position (A1, B1)

Interpretation:

- Player A should always choose A1
- Player B should always choose B1
- Neither player can improve their payoff by changing strategies
- Value of the game = 5

Illustration 4.4.3

Solve the game:

	B			
A		I	II	III
	I	-3	-2	-6
	II	3	0	2
	III	5	-2	-4

Find Row Minimums (for Maximin)

Solution

Row	Elements	Row Minimum
A I	-3, -2, -6	-6
A II	3, 0, 2	0
A III	5, -2, -4	-4

→ Maximin = max of row minimums = max(-6, 0, -4) = 0

Find Column Maximums (for Minimax)

Column	Elements	Column Maximum
B I	-3, 3, 5	5
B II	-2, 0, -2	0
B III	-6, 2, -4	2

→ Minimax = min of column maximums = min(5, 0, 2) = 0

Saddle point = AII, BII (value = 0)

4.4.7 Odds Method – Game Theory (2×n or m×2 Games)

The Odds Method is a shortcut technique used to solve 2 × n or m × 2 zero-sum games (games with 2 rows or 2 columns) without a saddle point, using mixed strategies.

When to Use the Odds Method:

- When the game is 2 × n or m × 2 in size.
- No saddle point exists (i.e., maximin ≠ minimax).
- You need to determine the optimal mixed strategy and value of the game.

For a 2 × n Game:

Payoff Matrix

	B1	B2	B3	...	Bn
A1	a1	a2	a3	...	an
A2	b1	b2	b3	...	bn

Steps (2 × n game):

1. Calculate Odds (O_1, O_2, ..., On) for each column:

   Oj=bj−aj(i.e., payoff of A2 – payoff of A1)

2. If all O_j are positive or all negative, convert them to positive values.
3. Add all Odds:

   Total Odds=∑Oj

4. Find probabilities for A1 and A2:

   Probability of **A1**:

   P1=∑Oj ∑Oj+

   Probability of **A2**:

   P2=1−P1

Find Value of the Game (V) by substituting the probabilities into the expected payoff equation.

Illustration 4.4.4

Solve the following payoff matrix, determine the Optimal strategies and value of the game.

(Mixed Strategy)

	B	
A	5	1
	3	4

Solution

This is a 2 × 2 game

Payoff Matrix (Player A's Payoffs):

	B1	B2
A1	5	1
A2	3	4

This is a zero-sum game, and we assume Player A is the row player (maximizing), and Player B is the column player (minimizing).

Step 1: Check for Saddle Point

Row Minima: A1 → min(5,1) = 1; A2 → min(3,4) = 3 → Maximin = 3

Column Maxima: B1 → max(5,3) = 5; B2 → max(1,4) = 4 → Minimax = 4

Since Maximin ≠ Minimax, there is no saddle point. So we proceed using Mixed Strategy (Odds Method).

Step 2: Use Odds Method

Let $a_{11} = 5$, $a_{12} = 1$, $a_{21} = 3$, $a_{22} = 4$

	B1	B2
A1	a1.1	a1.2
A2	a2.1	a2.2

	B1	B2
A1	5	1
A2	3	4

For Player A (Row Player):

$P(A1) = (a_{22} - a_{21}) / (a_{11} + a_{22} - a_{12} - a_{21})$

$= (4 - 3) / (5 + 4 - 1 - 3) = 1 / 5 = 0.2$

$P(A2) = 1 - 0.2 = 0.8$

For Player B (Column Player):

$P(B1) = (a_{22} - a_{12}) / (a_{11} + a_{22} - a_{12} - a_{21})$

$= (4 - 1) / (5 + 4 - 1 - 3) = 3 / 5 = 0.6$

$P(B2) = 1 - 0.6 = 0.4$

Step 3: Value of the Game (V)

$V = (a_{11} \cdot a_{22} - a_{12} \cdot a_{21}) / (a_{11} + a_{22} - a_{12} - a_{21})$

$= (5 \cdot 4 - 1 \cdot 3) / (5 + 4 - 1 - 3) = (20 - 3) / 5 = 17 / 5 = 3.4$

Final Answer:

Optimal Strategy for Player A: A1 = 0.2, A2 = 0.8

Optimal Strategy for Player B: B1 = 0.6, B2 = 0.4

Value of the Game = 3.4

4.4.8 Principle of Dominance

- Used in solving two-person zero-sum games, particularly when the payoff matrix is large

The Principle of Dominance is a simplification technique used in solving two-person zero-sum games, particularly when the payoff matrix is large. The Principle of Dominance states that if one strategy is better than (or equal to) another strategy in all situations, the inferior (dominated) strategy can be eliminated from the game without affecting the outcome. This allows us to reduce the size of the matrix, making it easier to solve using saddle point, graphical, or mixed strategy methods.

Types of Dominance:

1. **Row Dominance (Player A – Maximizer):**

If all elements in one row are less than or equal to the corresponding elements in another row, then the dominated row can be deleted.

Example: If row A1 is always less than or equal to row A2 → Eliminate A1.

2. **Column Dominance (Player B – Minimizer):**

If all elements in one column are greater than or equal to the corresponding elements in another column, then the dominated column can be deleted.

Example: If column B1 is always more than or equal to B2 → Eliminate B1.

3. Dominance by Linear Combination (Convex Dominance):

Sometimes a strategy is not dominated by a single row or column but is dominated by a combination (weighted average) of other strategies.

More complex but useful in larger matrices.

Purpose of Using Dominance:

- Reduce matrix size.
- Speed up calculations.
- Help focus only on relevant strategies.

Illustration 4.4.5

Consider the following payoff matrix for a two-person zero-sum game between Player A and Player B.

Apply the Principle of Dominance to simplify the given payoff matrix.

Identify and eliminate the dominated strategies (if any) for both players.

Determine the optimal strategies for both players and the value of the game, if possible.

	B1	B2	B3
A1	3	2	1
A2	5	3	2
A3	4	1	0

Solution

Compare A1 and A2:

$3 \leq 5$, $2 \leq 3$, $1 \leq 2$ → A1 is dominated by A2 → Eliminate A1

New reduced matrix:

	B1	B2	B3
A2	5	3	2
A3	4	1	0

Now compare A2 and A3:

- B1: 5 ≥ 4
- B2: 3 ≥ 1
- B3: 2 ≥ 0

A2 dominates A3 → Eliminate A3

Remaining Row: Only A2

Reduced matrix

	B1	B2	B3
A2	5	3	2

Now Player A has only one strategy, A2.

Column Dominance for Player B

Compare B1 and B2:

- A2: 5 ≥ 3 → B1 dominates B2 No
- A2: 5 ≥ 2 → B1 dominates B3 No
- A2: 3 ≥ 2 → B2 dominates B3 No

No column is strictly dominating, so no column can be removed.

Final Reduced Matrix:

	B1	B2	B3
A2	5	3	2

Pure Strategy Solution

Since Player A has only one strategy (A2) left, and Player B has 3 strategies, we check the minimum of row A2:

- A2 values: 5, 3, 2
- So min = 2

Solution:

- Player A's strategy: Play A2
- Player B's strategy: Play the column that minimizes Player A's gain, i.e., B3
- Value of the Game: 2

Player	Optimal Strategy
A	Pure Strategy A2
B	Pure Strategy B3
Value	2

Summarised Overview

This unit elaborates on key elements of game theory, starting with the payoff matrix, a tabular representation of outcomes for players based on their chosen strategies, particularly useful for two-person zero-sum games where each player is aware of all possible actions. The decision of a game involves selecting the best strategy to maximize gain or minimize loss, considering the payoff matrix, assumptions about opponents, and player rationality. Strategies are complete action plans, categorized as Pure Strategies (consistent action), Mixed Strategies (randomized choices with probabilities), Dominant Strategies (always yielding a better payoff regardless of opponent's action), and Dominated Strategies (always worse, and thus eliminated). Decision rules like Maximin (maximizing the minimum gain for the row player) and Minimax (minimizing the maximum loss for the column player) are also discussed. The concept of a Saddle Point is introduced as an equilibrium point in a payoff matrix where the value is the smallest in its row and largest in its column, indicating a stable solution and the value of the game. The Maximin-Minimax Principle states that if the maximin value equals the minimax value, a saddle point exists, leading to pure strategy solutions. Finally, the Principle of Dominance is presented as a simplification technique, allowing for the elimination of inferior (dominated) rows or columns (strategies) from the payoff matrix without affecting the game's outcome, thereby reducing complexity and facilitating solution finding.

Self-Assessment Question

1. What is the purpose of a payoff matrix in game theory, and what information does it convey to the players?
2. Define a saddle point in a payoff matrix
3. Describe the Maximin-Minimax Principle.
4. Explain the different types of strategy.
5. What do you mean by pay off matrix.

Assignments

1. What is the Principle of Dominance, and why is it useful in solving games?
2. Solve the following Payoff matrix, determine the optimal strategy and value of game

	B	
A	5	1
	3	4

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Unit 5

Value of the Game

Learning Outcomes

After completing this unit, the learner will be able to:

- Identify a Game with Pure Strategy
- Identify a Game with Mixed Strategy
- Calculate value of a game

Background

Game Theory is a mathematical tool used for analyzing competitive situations where the outcomes depend on the actions of two or more decision-makers (players). It is widely used in economics, political science, management, military strategy, and other fields to determine optimal decision-making strategies under conditions of conflict and cooperation. Pure strategies work when a clear, predictable outcome exists (a saddle point), while mixed strategies are necessary when players must remain unpredictable to secure an advantage or when no pure strategy solution exists.

Keywords

Pure Strategy, Mixed Strategy, Saddle Point

Discussion

Methods in Game Theory

1. Pure strategy game – saddle point
2. Mixed strategy games – no saddle point

Methods to find the optimal mixed strategies and value of the game

a. Algebraic method

b. Graphical method

c. Linear programming method

Situation	Type of Game	Method Used	Outcome
Saddle point exists	Pure Strategy	Saddle Point Method	Single optimal strategy
No saddle point (2×2)	Mixed Strategy	Algebraic Method	Probabilistic strategies
No saddle point (2×n or m×2)	Mixed Strategy	Graphical Method	Graph-based equilibrium
No saddle point (m×n)	Mixed Strategy	Linear Programming Method	LPP solution

4.5.1 Games with Pure Strategy

- Game in which players choose one specific strategy

A pure strategy game is a game in which players choose one specific strategy and use it consistently, without randomness. In other words, each player selects a single strategy and sticks to it throughout the game.

4.5.1.1 Characteristics of Pure Strategy Games:

1. Deterministic choice – no probability or mixing of strategies.
2. Players always play the same strategy every time.
3. Outcome is fixed once both players choose their strategies.
4. Saddle point exists in the payoff matrix.
5. The solution can be obtained through the Maximin–Minimax principle.

4.5.1.2 How to Identify a Game with Pure Strategy:

Find row minimums (for Player A – Maximizer).

Find column maximums (for Player B – Minimizer).

If max of row minimums = min of column maximums,

→ This common value is the saddle point.

→ The game has a pure strategy solution.

Illustration 4.5.1

Consider the following payoff matrix for a two-person zero-sum game between Player A and Player B

	B1	B2
A1	4	6
A2	2	3

- Find the Maximin and Minimax values of the game.
- Identify the optimal strategies for both players and the value of the game

Solution

- Row minimums:

A1 → min = 4

A2 → min = 2

→ Maximin = 4

- Column maximums:

B1 → max = 4

B2 → max = 6

→ Minimax = 4

Since Maximin = Minimax = 4,

The game has a pure strategy.

- Player A should play A1
- Player B should play B1
- Value of the Game = 4

Advantages of Pure Strategy Games:

- Easy to analyse and solve.
- No need to use complex probability or linear programming.
- Clear and predictable outcome.

Limitations:

- Not all games have pure strategy solutions.
- If Maximin ≠ Minimax → No saddle point → Need to use mixed strategies.

4.5.1.3 Algebraic Method

The Algebraic Method is used to find the optimal mixed strategies and the value of a 2×2 game when no saddle point exists (i.e., no pure strategy solution). This is typically applied to two-person zero-sum games.

When to Use:

- When the payoff matrix is 2 × 2
- And there is no saddle point (Maximin ≠ Minimax)

Steps of the Algebraic Method:

Step 1: Setup the Payoff Matrix

	B1	**B2**
A1	a11	a12
A2	a21	a22

Let Player A be the **row player** (maximizer), and Player B the **column player** (minimizer).

Step 2: Assume Mixed Strategy for Player A

Let:

- A plays A1 with probability p
- A plays A2 with probability 1 – p

Now compute expected payoffs to Player A for each of B's strategies:

- If B plays B1:

E1=a11p+a21(1−p)E_1 = a_{11}p + a_{21}(1 - p)E1=a11 p+a21(1−p)

- If B plays B2:

E2=a12p+a22(1−p)E_2 = a_{12}p + a_{22}(1 - p) E2=a12p+a22(1−p)

At equilibrium, A should be indifferent between what B plays:

$E1=E2E_1 = E_2E1=E2$

Solve for p to find A's optimal mixed strategy.

Assume Mixed Strategy for Player B

Let:

- B plays B1 with probability q
- B plays B2 with probability 1 – q

Now compute expected payoffs to Player A for each of A's strategies:

- If A plays A1:

$E1=a11q+a12(1-q)E_1 = a\_\{11\}q + a\_\{12\}(1 - q)$ $E1=a11q+a12(1-q)$

- If A plays A2:

$E2=a21q+a22(1-q)E_2 = a\_\{21\}q + a\_\{22\}(1 - q)E2=a21$ $q+a22(1-q)$

At equilibrium:

$E1=E2E_1 = E_2E1=E2$

Solve for q to find B's optimal mixed strategy.

Step 4: Find the Value of the Game

Substitute p or q into any expected payoff expression to compute the value (V) of the game.

4.5.2 Games with Mixed Strategies

45.2.1 Definition:

A mixed strategy is when a player does not rely on a single pure strategy, but instead assigns probabilities to two or more strategies and selects among them randomly. So, a game with mixed strategies is one where no saddle point exists, and players must randomize their choices to ensure optimal outcomes.

4.5.2.2 Why Use Mixed Strategies?

- Some games do not have a pure strategy solution (no saddle point).

- Players may want to be unpredictable to avoid being exploited.
- It helps in achieving a Nash equilibrium in such games. A Nash Equilibrium is a situation in a game where no player can improve their outcome by changing their strategy unilaterally, assuming the other player(s) keep their strategies unchanged.

4.5.2.3 Characteristics of Mixed Strategy Games:

Feature	Description
No Saddle Point	Maximin ≠ Minimax
Probabilistic Play	Players assign probabilities to each strategy
Solved using	Algebra, graphical method, odds method
Type of solution	Expected payoff or value of the game
Outcome	Not fixed – depends on probabilities

Illustration 4.5.2 (2 × 2 Game with No Saddle Point):

Find the value of the game.

	B1	B2
A1	5	1
A2	3	4

Solution

Step 1: Check for Saddle Point

- Row Minima: A1 → 1, A2 → 3 → Maximin = 3

Column Maxima: B1 → 5, B2 → 4 → Minimax = 4 → No saddle point, so use mixed strategy. (Algebraic Method).

Step 2: Player A's Mixed Strategy

Let Player A play A1 with probability p, and A2 with probability (1 – p).

Expected payoff if B plays B1: E = 5p + 3(1 - p) = 2p + 3

Expected payoff if B plays B2: E = p + 4(1 - p) = -3p + 4

Equating: 2p + 3 = -3p + 4 → 5p = 1 → p = 1/5 = 0.2

So, Player A's strategy: A1 = 0.2, A2 = 0.8

Step 3: Player B's Mixed Strategy

Let Player B play B1 with probability q, and B2 with probability (1 – q).

Expected payoff if A plays A1: E = 5q + 1(1 - q) = 4q + 1

Expected payoff if A plays A2: E = 3q + 4(1 - q) = -q + 4

Equating: 4q + 1 = -q + 4 → 5q = 3 → q = 3/5 = 0.6

So, Player B's strategy: B1 = 0.6, B2 = 0.4

Step 4: Value of the Game

V = 5p + 3(1 - p) = 5(0.2) + 3(0.8) = 1 + 2.4 = 3.4

Or V = p + 4(1 - p) = 0.2 + 3.2 = 3.4

Thus, the value of the game is 3.4

Final Answer:

Player A's strategy: (A1 = 0.2, A2 = 0.8)

Player B's strategy: (B1 = 0.6, B2 = 0.4)

Value of the Game: 3.4

Illustration 4.5.3

Consider the following payoff matrix for a two-person zero-sum game between Player A and Player B:

	B1	B2
A1	5	1
A2	3	4

- Find the Maximin and Minimax values of the game.
- Determine whether the game has a saddle point.
- If no saddle point exists, solve the game using the Algebraic Method to determine:
  - The optimal mixed strategies for both players (probabilities for each strategy).
  - The value of the game.

Solution

Step 1: No saddle point → use algebraic method.

Step 2: A's mixed strategy:

Let A play A1 with p, A2 with 1 − p

If B plays B1:

$E=5p+3(1-p)=2p+3$

If B plays B2:

$E=1p+4(1-p)=-3p+4$

Equating:

$2p+3= -3p+4 \Rightarrow 5p=1 \Rightarrow p=0.2$

So, A1 = 0.2, A2 = 0.8

Step 3: B's mixed strategy:

Let B play B1 with **q**, B2 with $\mathbf{1-q}$

If A plays A1:

$E=5q+1(1-q)=4q+1$

If A plays A2:

$E=3q+4(1-q)=-q+4$

Equating:

$4q+1=-q+4 \Rightarrow 5q=3 \Rightarrow q=0.6$

So, B1 = 0.6, B2 = 0.4

Step 4: Value of the Game:

$V=5(0.2)+3(0.8)=1+2.4=3.4$

Final Answer

Player	Mixed Strategy	Probabilities
A	(A1, A2)	(0.2, 0.8)
B	(B1, B2)	(0.6, 0.4)
Value of the Game		3.4

4.5.3 Advantages of Game Theory

i. Game theory offers valuable insights into complex situations where conflicts of interest are involved and may not be easily observed.

ii. It provides a structured approach for analysing decision-making in environments where the actions of one firm or player directly affect others (interdependence).

iii. In the case of two-person zero-sum games, it introduces a scientific and quantitative method to help players determine their optimal strategies.

4.5.4 Limitations of Game Theory

i. The assumption that all players are fully aware of their own payoffs and those of their opponents is often unrealistic.

ii. Solving games involving mixed strategies becomes highly complex, especially when the payoff matrix is large.

iii. Not all competitive scenarios can be effectively addressed or modelled using game theory techniques.

iv. Assumed that the two persons involved in the game have equal intelligence, in practical it may not so.

Summarised Overview

Game theory is a mathematical framework used to analyze strategic interactions between rational decision-makers, particularly in competitive or conflicting situations. In games with pure strategies, each player selects and consistently follows a specific strategy, and such games are solved using the Maximin–Minimax principle when a saddle point exists in the payoff matrix. However, when no saddle point is found (i.e., Maximin ≠ Minimax), players must use mixed strategies, assigning probabilities to their available choices to ensure optimal outcomes. These mixed strategy games are commonly solved using the algebraic method, especially in 2×2 games, by equating expected payoffs to find equilibrium probabilities for each strategy. While game theory provides a systematic and logical approach to decision-making, especially in two-person zero-sum games, it also has limitations, such as unrealistic assumptions of perfect information and increasing complexity in larger matrices.

Self-Assessment Question

1. If the Maximin ≠ Minimax, what type of strategy should be used?
2. What is meant by the value of the game?
3. In a 2×2 payoff matrix, what method is commonly used if there is no saddle point?
4. Explain the advantages and limitations of game theory.
5. Explain games with mixed strategies.

Assignments

1. Define pure strategy and mixed strategy in game theory. Give suitable examples.
2. Explain the steps involved in identifying a game with a pure strategy solution.
3. Use the algebraic method to solve the following game

	B1	B2
A1	2	4
A2	3	1

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