Mathematical Economics

UNDER GRADUATE PROGRAMME IN ECONOMICS

Self Learning Material

B21EC02DC

SREENARAYANAGURU
OPEN UNIVERSITY

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The State University for Education, Training and Research in Blended Format, Kerala

SREENARAYANAGURU OPEN UNIVERSITY

Vision

To increase access of potential learners of all categories to higher education, research and training, and ensure equity through delivery of high quality processes and outcomes fostering inclusive educational empowerment for social advancement.

Mission

To be benchmarked as a model for conservation and dissemination of knowledge and skill on blended and virtual mode in education, training and research for normal, continuing, and adult learners.

Pathway

Access and Quality define Equity.

Mathematics for Economics

Course Code: B21EC02DC

Semester - II

Bachelor of Arts
Economics
Self Learning Material

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Mathematics for Economics
Semester II

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DOCUMENTATION

Academic Committee

Dr. Sanathanan Velluva
Dr. Ranjith Mathew Abraham
Dr. Sindhu Pratap
Dr. Jisha K.K.
Dr. Jayasree Paul
Dr. Shiby M. Thomas
Dr. Suchithra Devi S.
Biji Abraham
Dr. Priyesh C.A.
Dr. Swathy Varma
Dr. Chacko Jose P.
Dr. Muneer Babu. M
Sandhu John Sajan
Dr. Rajeev S.R.
Dr. Ratheesh C.

Development of the Content

Dr. G. Hari Prakash, Nijil Jacobi, Dr. Sindhu M., Dr. Sanoop M.S.

Review

Content : Dr. Christabell P.J.
Format : Dr. I.G. Shibi
Linguistics : Dr. Anitha C.S.

Edit

Dr. Christabell P.J.

Scrutiny

Dr. Suraj G., Dr. Suchitra K.R., Dr. Lakshmy Ravi, Yedu T. Dharan, Muneer K., Soumya V.D., Hima Chandran J.

Co-ordination

Dr. I.G. Shibi and Team SLM

Design Control

Azeem Babu T.A.

Production

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MESSAGE FROM VICE CHANCELLOR

Dear

I greet all of you with deep delight and great excitement. I welcome you to the Sreenarayanaguru Open University.

Sreenarayanaguru Open University was established in September 2020 as a state initiative for fostering higher education in open and distance mode. We shaped our dreams through a pathway defined by a dictum 'access and quality define equity'. It provides all reasons to us for the celebration of quality in the process of education. I am overwhelmed to let you know that we have resolved not to become ourselves a reason or cause a reason for the dissemination of inferior education. It sets the pace as well as the destination. The name of the University centres around the aura of Sreenarayanaguru, the great renaissance thinker of modern India. His name is a reminder for us to ensure quality in the delivery of all academic endeavours.

Sreenarayanaguru Open University rests on the practical framework of the popularly known "blended format". Learner on distance mode obviously has limitations in getting exposed to the full potential of classroom learning experience. Our pedagogical basket has three entities viz Self Learning Material, Classroom Counselling and Virtual modes. This combination is expected to provide high voltage in learning as well as teaching experiences. Care has been taken to ensure quality endeavours across all the entities.

The university is committed to provide you stimulating learning experience. The UG programme in Economics is designed at par with that of the quality academic programme of the state universities in the country. The latest methodologies of exposition of economic ideas and concepts have been well embedded in to the curriculum. It creates interest among students for a deeper understanding of the discipline. The curriculum covers the theories and the historical evidences as well. Due emphasis is given to Indian experiences in the economic transformation. This would help learners in preparing for the competitive examinations, in case they opt for. We expect that the students on successful completion of the programme will be equipped to handle the key areas of the discipline. We assure you that the university student support services will closely stay with you for the redressal of your grievances during your studentship.

Feel free to write to us about anything that you feel relevant regarding the academic programme.

Wish you the best.

Regards,
Dr. P.M. Mubarak Pasha 01.12.2023

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1

BLOCK

Basic Mathematical Concepts: Arithmetic and Algebra

UNIT 1

Number System and Arithmetic Operations

Learning Outcomes

After completing this unit, the learner will be able to:

- demonstrate proficiency in number system
- perform accurate arithmetic operations
- apply mathematical concepts to economic scenarios
- critically evaluate and communicate results

Prerequisites

Imagine, you are living in a peaceful village nestled at the foot of a grand mountain. In this village, each person had a unique job, and their livelihoods depended on trading goods and services with one another. But there was a problem - the villagers had no way to count or measure what they traded. Chaos reigned as disagreements arose over who owed what to whom.

One day, a wise traveller arrived in the village. This traveller possessed a magical gift - the knowledge of numbers and arithmetic operations. With a piece of chalk and a board, the traveller began to draw strange symbols that looked like this: 1, 2, 3... and so on. The villagers gathered around, curious and eager to learn.

The traveller taught them how to add, subtract, multiply, and divide these numbers. Suddenly, the villagers could calculate the worth of their goods and services accurately. Trade flourished, and prosperity returned to the village.

As time passed, the villagers began to explore the world of numbers even further. They discovered not only whole numbers but also fractions, decimals, and even strange creatures known as irrational numbers. Each of these numbers had its unique characteristics and uses.

With their new found knowledge, the villagers were able to understand economic concepts, analyse data, and make informed decisions. They could calculate interest rates, inflation, and budget their resources wisely. The once-confused villagers had become skilled economists, using numbers as their tools to navigate the complexities of their world.

Keywords

Real numbers, Closure, Associative, Identity, Addition, Multiplication

Discussion

1.1.1 Introduction to Mathematics for economics

Mathematics is the language of economics. Studying mathematics is crucial for learners of Economics as it helps to build a strong foundation in analytical thinking and problem-solving. If you are studying the concept of supply and demand, mathematics helps you to create equations that represent the relationship between demand and supply. Learning matrices helps to make sense of real-life economic situations. For example, think of a grocery store manager who wants to figure out the best way to stock their shelves with various products while staying within a budget. Matrices can help them organise and analyse data, making it easier to make smart decisions about what to buy and how much, just like you do when you shop for groceries. By using algebra, you can calculate how changes in prices and quantities affect the market equilibrium. This mathematical understanding is vital for making effective economic policies and strategies.

Moreover, in financial economics, mathematics plays a significant role. Calculating interest rates, risk assessments, and making investment decisions all rely on mathematical techniques. For instance, when analysing the returns on an investment, you use concepts like compound interest and probability to make wise choices. In total, mathematics equips learners of economics with the tools they need to interpret data, build models, and make informed decisions, making it a fundamental part of their education. By mastering these simple mathematical concepts like algebra, matrices, calculus etc you'll be better prepared for academic success and career opportunities within the field of economics. You will also find that mathematics can be engaging and rewarding, even if you've struggled with it in the past.

1.1.2 Real Number System

To discuss and study any subject everyone agrees on a “common language”. In a sense of language study of mathematics ensure that all understand exactly the same thing when we see a given symbol. Most of the scientist use numbers to measure, compare and classify their information. For example, the Saffir- Simpson Hurricane Scale uses symbols like 1, 2, 3, 4 and 5 to classify hurricanes according to the amount of damage they are likely to cause.

The numbers 1, 2, 3 ... used for counting, form the set of integers Z, Z comes from German word "Zahlen" which means “to count". These numbers only indicate the profit earned by a company. To indicate loss for company, need to introduce negative integer. It is denoted by z^{-}. When there is no profit (and no loss) we introduce 0. The positive integers are denoted by z+. Hence the set of all integers include positive integers, zero and negative integers. The set of non-negative integers defines the set of whole numbers and denoted by W.

$$W = z^{+} \cup \{0\} \text{ and } z = z^{+} \cup \{0\} \cup z^{-1}$$

$$z^{+} = \{1, 2, 3, 4 \ldots\ldots.\}$$

$$z^{-} = \{-1, -2, -3, \ldots\ldots\}$$

W = {0, 1, 2, 3, …….}

Indeed, the representation of numbers as integers alone can be insufficient in many practical scenarios. For instance, when we need to distribute a jackfruit among three people, each person would receive one-third of the fruit. Similarly, when dividing three oranges among five individuals, they would each receive three-fifths of the oranges. In these cases, we introduce new symbols like $\frac{1}{3}$ and $\frac{3}{5}$, which take the form of $\frac{p}{q}$, where both p and q are integers. These fractional representations allow us to precisely describe and work with quantities that cannot be expressed as whole numbers, offering a more accurate and versatile tool for various real-world situations. Any number which can be expressed in the form $\frac{p}{q}$ where p and q are integers is called rational numbers.

Examples of rational numbers.

i. $\frac{7}{8}$, $\frac{6}{5}$, $\frac{-6}{5}$, $\frac{2}{3}$, -1, $\frac{-3}{4}$,……

ii. 5, $\frac{2}{3}$, $\frac{5}{7}$, $\frac{-1}{8}$, -6, …….

iii. 0, -20, $\frac{-2}{7}$, $\frac{3}{7}$, $\frac{5}{7}$, -5,…….

Note. 1. Every integer is not natural number. Example. -8, -3

2. Every natural number is an integer.

3. Every integer is a rational number.

Hence, we have seen three fundamental number systems, the natural number system, the system of integers and the rational number system. The rational number system is inadequate for sophisticated uses of mathematics since there exists numbers that are not rational. Hence there are numbers which cannot be expressed in the form of the ratio of two integers. These are called irrational numbers. Example $\sqrt{2}$ is an irrational number. During 476 - 550 AD Greek Genius Archimedes expressed digits in the expansion of decimals.

Example of decimal form of numbers.

i. $\frac{1}{10}$ = 0.1000…

ii. $\frac{3}{4}$ = 0.75

iii. $\frac{-4}{5}$ = -0.8

iv. $\frac{2}{3}$ = 0.666…

v. $\frac{1}{11}$ = 0.090909

It's really important to know about equations and logics for doing any mathematical problems. Equations are like the road signs that show how numbers are related. Let’s go through the basic rules and equations before moving in to the topic Arithmetic operations.

I. Symbols, Formulae, Hints and results

(1) N - Natural numbers or Positive integers

(2) Z^+ - Positive integers

(3) Z^- - Negative integers

(4) Z - Integers $Z^{-1} \cup \{0\} \cup Z^+$

(5) W - Whole numbers (non-negative integers) = $Z^+ \cup \{0\}$

(6) Q - Rational

(7) S - Irrationals

(8) R - Real numbers

(9) $\in$ - an element of

(10) $\exists$ - there exists

(11) $\forall$ - for all values of

(12) ș - such that

(13) $\Longrightarrow$ - implies

(14) $\not\Rightarrow$ - does not implies

(15) @- at the rate of

(16) $<$ less than, $\leq$ less than or equal to

(17) $>$ greater than, $\geq$ greater than or equal to

(18) $\subseteq$ subset of

(19) $\emptyset$ - null set

(20) LHS – Left Hand Side

(21) RHS – Right Hand Side

II Working rule

1. Multiplication and division do before addition and subtraction.
2. Common factor of numerator and denominator can be cancelled.
3. Addition, subtraction or comparison of two or more fractions can be done by making the denominator same (use LCM).
4. Percentage. To convert a fraction to percentage made the denominator 100.
5. Perform all operations with in parenthesis.
6. If a and b real numbers.
   (i) $a\ (b + c) = ab + ac$
   (ii) $ab = ac \implies b = c$
   (iii) $a + (b + c) = (a + b) + c$
   (iv) $a.(bc) = (ab).c$
   (v) $a + b = b + a$
   (vi) $ab = ba$
7. Dividend = (Divisor $\times$Quotient) + Remainder

8. Product of two numbers. = G.C.M. × L.C.M

9. Average = $\frac{\text{sum}}{\text{number}}$

10. Ratio of a and b = $\frac{a}{b} = \frac{ma}{mb}$

11. Percentage: Percentage of a on b is $\frac{a}{b} \times 100$

12. Simple Interest (S.I.) = $\frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100}$

13. Amount = Principal + Interest

14. Principal = $\frac{\text{Interst} \times 100}{\text{Rate} \times \text{Time}}$

15. Rate % = $\frac{\text{Interst} \times 100}{\text{Principal} \times \text{Time}}$

16. Compound Interest = P $\left[(1 + \frac{r}{100})^n - 1\right]$ where P is the principal, r - rate, n - period

III. Results

1. Every real number except '0' has a reciprocal.
2. Product of a number and its reciprocal is one
3. The value will not change by multiplying the numerator and denominator by any non-zero numbers.
4. The value will not change by dividing numerator and denominator by same non-zero quantity.
5. $|a| = a$ if $a \geq 0$ and $-a$ if $a < 0$

1.1.3 Arithmetic Operations

In the study of algebra, arithmetic vocabularies are introduced. In the multiplication of two real numbers, each number is called the factor, and result is called the product.
Usually, multiplication is indicated by a dot '.' or parenthesis.

(9)(5) = 45. Here 9 and 5 are factors and 45 is the product. The symbol '×' is avoided since it may confuse with other algebraic symbols.
Consider the division terminology ÷
When 54 ÷ 6= 9. Hence 54 is the divident, 6 is the divisor and 9 is the quotient. Instead of 54 ÷ 6 we usually write $\frac{54}{6}$. By interchanging the factors in multiplication product value is unaltered (9)(5) = 45 = (5)(9).
But changing the dividend and divisor quotient may altered. 54÷6 = 9 ≠ 6 ÷ 54

Remark. 1. Addition can be performed in any order.

2. Multiplication can be performed in any order.

3. Subtraction and division must be performed in given order.

Example. (7)(8) = 56 = (8)(7)

$7 + 8 \neq 8 - 7$

$\frac{7}{8} \neq \frac{8}{7}$

Example. $\frac{30}{10} - \frac{40}{10} + (7)(5) + 4$

Ans. $\frac{30}{10} - \frac{40}{10} + (7)(5) + 4$

$= 3 - 4 + 35 + 4 = 38$

Remark 2. Always do multiplication and division before addition and subtraction.

Example. $7 + (8)(4) + \frac{15}{3} - 2$

$= 7 + 32 + 5 - 2 = 42$

Illustration 1.1.1

A rental car costs F dollars per day in fixed charges and b dollars per kilometre. How much dollars must a customer pay to drive x kilometers in 1 day?

Solution

Fixed Cost per day = F dollars

Cost per kilometre = b dollars

Cost for x kilometers = $x \times b$

Total cost = Fixed Cost per day + Cost for x kilometers $= F + x \times b$

Illustration 1.1.2

A company has fixed cost of F dollars per year and variable cost of C dollars per unit (i) Find an expression for the total cost per unit (total average cost) incurred by the company if it produces x units in one year?

(ii) Also find the cost of the 20 unit produced by the company in two years when the fixed annual cost is 100 dollars and unit cost is 5 dollars.

Solution

(i) Fixed cost of the company = F

Cost of x units = cx

$\therefore$ Total cost = $F + cx$

(ii) Cost of 20 units in two years = $F + cx$

$= 100 + 5 \times 20$

= 200 dollars

Illustration 1.1.3

A square tin plate 18 cm wide is to be made into an open box by cutting out equally sized square of (i) width x units in each corner and then folding over the edges. Find the volume of the resulting box. (Draw a figure). (ii) of width 6 units in each corner and then folding over the edge to resulting box.

Solution

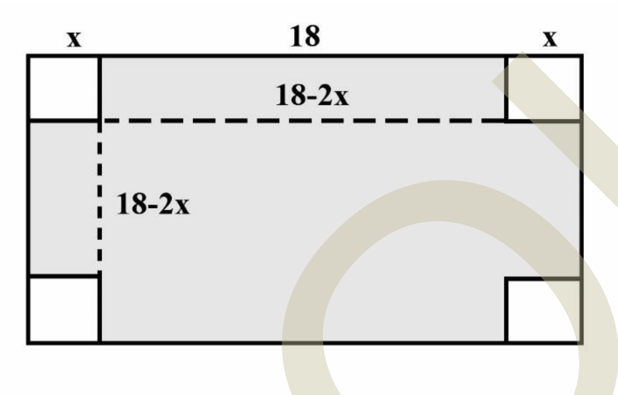


Fig (i)

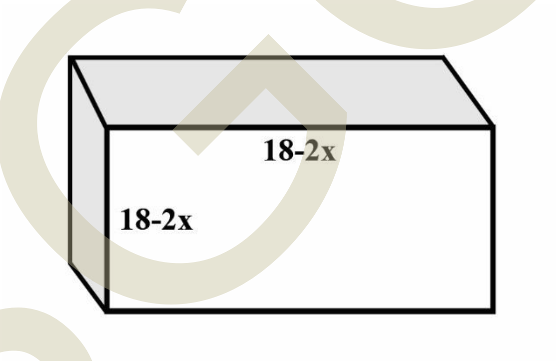


Fig (ii)

By cutting each corner a square of side x every remaining cutting portion = 18 − x. When the edges are folded upwards as shown in figure (i) we will get an open rectangular box with sides $18 - 2x$ and $18 - 2x$ having height x.

The volume of this rectangular box = $(18 - 2x)(18 - 2x) \times x$

$= [18^2 - 2 \times 18 \times 2x + (2x)^2] \times x$

$= [324 - 72x + 4x^2] \times x$

$= [324x - 72x^2 + 4x^3]$

$= 4x[81 - 18x + x^2]$

$= 4x \times (9 - x)^2$

(ii) It is given $x = 6$. As above when its corners are cutting out and folded upwards the side edges as shown in figure (ii) we will get an open cube with side 6 units.

Its volume = 6 × 6 × 6 = 6^3 = 216 cubic unit

If we put x = 6 in previous problem

Volume $= 4 \times x \times (9 - x)^2$

$= 4 \times 6 \times (-3)^2$

= 24 × 9

= 216 cubic units

Illustration 1.1.4

A textile owner purchases 200m material for shirt, 300m for saree, and 150m for pant. If the cost of goods per meter is 50, 150 and 200 respectively, what is the total expenditure?

Solution:

Ans. Total expenditure = $200 \times 50 + 300 \times 150 + 150 \times 200$

$= 10000 + 45000 + 30000$

= Rs.85000

1.1.4 Properties of Arithmetic Operations $*$ on a set

1. **Closure property**

   If $*$ is an operation defined on a set S then S is closed with respect to $*$

   If $a, b \in S \Rightarrow a * b \in S$

   Example.1. Let N be the given set and the operation $*$ is defined as the ordinary addition. We know that the sum of any two natural numbers is also a natural number.

   i.e., for $a, b \in N \Rightarrow a + b \in N$

   i.e., N is closed with respect to '+'.

   Example.2. Let N be the given set and the operation $*$ is defined as the ordinary multiplication. We know the product of two natural numbers is a natural number.

   i.e., $a, b \in N \Rightarrow a * b \in N$

   $\therefore N$ is closed with respect to multiplication.

   Example.3. Let Q be the set of rational numbers. Define an operation $*$ as follows

   For any $a, b \in Q, a * b = \sqrt{a.b}$

   Here $a * b \notin Q$ since when a = 1, b = 2, $a * b = \sqrt{1 \times 2} = \sqrt{2} = \notin$ Q

   Hence the defined $*$ is not closed in the given set Q.

2. **Associative property**

   Let S be any set of numbers and $*$ is an operation defined on S. Then $*$ is associative on S if $a * (b * c) = (a * b) * c \quad \forall\, a, b, c \in S$

Example.1. Let N be the set of natural numbers with an operation $*$ is the ordinary addition defined on N. Then we have

$$a + (b + c) = (a + b) + c \qquad \forall\, a, b, c \in N$$

Hence '+' is associative in N.

Example.2. Consider the set of natural number N in which an operation $*$ is the multiplication defined on N. Then,

$$a * (b * c) = (a * b) * c \qquad \forall\, a, b, c \in N$$

$\therefore$ multiplicative is associative in N.

3. Existence of Identity

Let S be any set in which an operation $*$ is defined. If $\exists$ an element, e $\in$ S such that a $*$ e = a = e $*$ a then e is called the identity in S.

Example. 1. In the set of integers if the operation defined is ordinary addition $\exists\ 0 \in$ Z such that $a + 0 = a = 0 + a \quad \forall\, a \in Z$

Example.2. In the set of natural numbers N consider the operation $*$ as ordinary addition defined on N. The identity element under the operation addition is 0 which does not belong to the set N. So, in the set N under the operation addition identity does not exist.

Example.3. Consider the set of natural numbers N in which the operation $*$ is ordinary multiplication. Then for any $a \in N,\ \exists\ 1 \in N$ such that

$a * 1 = a = 1 * a$

Hence identity exists in N when the operation is ordinary multiplication.

4. Existence of inverse

Let S be any set in which an operation $*$ is defined with identity element e. Then for any $a \in S,\ \exists\ a' \in S$ such that $a * a' = e = a' * a$ Inverse exists in S with respect to operation $*$.

Example. 1. Let Z be the set integers. Let $*$ is ordinary addition defined on Z. Then 0 $\in$ Z is the additive identity.

For any $a \in Z,\ \exists - a \in Z$ such that $a + (-a) = 0 = (-a) + a$. Hence inverse exists in Z with respect to ordinary addition.

Example.2. Let Q be the set of non zero rational and an operation $*$ be ordinary multiplication defined on Q. Then 1 is the identity in Q. For any a $\in$ Q, $\exists\ \frac{1}{a} \in$ Q such that a $* \frac{1}{a} = 1 = \frac{1}{a} * a$. Hence inverse with respect to ordinary multiplication exists in Q.

5. **Commutative Property**

Let S be any set and $*$ is an operation defined on S. Then $*$ is commutative on S if for any $a, b \in S,\ a * b = b * a$

Example. 1. Let N be the set of natural numbers in which the operation $*$ is the ordinary addition. Then for any $a, b \in N,\ a + b = b + a$

Ordinary addition in any set is commutative. Here '+' is commutative in N.

Example 2. Let N be the set of natural numbers in which the operation $*$ is defined as ordinary multiplication. We know ordinary multiplication in any set is commutative. Here for any $a, b \in N,\ a * b = b * a$

Hence ordinary multiplication is commutative in N.

6. **Distributive Property**

Let S be any set. $*$ and $\bullet$ are two operations defined on S. Then $*$ is distributive over $\bullet$ on S if $a * (b \cdot c) = (a * b) \cdot (a * c) \quad \forall\, a, b, c \in S$

Example. 1. Let N be the set of natural numbers. Define ordinary multiplication $\times$ and ordinary addition $+$ defined on S.

Then $\times$ is distributive over $+$ in N if

$\mathrm{a} \times (\mathrm{b} + \mathrm{c}) = (\mathrm{a} \times \mathrm{b}) + (\mathrm{a} \times \mathrm{c}) \quad \forall\, \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{N}$

Illustration 1.1.5

Example.3. Consider the set of rational numbers Q in which an operation $*$ is defined such that a * b = $\frac{a-b}{2}$. Prove that $*$ is not associative.

Solution

Given, a * b = $\frac{a-b}{2}$

If $a * (b * c) = (a * b) * c$, then it is associative.

Here LHS = $a * (b * c)$ and RHS = $(a * b) * c$

According to the operation defined,

$$\text{LHS} = a * (b * c) = a * \frac{\mathrm{b}-c}{2}$$

$$= \frac{a - \frac{\mathrm{b}-c}{2}}{2}$$

$$= \frac{\frac{2a-b+c}{2}}{2}$$

$$= \frac{2a-\mathrm{b}+c}{4}$$

$$\text{RHS} = (a * b) * c = \frac{\mathrm{a}-b}{2} * c$$

$$= \frac{\frac{\mathrm{a}-\mathrm{b}}{2} - c}{2}$$

$$= \frac{\frac{a-b-2c}{2}}{2}$$

$$= \frac{a-\mathrm{b}-2c}{4}$$

Let us check the above property by putting values for a, b and c.

Let a =2, b =1, c =0.

Then, $\text{LHS} = \frac{2a-\mathrm{b}+c}{4} = \frac{2\times2-1+0}{4} = \frac{4-1}{4} = \frac{3}{4}$

$$\text{RHS} = \frac{a-\mathrm{b}-2c}{4} = \frac{2-1-2\times0}{4} = \frac{2-1-0}{4} = \frac{1}{4}$$

When we check, LHS ≠ RHS

$$\therefore (a*b)*c \neq a*(b*c)$$

Here the defined * is not associative in Q.

Illustration 1.1.6

Let Q be the set of rational numbers in which an operation $*$ is defined follows:

$a*b = a - \frac{b}{2}$. Check whether operation $*$ is commutative.

Solution

Given $a*b = a - \frac{b}{2}$

If , $a*b = b*a$, then it is communicative.

Here LHS = $a*b$ and RHS = $b*a$

According to the operation defined,

$$\text{LHS} = a*b = a - \frac{b}{2}$$

$$= \frac{2a-b}{2}$$

$$\text{RHS} = b*a = b - \frac{a}{2}$$

$$= \frac{2b-a}{2}$$

Let us check the above property by putting values for a and b

Let a = 2 and b =1,

Then $\text{LHS} = \frac{2\times2-1}{2} = \frac{4-1}{2} = \frac{3}{2} = 1.5$

$$\text{RHS} = \frac{2\times1-2}{2} = \frac{2-2}{2} = \frac{0}{2} = 0$$

When we check, LHS ≠ RHS

$$\therefore a*b \neq b*a$$

Here the defined * is not communicative in Q.

Illustration 1.1.7

Let Q be the set of rational numbers. Define two operators $*$ and $\bullet$ on Q such that $x * y = x + \frac{y}{2}$, $x \bullet y = \frac{xy}{2}$ $\forall\ x, y \in Q$. Prove that the operation $*$ is not distributive over.

Solution

Given, $x * y = x + \frac{y}{2}$ and $x \bullet y = \frac{xy}{2}$

If $a * (b \bullet c) = (a * b) \bullet (a * c)$, then it is distributive.

Here LHS = $a * (b \bullet c)$ and RHS = $(a * b) \bullet (a * c)$

According to the operation defined,

$$\text{LHS} = a * (b \bullet c) = a * \frac{bc}{2}$$

$$= a + \frac{\frac{bc}{2}}{2}$$

$$= a + \frac{bc}{4} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

$$\text{RHS} = (a * b) \bullet (a * c) = \left(a + \frac{b}{2}\right) \bullet \left(a + \frac{c}{2}\right)$$

$$= \frac{\left(a + \frac{b}{2}\right)\left(a + \frac{c}{2}\right)}{2}$$

$$= \frac{(2a + b)(2a + c)}{8} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

Let us check the above property by putting values for a, b and c.

Let a = 3, b = 2, c = 1

Then (1) becomes, $a + \frac{bc}{4} = 3 + \frac{2\times1}{4} = 3 + \frac{2}{4} = \frac{12+2}{4} = \frac{14}{4} = 3.5$

(2) becomes, $\frac{(2a + b)(2a + c)}{8} = \frac{(2\times3 + 2)(2\times3 + 1)}{8} = \frac{(8)(7)}{8} = \frac{56}{8} = 7$

When we check, LHS ≠ RHS

$a * (b \bullet c) \neq (a * b) \bullet (a * c)$

Hence the defined operation is not distributive.

Recap

- Real numbers include rational and irrational numbers
- Rational numbers - expressed as the ratio of two integers
- Irrational numbers - cannot be expressed as a simple fraction of two integers
- Real numbers can be positive, negative, or zero
- Integers include all positive and negative whole numbers as well as zero
- Whole numbers include all non-negative integers (0, 1, 2, 3, ...).
- Closure Property - if an operation '*' is defined on a set S, then for all a and b in S, the result of a * b is also in S
- Associative Property - for all a, b, and c in S, the operation '*' is associative if (a * b) * c = a * (b * c)
- Commutative Property - for all a and b in S, the operation '*' is commutative if a * b = b * a
- Existence of inverse - if a is an element of S, the existence of inverse is defined by $a * a' = a' * a$ = e

Objective Questions

1. What is the square root of 144?
2. Which set of numbers includes all real numbers?
3. What is the inverse of 7 under multiplication?
4. What property states that a * b = b * a for all real numbers a and b?
5. Which property states that (a * b) * c = a * (b * c) for all real numbers a, b, and c?
6. What is the identity element for addition in the real numbers?
7. If you multiply any real number by 1, what is the result?
8. What will be the sum of any two natural numbers?
9. What is the order of operations implied when multiplication, subtraction, and division are given in an equation?
10. What method is commonly employed to facilitate addition, subtraction, or comparison of two or more fractions?

Answers

1. 12
2. Complex numbers
3. $\frac{1}{7}$
4. Commutative Property
5. Associative Property
6. 0
7. The original number
8. A natural number
9. Division, Multiplication and Subtraction
10. By making their denominators the same

Assignments

1. Differentiate between natural numbers, whole numbers, integers, and real numbers. Provide examples for each category.
2. Define and explain the commutative property of addition and multiplication. Provide examples to illustrate both properties.
3. Discuss the associative property of addition and multiplication. How does it affect the grouping of numbers in mathematical operations?
4. A square tin plate 24 cm wide is to be made into an open box by cutting out equally sized squares of (i) width y units in each corner and then folding over the edges. Calculate the volume of the resulting box. (Include a diagram) (ii) of width 8 units in each corner and then folding over the edges to create the resulting box.

Suggested Readings

1. "Discrete Mathematics and its Applications" by Kenneth H. Rosen, McGraw-Hill publications, 2017
2. "A First Course in Discrete Mathematics" by Ian Anderson, Springer publications, 2000
3. "Number Theory" by George E. Andrews, Dover publications,1994

References

1. "Elementary Number Theory" by David M. Burton, McGraw-Hill publications, 2017
2. "Number Systems and the Foundations of Analysis" by Elliott Mendelson, Dover Publications, 2008
3. "Introduction to the Theory of Numbers" by Ivan Niven and Herbert S. Zuckerman, John Wiley & Sons publications, 1991

UNIT 2

Exponents - Factorization- Fractions - Proportions - Ratios

Learning Outcomes

After completing this unit, the learner will be able to:

- understand exponents and their applications
- perform factorization and simplification
- apply fractions, proportions, and ratios in economic analysis
- utilize mathematical techniques for economic decision-making

Prerequisites

In a little town named Baker's Haven, there was a lovely bakery owned by a kind baker named Mrs. Jincy. She made the yummiest cakes, pies, and pastries that everyone in town loved. One day, lots of people wanted Mrs. Jincy tasty lemonade. She used a special method of cooking using ratios and proportions. She needs 2 cups of water for every 1 cup of lemon juice and 1/2 cup of sugar to make it taste perfect. This ratio (2:1:0.5) ensures a balanced and tasty drink. She makes more lemonade, she uses proportions: For 4 cups of lemon juice, you'll need 8 cups of water (4 cups × 2) and 2 cups of sugar (4 cups × 0.5). Using the ratio (2:1:0.5) as a trick helps make delicious lemonade. As Mrs. Jincy's bakery got busier, she had to be careful with her ingredients. She used fractions, like parts of a whole, to measure things just right. For example, if a recipe needed half a cup of sugar or three-fourths of a teaspoon of vanilla, she used these fractions for cooking. As the bakery flourished, they faced the challenge of scheduling different baking batches. Sometimes cakes took 2 hours to bake, and cookies took 1 hour.

To figure out when both would be ready at the same time, Mrs. Jincy used LCM, which stands for Least Common Multiple. Imagine if cakes needed to be baked every 2 hours and cookies every 1 hour. The LCM of 2 and 1 is 2. This means every 2 hours, both cakes and cookies would be ready. So, Mrs. Jincy used this trick to plan her baking schedule. If Mrs. Jincy had 18 cups of flour and 24 cups of sugar, she needed to figure out the largest amount of cupcakes she could make using both resources equally. The HCF of 18 and 24 is 6. This means she could make 6 batches of cupcakes, using 3 cups of flour and 4 cups of sugar for each batch, and both flour and sugar would be used up evenly. Now, imagine Mrs. Jincy needed 3 eggs for each cake, and she had to make 5 cakes. Instead of counting 3 eggs 5 times (which is 3 + 3 + 3 + 3 + 3), she used exponents. It became 3 × 5, meaning 3 multiplied by itself 5 times (3 × 3 × 3 × 3 × 3), which equals 243 eggs. This trick helped them calculate ingredients quickly and accurately, making sure every cake was just perfect.

In the end, Mrs. Jincy and her bakers in Baker's Haven showed us that math isn't just for school. By using ratios, proportions, fractions, exponents, factorization, LCM, and HCF, they made amazing treats. These math tricks helped them bake the right amount, at the right time, and with the perfect taste. Their story teaches us that math is everywhere, even in the sweetest moments of life.

Keywords

Prime numbers, LCM, HCF, Fractions, Factorisation, Ratio, Proportions

Discussion

1.2.1 Exponents

Certain arithmetic operations which are long and laborious can be carried out simply and accurately by using laws of indices. Suppose we have to multiply 10 seven times. We can write it as 10^7. Here 10 is the base and 7 is the power. In doing the operation (5 × 5 × 5 × 5) (7 × 7 × 7) we express it as $5^4 \times 7^3$.

When the base is same, laws of indices are used. Consider $(4 \times 4 \times 4 \times 4 \times 4) \times (4 \times 4) = 4^5 \times 4^2$. In such cases (with same base), add the power for the base. Here, it can be written as 4^7.

First law of exponent

$a^m \times a^n = a^{m+n}$ where a is the base and m and n are powers.

Second law of exponent

$\frac{a^m}{a^n} = a^{m-n}$ where a is the base and m and n are powers.

Third law of the exponent

$(a^m)^n = a^{mn}$, where a is the base and m and n are powers.

Hence the result follows.

(1) $\frac{a^n}{a^n} = a^{n-n} = a^0 = 1$

(2) $a^{-m} = a^{0-m} = \frac{a^0}{a^m} = \frac{1}{a^m}$

(3) $a^{-1} = \frac{1}{a}$

Illustration 1.2.1

Simplify

1. $10^3 \times 10^6$
2. $\frac{7^9}{7^3}$
3. $\left(\frac{8}{5}\right)^3$
4. $(7^6)^5$
5. $\frac{8^5 \times 7^6}{8^2 \times 7^3}$
6. $\frac{3^3 \times 3^5}{3^2 \times 3^6}$

Solution

1. $10^3 \times 10^6 = 10^{3+6} = 10^9$

2. $\frac{7^9}{7^3} = 7^{9-3} = 7^6$

3. $\left(\frac{8}{5}\right)^3 = \frac{8^3}{5^3} = \frac{(2^3)^3}{5^3} = \frac{2^9}{5^3}$

4. $(7^6)^5 = 7^{6\times5} = 7^{30}$

5. $\dfrac{8^5 \times 7^6}{8^2 \times 7^3} = 8^{5-2} \times 7^{6-3}$

$= 8^3 \times 7^3$

$= (2^3)^3 \times 7^3$

$= 2^9 \times 7^3$

6. $\dfrac{3^3 \times 3^5}{3^2 \times 3^6} = \dfrac{3^{3+5}}{3^{2+6}}$

$= \dfrac{3^8}{3^8} = 3^0 = 1$

Illustration 1.2.2

Find (1) $\sqrt[2]{25}$ (2) $5\sqrt[3]{81}$ (3) $10\ \sqrt[31]{1000}$

Solution

(1) $\sqrt[2]{25} = (25)^{\frac{1}{2}}$

$= (5^2)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}}$

$= 5$

(2) $5\ \sqrt[3]{81} = 5\,.\,(81)^{\frac{1}{3}}$

$= 5\ (3^4)^{\frac{1}{3}}$

$= 5(3^{\frac{4}{3}})$

(3) $10\ \sqrt[3]{1000} = 10\ (1000)^{\frac{1}{3}}$

$= 10 \times (10^3)^{\frac{1}{3}}$

$= 10 \times 10 = 10^2 = 100$

Illustration 1.2.3

If $x^y = y^x$ and $x = 2y$ show that $y = 2$

Solution

Given $x = 2y$

$x^y = y^x$

$\Longrightarrow (2y)^y = (y)^{2y}$

$\Longrightarrow 2^y\,.\,y^y = y^2.\,y^y$

$\Longrightarrow 2^y = y^2$

$\Longrightarrow y = 2$ Hence the result.

Illustration 1.2.4

If $m = a^x$; $n = a^y$ and $m^y . n^x = a^{\frac{2}{z}}$. Show that $xyz = 1$.

Solution

$m = a^x \Longrightarrow m^y = (a^x)^y = a^{xy}$

$n = a^y \Longrightarrow n^x = (a^y)^x = a^{xy}$, then $m^y n^x = a^{xy}\, a^{xy} = a^{2xy}$

But given $m^y . n^x = a^{\frac{2}{z}}$

$\therefore 2xy = \frac{2}{z} \Longrightarrow xyz = 1$

Illustration 1.2.5

If $a^x = b^y = c^z$ and $b^2 = ac$, show that $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}$.

Solution

Given $a^x = b^y = c^z$

Let $a^x = b^y = c^z$ = k, then $a = k^{\frac{1}{x}}$, $b = k^{\frac{1}{y}}$, $c = k^{\frac{1}{z}}$

$b^2 = ac \Longrightarrow k^{\frac{2}{y}} = k^{\frac{1}{x}} . k^{\frac{1}{z}}$

$\Longrightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

1.2.2 Factorisation

Factorization is a fundamental concept in mathematics that involves breaking down a number or an algebraic expression into its constituent factors, which are smaller numbers or expressions.

Factors of a number are the numbers that divide the original number exactly without leaving a remainder. For example, the factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.
In algebra, factorization involves expressing algebraic expressions as a product of simpler algebraic expressions or factors.

There are various methods to factorise numbers and algebraic expressions.

1. Trial and Error Method-Testing different factors to find the ones that multiply to the given number or expression.

2. Common Factor-Identifying and factoring out common factors in an expression.

3. Difference of Two Squares-Factorising expressions of the form $a^2 - b^2$.

4. Quadratic Factorisation-Factorising quadratic expressions of the form $ax^2 + bx + c$.

A polynomial is an expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \ldots + a_n$, where $a_0\ a_1, a_2 \ldots\ldots$ are independent of x and when $a_0 \neq 0$ this polynomial is of degree n. Expressing polynomial as a product of two or more factors is called the factorisation of polynomials.

1.2.2.1 Factorisation of Quadratic polynomial

Step 1. Consider the quadratic polynomial $ax^2 + bx + c$

Step 2. Find two numbers such that their product is ac and sum is b.

Step 3. Split the middle term using these two numbers.

Step 4. Take the common factors out and simplify.

Factorization of general quadratic polynomial $x^2 + bx + c$ is done as follows.

Find two integers p and q such that $p + q = b$ and $pq = c$

Then equation becomes

$$x^2 + (p + q)\,x + pq = x^2 + px + qx + pq = x(x + p) + q(x + p)$$
$$= (x + p)(x + q)$$

Factor Theorem

Let f(x) is a polynomial of degree n >1 and 'a' be any real number.

(i) If $f(a) = 0$, then $x - a$ is a factor of $f(x)$

(ii) If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$

Important Identities

$a^2 - b^2 = (a + b)(a - b)$

$a^2 + 2ab + b^2 = (a + b)(a + b)$

$a^2 - 2ab + b^2 = (a - b)(a - b)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

$(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$

$(x^3 - y^3) = (x - y)(x^2 + xy + y^2)$

Illustration 1.2.6

Factorise

(i) $ab + bc + ax + cx$

(ii) $x^2 + 3x + x + 3$

(iii) $6ab - b^2 + 12ac - 2bc$
(iv) $a^2 + b - ab - a$

Solution

(i) $ab + bc + ax + cx = b(a+c) + x(a+c)$
$= (a + c)(b + x)$

(ii) $x^2 + 3x + x + 3 = x(x + 3) + (x + 3)$
$= (x + 3)(x + 1)$

(iii) $6ab - b^2 + 12ac - 2bc = 6ab + 12ac - b^2 - 2bc$
$= 6a(b + 2c) - b(b + 2c)$
$= (b + 2c)(6a - b)$

(iv) $a^2 + b - ab - a = a(a-b) - (a-b)$
$= (a-b)(a-1)$

Illustration 1.2.7

Factorise (i) $x^2 - 1 - 2a - a^2$
(*ii*) $1 + 2ab - a^2 - b^2$

Solution

(i) $x^2 - 1 - 2a - a^2 = x^2 - (1 + a^2 + 2a)$
$= x^2 - (1 + a)^2$
$= [x + (1 + a)][x - (1 + a)]$
$= (x + a + 1)(x - a - 1)$

(ii) $1 + 2ab - a^2 - b^2 = 1 - (a^2 - 2ab + b^2)$
$= 1 - (a - b)^2$
$= [1 + (a - b)][1 - (a - b)]$
$= (1 + a - b)(1 - a + b)$

Illustration 1.2.8

Factorise $a(a-1) - b(b-1)$

Solution

$a(a-1) - b(b-1) = a^2 - a - b^2 + b$
$= (a^2 - b^2) - (a - b)$
$= (a + b)(a - b) - (a - b)$
$= (a - b)[a + b - 1]$

Illustration 1.2.9

Show that $x - 3$ is a factor the polynomial, $f(x) = x^3 + x^2 - 17x + 15$.

Solution

Given $f(x) = x^3 + x^2 - 17x + 15$ ________ (1)

$(x - 3)$ is a factor of equation (1), if $f(3) = 0$

From equation (1), $f(3) = 3^3 + 3^2 - 17 \times 3 + 15$

$= 27 + 9 - 51 + 15 = 0$

$\therefore x - 3$ is a factor of equation (1)

Illustration 1.2.10

Show that $x + 5$ is a factor of polynomial, $f(x) = x^3 + x^2 + 3x + 115$.

Solution

Given $f(x) = x^3 + x^2 + 3x + 115$ ________________ (1)

$= (-5)^3 + (-5)^2 + 3 \times (-5) + 115$

$= -125 + 25 - 15 + 115$

$= -140 + 140 = 0$

$\therefore$ By factor theorem $x + 5$ is a factor of equation (1).

Illustration 1.2.11

Find the value of 'a' so that $x - a$ is a factor of

$f(x) = x^5 - a^2x^3 + 2x + a - 3$

Solution

Given $f(x) = x^5 - a^2x^3 + 2x + a - 3$

$f(a) = a^5 - a^2a^3 + 2a + a - 3$

$= a^5 - a^5 + 3a - 3$

$= 3a - 3$

$x - a$ is a factor $\Leftrightarrow f(a) = 0$

$\Leftrightarrow 3a - 3 = 0$

$\Leftrightarrow 3a = 3$

$\Leftrightarrow a = 1$

Illustration 1.2.12

Factorise $x^2 - 64$

Solution

$x^2 - 64 = (x + 8)(x - 8)$

$(x + 8)$ and $(x - 8)$ are factors.

Illustration 1.2.13

Factorise $2x^2 - x - 10$

Solution

Find 2 numbers whose product is -10 × 2 = -20 and sum = -1

The numbers are 4 and -5

$2x^2 - x - 10 = 0 \Longrightarrow 2x^2 + 4x - 5x - 10$

$\Longrightarrow 2x(x+2) - 5(x+2)$

$\Longrightarrow (2x-5)(x+2)$

Factors are $2x - 5$ and $x + 2$

Illustration 1.2.14

Factorise $14x^2 + 13x + 3$

Product 42, sum 13 → Numbers 7,6

Solution

$14x^2 + 13x + 3$

$\Longrightarrow 14x^2 + 7x + 6x + 3$

$\Longrightarrow 7x(2x+1) + 3(2x+1)$

$\Longrightarrow (2x+1)(7x+3)$

Illustration 1.2.15

Factorise $15x^2 - 18x - 10$

Solution

Product = -150, Sum = -18 → Numbers -25,6

$15x^2 - 18x - 10$

$\Longrightarrow 15x^2 - 25x + 6x - 10$

$\Longrightarrow 5x(3x-5) + 2(3x-5)$

$\Longrightarrow (3x-5)(5x+2)$

Illustration 1.2.16

Factorise $x^2 + 9x + 18$

Solution

$p + q = 9, pq = 18$

$p = 6, q = 3$

$x^2 + 9x + 18 = (x+6)(x+3)$

Illustration 1.2.17

Factorise $x^2 + 5x - 24$

Solution

$p + q = 5, pq = -24$

$p = 8, q = -3$

$x^2 + 5x - 24 = (x + 8)(x - 3)$

Illustration 1.2.18

Factorise $x^2 - 4x - 21$

Solution

$p + q = -4, pq = -21$

$p = -7, q = 3$

$x^2 - 4x - 21 = (x - 7)(x + 3)$

Illustration 1.2.19

Factorise $6x^2 + 7x - 3$

Solution

$p + q = 7, pq = -18$

$p = 9, q = -2$

$6x^2 + 7x - 3 = 6x^2 + 9x - 2x - 3$

$= 3x(2x + 3) - (2x + 3)$

$= (2x + 3)(3x - 1)$

Illustration 1.2.20

Factorise $2x^2 - 7x - 39$

Solution

$p + q = -7, pq = -78$

$p = -13, q = 6$

$2x^3 - 7x - 39 = 2x^3 - 13x + 6x - 39$

$= x(2x - 13) + 3(2x - 13)$

$= (x + 3)(2x - 13)$

Illustration 1.2.21

Factorise $2x^2 - \frac{5}{6}x + \frac{1}{12}$

Solution

$\frac{24x^2 - 10x + 1}{12} = \frac{1}{12}(24x^2 - 10x + 1)$

$p + q = -10, pq = 24$

$p = -6, q = -4$

$\frac{1}{12}(24x^2 - 10x + 1) = \frac{1}{12}(24x^2 - 6x - 4x + 1)$

$= \frac{1}{12}[6x(4x - 1) - (4x - 1)]$

$= \frac{1}{12}(4x - 1)(6x - 1)$

Illustration 1.2.22

Factorise $2x + 2y + ax + ay$

Solution

$2x + 2y + ax + ay = 2(x + y) + a(x + y)$

$= (x + y)(2 + a)$

Illustration 1.2.23

Factorise $x^3 + 3x^2 - 6x - 18$

Solution

$x^3 + 3x^2 - 6x - 18$

$= x^2(x + 3) - 6(x + 3)$

$= (x + 3)(x^2 - 6)$

1.2.2.1 L.C.M. (Least Common Multiple)

The Least Common Multiple (LCM) of two or more numbers is the smallest positive integer that is a multiple of each of the numbers.

To find the LCM, factorise each number into its prime factors and take the highest power of each prime factor that appears in any of the factorisations.

Example. 1. Find L.C.M of 3 and 12.

Multiple of 3 : 3, 6, 9, **12**

Multiple of 12 : **<u>12,</u>** 24, 36, …..
Least common factor is 12
Another method to find the LCM is Ladder Method

$$\begin{array}{r|l} 3 & 3, 12 \\ \hline & 1, 4 \end{array}$$

L.C.M = 3 × 1 × 4 = 12

1.2.2.2 Highest Common Factor (H.C.F)

The Highest Common Factor (HCF) of two or more numbers is the largest positive integer that divides each of the numbers without leaving a remainder.

When factorizing numbers, you identify the prime factors of each number and determine the common prime factors they share. The product of these common prime factors gives you the HCF. H.C.F is also called the Greatest Common Division (G.C.D)
There are three methods to find the H C F.

1) Ladder Method
2) Division method
3) Prime factorisation method

Division method

Step I. Divide the largest number by the smallest number
Step II. Take the divisor as new dividend and remainder as new divisor. Divide the first divisor by the first remainder.
Step III. Proceed this process till the remainder is zero. Then last divisor will be the H.C.F of the given numbers.

Prime factorisation method

Step I. Factorize each number into its prime factors.
Step II. Identify the common prime factors and multiply them to obtain the HCF.

Illustration 1.2.24

Find L.C.M of 6, 8, 12
Solution

$$
\begin{array}{r|l}
2 & 6, 8, 12 \\ \hline
2 & 3, 4, 6 \\ \hline
3 & 3, 2, 3 \\ \hline
 & 1, 2, 1
\end{array}
$$

L.C.M. = 2 × 2 × 3 × 1 × 2 × 1

= 24

Illustration 1.2.25

Find the L.C.M of 75, 45

Solution

$$
\begin{array}{r|l}
3 & 75, 45 \\ \hline
5 & 25, 15 \\ \hline
 & 5, 3
\end{array}
$$

L.C.M = 3 × 5 × 5 × 3 = 225

Illustration 1.2.26

Using division method, find the H.C.F of 24 and 15.

Solution

$$
\begin{array}{l}
15 \,\big)\, 24 \,\big(\, \\
\quad\;\; 15 \\ \hline
\quad 9 \,\big)\, 15 \,\big(\, 1 \\
\qquad\;\; 9 \\ \hline
\qquad 6 \,\big)\, 9 \,\big(\, 1 \\
\qquad\quad\; 6 \\ \hline
\qquad\quad 3 \,\big)\, 6 \\
\qquad\qquad\;\; 6 \\ \hline
\qquad\qquad\;\; 0
\end{array}
$$

H.C.F = 3

Ladder method

$$\begin{array}{r|l} \underline{3} & 24, 15 \\ \hline & 8, 5 \end{array}$$

H.C.F = 3

Illustration 1.2.27

Find H.C.F by division method of 492, 248

Solution

$$\begin{array}{r|l|l} 248 & 492 & 1 \\ & 248 & \\ \hline \end{array}$$

$$\begin{array}{r|l|l} 244 & 248 & 1 \\ & 244 & \\ \hline \end{array}$$

$$\begin{array}{r|l|l} 4 & 244 & 6 \\ & 244 & \\ \hline & \text{X} & \end{array}$$

H.C.F = 4

Illustration 1.2.28

Find H.C.F of 24, 36, 64 via factorisation method

Solution

$$24 = 2 \times 2 \times 2 \times 3$$

$$36 = 2 \times 2 \times 3 \times 3$$

$$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

H.C.F = 2 × 2 = 4

Illustration 1.2.29

Find H.C.F and L.C.M of 510, 765, 408.

Solution

510 = 2 × 3 × 5 × 17

765 = 3 × 3× 5 ×17

408 = 2 × 2 × 2× 3 × 17

L.C.M = $2^3 \times 3^2 \times 5 \times 17 = 6120$

H.C.F $= 17 \times 3 = 51$

Illustration 1.2.30

Find H.C.F of 32, 56, 46 using (1) Prime factorization method (2) ladder method (3) division method.

Solution

(1) Prime factorization method

$32 = 2 \times 2 \times 2 \times 2 \times 2$

$$\begin{array}{r|l} 2 & 32 \\ \hline 2 & 16 \\ \hline 2 & 8 \\ \hline 2 & 4 \\ \hline & 2 \end{array}$$

$56 = 2 \times 2 \times 2 \times 7$

$$\begin{array}{r|l} 2 & 56 \\ \hline 2 & 28 \\ \hline 2 & 14 \\ \hline & 7 \end{array}$$

$46 = 2 \times 23$

$$\begin{array}{r|l} 2 & 46 \\ \hline & 23 \end{array}$$

H.C.F = 2

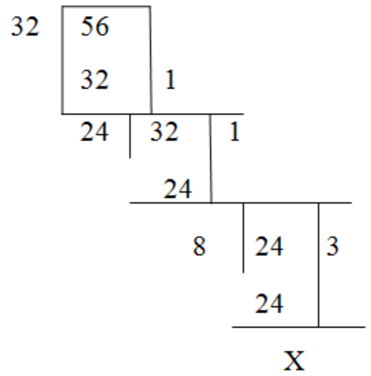
(2) Ladder method

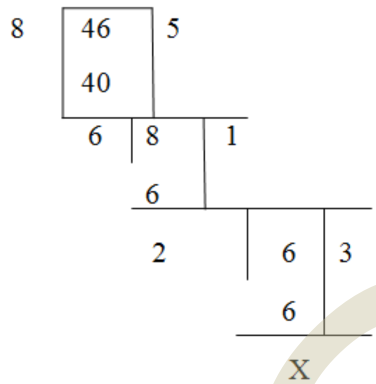
$$\begin{array}{r|l} 2 & 32, 46, 56 \\ \hline & 16, 23, 28 \end{array}$$

H.C.F = 2

(3) Division method

a. 56





H.C.F = 2

Illustration 1.2.31

Find the H.C.F of 70, 80, 120, 150 by (1) Prime factorization (2) Ladder method (3) Division method.

Solution

(1) Prime factorization

70 = 2 × 5 × 7

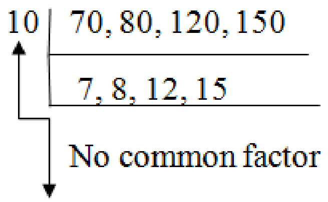
80 = 2 × 2 × 2 × 2 × 5

120 = 2 × 2 × 2 × 5 × 3

150 = 2 × 5 × 5 × 3

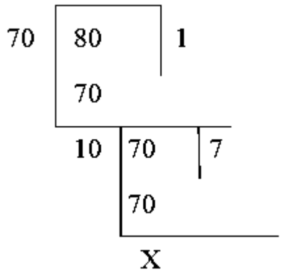
H.C.F = 2 × 5 = 10

(2) Ladder method

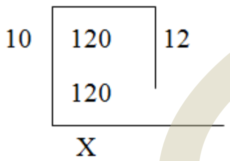


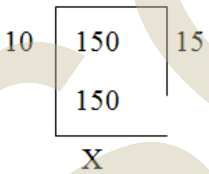
H C F is 10

(3) Division method



H.C.F last divisor = 10





H C F is 10.

1.2.2.3 Percentage

Percentage means per hundred or hundreds. Any fractional number can be expressed in percentage. A number which is expressed in percentage can affix the sign '%'. Hence the percentage sign following a number means "place the number over hundred".

A number whose denominator is 100 is converted to decimal form by multiplying it by 10.

Example. (1) 65 % = $\frac{65}{100}$ = 0.65

(2) 7 % = $\frac{7}{100}$ = 0.07

1.2.2.4 Prime Number

A prime number is a positive integer which has no proper factors.

Example. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 91, 97, ….

1.2.2.5 Relatively Prime

Two integers are called relatively prime or prime to each other if they contain no common prime factors.

Example, (1) 72 and 35 (2) 45 and 49

1.2.2.6 Composite Numbers

A composite number is a positive integer which has at least one proper factor.

Example. 10, 12, 16, 21, …..

Illustration 1.2.32

The relation between temperature in degree Celsius and Fahrenheit is given by $F = \frac{1}{5}(160 + 9C)$. Suppose on the day of January 21st 2021, temperature in Oslo was 41^0 F. While in Los Angeles it was $59^{\ 0}$ F. Compare the temperature of these places in degree Celsius.

Solution

Let C_1 denote the temperature in Oslo and C_2 denote the temperature in Los Angeles in degree celsius.

Then on January 21st 2021

$$F = \frac{1}{5}(160 + 9C)$$

$$5F = (160 + 9C)$$

$$9C = 5F - 160$$

$$C = \frac{5}{9}(F - 32)$$

When $F = 41^0$ F, $C = \frac{5}{9}(41 - 32) = 5^0C$

When $F = 59^0$ F, $C = \frac{5}{9}(59 - 32) = 15^0C$

Hence C_2 is three times of C_1.

Illustration 1.2.33

Write the following as fraction

(i) 25% (ii) 142%

Solution

(i) 25% = $\frac{25}{100} = \frac{1}{4}$

(ii) 142% = $\frac{142}{100} = \frac{71}{50}$

Illustration 1.2.34

What is 30% of 15?

Solution

30% = $\frac{30}{100}$ = 0.3

30% of 15 = 0.3 × 15 = 4.5

1.2.2.7 Rationalisation

The process of converting the irrational denominator to a rational number by a suitable number is called rationalization.

For example, Simplify $\frac{3}{\sqrt{5}}$ by rationalizing the denominator?

Ans. $\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{(\sqrt{5})(\sqrt{5})}$, here $\sqrt{5}$ is multiplied with numerator and denominator.

$= \frac{3\sqrt{5}}{5}$

<u>Rationalizing factor</u>

If the product of two irrational number is rational, then each one is called rationalizing factor of other.

a + $\sqrt{b}$ and a $-$ $\sqrt{b}$ are rationalizing (RF) of the other.

Illustration 1.2.35

Rationalise the denominator of $\frac{6}{6-(\sqrt{2})}$

Solution

$$\frac{6}{6-(\sqrt{2})} = \frac{(6)(6+\sqrt{2})}{(6-\sqrt{2})(6+\sqrt{2})}$$

$$= \frac{36+6\sqrt{2}}{36-2}$$

$$= \frac{36+6\sqrt{2}}{34}$$

Illustration 1.2.36

Rationalize the denominator of $\frac{2}{\sqrt{3}-\sqrt{5}}$

Solution

Ans. $\frac{2}{\sqrt{3}-\sqrt{5}} = \frac{(2)(\sqrt{3}+\sqrt{5})}{(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})}$

$= \frac{2\sqrt{3}+2\sqrt{5}}{3-5}$

$= \frac{2\sqrt{3}+2\sqrt{5}}{-2} = \frac{2(\sqrt{3}+\sqrt{5})}{-2}$

$= -(\sqrt{3}+\sqrt{5})$

Illustration 1.2.37

If a and b are rational numbers and $\frac{2+\sqrt{3}}{2-\sqrt{3}} = a + b\sqrt{3}$, find a and b?

Solution

Ans. $\frac{2+\sqrt{3}}{2-3} = \frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$

$= \frac{4+2\sqrt{3}+2\sqrt{3}+3}{4-3}$

$= 7 + 4\sqrt{3}$

$\therefore$ a = 7, b = 4

Illustration 1.2.38

Rationalize the denominator of $\frac{5}{\sqrt{3}-\sqrt{5}}$

Solution

$\frac{5}{\sqrt{3}-\sqrt{5}} = \frac{5(\sqrt{3}+\sqrt{5})}{(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})}$

$= \frac{5(\sqrt{3}+\sqrt{5})}{3-5}$

$= \frac{-5}{2}(\sqrt{3}+\sqrt{5})$

1.2.3 Arithmetic of Fractions

Fractions are a fundamental concept in mathematics that represents a part of a whole or a ratio of one quantity to another. They are essential for understanding and working with quantities that are not whole numbers. A fraction consists of two main components: the numerator and the denominator. The numerator is the top part of the fraction and the denominator is the bottom part of the fraction.

A fraction is typically written in the form $\frac{a}{b}$, where "a" is the numerator and "b" is the denominator.

For example, in the fraction $\frac{3}{4}$, 3 is the numerator and 4 is the denominator.

In certain cases, both the numerator and denominator can themselves be expressed as the product of two or more factors. When this occurs, we can determine the product of the factors for the numerator and denominator separately. The result of multiplying the factors of the numerator forms the new numerator, while the product of the factors of the denominator constitutes the new denominator.

Example. $\frac{(7 \times 8)(3)}{(4)(3)} = \frac{168}{12}$

Multiply (a) $\frac{5}{2}$ and $\frac{1}{7}$?

(b) $\frac{2}{5}$ and $\frac{11}{3}$

Ans. (a) $\frac{(5)(1)}{(2)(7)} = \frac{5}{14}$

(b) $\frac{(2)(11)}{(5)(3)} = \frac{22}{15}$

Some common types of fractions are

1.2.3.1 Reciprocal

Reciprocal of a fraction with numerator not equal to zero is the fraction obtained by interchanging numerator and denominator.

Find the reciprocal of the following

(a) $\frac{2}{3}$ (b) $\frac{-5}{6}$ (c) 10

Ans: (a) $\frac{2}{3}$ has reciprocal $\frac{3}{2}$

(b) $\frac{-6}{5}$ is the reciprocal of $\frac{-5}{6}$

(c) $\frac{1}{10}$ is the reciprocal of 10

Remark 1. 0 has no reciprocal

2. 1 is the reciprocal of itself

3. Multiplying a fraction by its reciprocal we will get 1

Example. $\left(\frac{3}{5}\right)\left(\frac{5}{3}\right) = 1$

1.2.3.2 Types of Fractions

Fractions can be expressed in various forms based on their numerator and denominator. In this section, we'll go over each of them.

1.Proper Fraction

A proper fraction is a fraction where the numerator is less than the denominator, such as $\frac{1}{2}$ or $\frac{3}{4}$.

2.Improper Fraction

An improper fraction is a fraction where the numerator is equal to or greater than the denominator, such as $\frac{5}{4}$ or $\frac{7}{3}$.

3.Mixed Fraction

A mixed number combines a whole number and a fraction.

For example, $1\frac{1}{2}$ is a mixed number, where 1 is the whole number and $\frac{1}{2}$ is the fraction.

4. Equivalent Fraction

For getting equivalent fraction multiply or divide the fraction by the same fraction.

Example. 1. $\frac{7}{2} = \frac{(7)(5)}{(2)(5)}$

$$= \frac{(7)\left(\frac{3}{5}\right)}{(2)\left(\frac{3}{5}\right)}$$

$$= \frac{(7)\left(\frac{-1}{2}\right)}{(2)\left(\frac{-1}{2}\right)}$$

5. Division of Fractions

By forming the reciprocals, we can find the division of fractions.

Example. (1) $\frac{5}{6} \div \frac{2}{3} = \left(\frac{5}{6}\right)\left(\frac{3}{2}\right)$

$$= \frac{(5)(3)}{(6)(2)}$$

$$= \frac{15}{12}$$

(2) $-\frac{7}{5} \div \frac{5}{11} = \left(\frac{-7}{5}\right)\left(\frac{11}{5}\right)$

$$= \frac{(-7)(11)}{(5)(5)} = \frac{-77}{25}$$

1.2.4 Ratio and Proportions

A ratio is a way of comparing two or more quantities or values. It expresses the relationship between these quantities in terms of how many times one quantity is contained within another. Ratios are often written in the form of a fraction or using a colon (" : ") such as

$\frac{a}{b}$ or a:b. The first term in a ratio is called antecedent and the second term is called consequent. Ratio has no units. We can multiply and divide the terms of the ratio by a non-zero number.

For example, if you have a basket with 2 apples and 3 oranges, the ratio of apples to oranges is 2:3, which means there are 2 apples for every 3 oranges.

1.2.4.1 Types of Ratios

Ratios can be divided into simple ratio, compound ratio, continued ratio and equivalent ratio. Each of them can be explain one by one.

Simple Ratio

A comparison of two quantities without specifying a particular unit of measurement is a simple ratio.
For example, 2:3 is a simple ratio.

Compound Ratio

The concept of compound ratios involves combining two or more simple ratios into a single ratio, which can be expressed as a product of these individual ratios.
The compound Ratio of $a:b, c:d, e:f$ can be written in fraction as $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$ which is essentially the result of multiplying these simple ratios together.
For example, 2:3 and 4:5 combined as $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

Continued Ratio

A continued ratio, also known as a continued fraction, is a mathematical expression that represents a number as an infinite sequence of fractions.
For example, 2:3:4:5…… is a continued ratio.

Equivalent Ratios

Ratios that represent the same relationship between quantities are considered equivalent. For example, 2:3 is equivalent to 4:6 because they both represent the ratio of two parts to three parts.

Illustration 1.2.39

Find the ratio of 650 grams and 26 kg.
Solution
Since the ratio is always between the quantities of the same kind, we must convert them into same kind.

Since 26 kg =26000 gm, required ratio is

$$650{:}\,26000 = \frac{650}{26000} = \frac{1}{40} = 1{:}\,40$$

Illustration 1.2.40

In a mixture of gold and copper are in the ratio 7:3. How much are each in a mixture of 120 grams?

Solution

Quantity of gold in the mixture = $\frac{7}{(7+3)} \times 120 = \frac{7}{10} \times 120 = 84\ grams$

Quantity of copper in the mixture= $\frac{3}{(7+3)} \times 120 = \frac{3}{10} \times 120 = 36\ grams$

Required ratio = $84{:}\,36$, $= \frac{84}{36} = \frac{7}{3}$

Illustration 1.2.41

Income of Rahim is Rs 12000 per month and that of Amith is Rs 191520 per annum. If the monthly expenditure of each of them is Rs 9960 per month, find the ratio of their savings.

Solution

Savings of Rahim per month = $Rs\ (12000 - 9960) = Rs\ 2040.$

Monthly income of Amith = $Rs\ \frac{191520}{12} = Rs.15960$

Savings of Amith per month = $Rs\ (15960 - 9960) = Rs\ 6000$

Therefore, ratio of savings of Rahim and Amith = 2040: 6000 = 17:50

Illustration 1.2.42

If a bus travels 160 km in 4 hours and a train travels 320 km in 5 hours at uniform speeds, then find the ratio of the distances travelled by them in one hour?

Solution

Ratio of bus speed: train speed = $\frac{160}{4} : \frac{320}{5} = 40{:}\,64 = \frac{40}{64} = 5{:}8$

Illustration 1.2.43

Find the greatest ratio among the ratios $2:3,\ 5:8,\ 75:12\ and\ \ 40:25$.

Solution

$$2:3,\ 5:8,\ 75:12,\ 40:25 = \frac{2}{3},\frac{5}{8},\frac{75}{12},\frac{40}{25}$$

$$= \frac{2}{3},\frac{5}{8},\frac{25}{4},\frac{8}{5}$$

LCM = 120

$\therefore \frac{80}{120}, \frac{75}{120}, \frac{750}{120}, \frac{192}{120}$

$= \frac{5}{8}, \frac{2}{3}, \frac{25}{4}, \frac{8}{5}$

$\therefore$ the greatest ratio = 75:12

i.e., $\frac{75}{12}$

Illustration 1.2.44

Find the compound Ratio of $2:3,\ 4:9, 10:13$

Solution

Compound Ratio = $2:3, 4:9, 10:13 = \frac{2}{3} \times \frac{4}{9} \times \frac{10}{13} = \frac{80}{351}$

Illustration 1.2.45

The ratio between present ages of Mohan and Vivek is 5:8. After 4 years, the ratio between their ages will be 2:3. What is the present age of Vivek?

Solution:

Let present ages of Mohan and Vivek are 5x and 8x respectively

Given $\frac{(5x+4)}{(8x+4)} = \frac{2}{3}$, cross multiplying

$15x + 12 = 16x + 8$

$16x + 8 - 15x - 12 = 0$

$x = 4$

Vivek present ages $= 8x = 8 \times 4 = 32$ years.

Proportions

A proportion is an equation that states that two ratios are equal. If the numbers $a, b, c, and\ d$ are in proportion we can express $a:b = c:d$. It can also be expressed as $a:b::c:d$.

In any proportion, the first and fourth terms are called the extremes while the second and the third terms are called means. In the proportion $a:b::c:d$, a and d are extremes and b and c are means.

Product of extremes = Product of means

i.e., if $a:b = c:d$, then $a \times d = b \times c$

To solve a proportion, you can use the "cross-multiplication" method. It involves multiplying the extremes (the first and last terms) and the means (the second and third terms) and setting them equal to each other.

For example, Solve for x in the proportion $3:5::x:15$

$$3:5::x:15 \rightarrow \frac{3}{5} = \frac{x}{15}$$

Cross-multiplying: $3 \times 15 = x \times 5$

$$45 = x \times 5$$

$$x = \frac{45}{5} = 9$$

Mean proportion

If $a:x = x:b$ then x is called Second Proportion (Mean proportion) of a and b

$a:x = x:b \Longrightarrow x^2 = ab \Longrightarrow x = \sqrt{ab}$

Direct proportion

Two quantities are said to be in direct proportion if increase in one quantity follows the increase in other quantity or decrease in one quantity follows the decrease in other quantity in the same ratio.

Example: Number of articles bought and total cost of article. That is more articles, more cost and less articles, less cost.

Indirect proportion

Two quantities are said to be in indirect or inverse proportion, if increase in one quantity follows decrease in the other quantity in the same ratio and vice versa.

Example: Number of workers and days taken to do a certain work. i.e., more workers, less days and less workers days, more workers days.

If the two quantities x, y are indirectly proportional to each other, then $x = \frac{k}{y}$ or $\frac{x_1}{y_1} = \frac{x_2}{y_2}$

Illustration 1.2.46

What must be added to each of the numbers 10,18,22,38 so that they become in proportion.

Solution

Let x be the number to be added to each of the numbers.

$(10 + x):(18 + x)::(22 + x):(38 + x)$

$(10 + x)(38 + x) = (18 + x)(22 + x)$

$380 + 48x + x^2 = 396 + 40x + x^2$

$48x - 40x = 396 - 380$

$8x = 16$

$x = 2$

Illustration 1.2.47

The profit of company will be proportional to the supply of commodities. If during the year, the number of units sold is 50 and the profit is 25 dollars, what is the profit in selling 60 units?

Solution

Let p be the profit in selling r units

Given p = 25 dollar when r = 50 units

So that, $\frac{p}{r} = \frac{25}{50} = \frac{1}{2}$

The proportion value of p when r = 60, we get

$\frac{1}{2} = \frac{r}{60}$

$\therefore$ r = $\frac{60}{2}$ = 30

Illustration 1.2.48

The ratio between two quantities is 9:8. If the first quantity is 45. Find the other quantity.

Solution

Suppose the second quantity be x, then $9:8 = 45:x$

In a proportion, product of the means = Product of extremes

$\frac{9}{8} = \frac{45}{x}$, 9$x$ = (45)(8)

$\therefore x = \frac{45\times8}{9}$, x = 40

Illustration 1.2.49

If $a:b = 2:3, b:c = 4:5, c:d = 6:7$ find $a:b:c:d$

Solution

$a:b = 2:3 ------ (1)$, $b:c = 4:5 -------- (2)$

Here values of b in both ratios is different. So make both values of b equal.

Multiply (1) by 4 and (2) by 3 we get

$a:b = 8:12$ and $b:c = 12:15$

$i.e., a:b:c = 8:12:15$ --------------(3)

$c:d = 6:7$ ------(4)

make c part equal Multiply (3) by 2 and (4) by (5)

$a:b:c = 16:24:30$ and $c:d = 30 : 35$

Hence $a:b:c:d = 16:24:30:35$

Illustration 1.2.50

A certain amount was divided between A and B in the ratio 4:3. If B's share was Rs 2400, what is the total amount?

Solution

Let their shares be $4x$ and $3x$. Then, $3x = 2400 \rightarrow x = 800$

Total amount $= 4x + 3x = 7x = 7 \times 800 = 5600$

Illustration 1.2.51

If 72 men complete a work in 24 days, how many men will complete the work in 36 days?

Solution

Men	Days
72 (x_1)	24 (y_1)
x (x_2)	36 (y_2)

Let x men complete the work in 36 days. Compare *men* with *days*. If there are *more men*, it takes *less days* to complete the work. (*Indirect variation*)

$$72 : x = 36 : 24 \implies 72 \times 24 = 36x \implies x = 72 \times \frac{24}{36} \implies 48$$

Illustration 1.2.52

If 300 men can complete a work in 16 days, how many men would do 1/5 of the work in 15 days?

Men	Work	Days
300 (x_1)	1(y_1)	16 (z_1)
x (x_2)	1/5 (y_2)	15 (z_2)

Solution:

Compare *men* with *work* and *days*.

More *men* can do *more work*. (*Direct proportion*)

If there are *more men*, it takes *less days* to complete the work. (*Indirect proportion*)

Hence, it is a Combined proportion: $\frac{x_1}{x_2} = \frac{y_1}{y_2} \times \frac{z_2}{z_1}$

$$\frac{300}{x} = \frac{1}{\frac{1}{5}} \times \frac{15}{16} \implies \frac{300}{x} = \frac{15 \times 5}{16} \implies x = 64$$

Recap

- Fractions - ratio of one quantity to another. A fraction consists of two parts 1) the numerator and the denominator
- Reciprocal of a fraction with a numerator not equal to zero is the fraction obtained by interchanging the numerator and denominator
- 0 has no reciprocal, 1 is the reciprocal of itself, and multiplying a fraction by its reciprocal we will get 1
- A proper fraction is a fraction where the numerator is less than the denominator
- Improper fraction is a fraction where the numerator is equal to or greater than the denominator
- A mixed number combines a whole number and a fraction
- A prime number is a positive integer which has no proper factors
- Two integers are called relatively prime or prime to each other if they contain no common prime factors
- A composite number is a positive integer which has at least one proper factor
- The Least Common Multiple of two or more numbers is the smallest positive integer that is a multiple of each of the numbers.
- The Highest Common Factor (HCF) of two or more numbers is the largest positive integer that divides each of the numbers without leaving a remainder
- H.C.F. is also called the Greatest Common Division (G.C.D). There are three methods to find the HCF. They are Ladder Method, division method, Prime factorisation method
- The rationalization process of converting the irrational denominator to a rational number by a suitable number
- Factorization is an algebraic expression into its constituent factors, which are smaller numbers or expressions
- Ratio - relationship between these quantities in terms of how many times one quantity is contained within another
- Ratios are often written in the form of a fraction or using a colon (" : ") such as a/b or a:b
- Ratio has no units. We can multiply and divide the terms of the ratio by a non-zero number

Objective Questions

1. Write the denominator of percentage of a fraction.
2. Define the sum of money stated in the bill.
3. Are the prime numbers odd or even?
4. How many different symbols did the Romans used in writing numbers.
5. Is the principle of place value used in Roman system.
6. Write one-fifth of an amount in three different ways.
7. Find $\frac{5}{8} \times \frac{7}{2}$
8. Find $(3 - 5)(7 + 2)$
9. Evaluate $\frac{7}{3} \div \frac{28}{9}$
10. Evaluate $\frac{p-q}{p-q}$, when $p = 5$, $q = 2$.
11. Let $f(x) = x^2 + 1$, Compute $f(-1)$.
12. If cost of producing 'a' unit of a commodity is, $C(a) = 5a + 10$. Find the cost of producing 35 units.
13. Find 207×193
14. Evaluate 305×308

Answers

1. 100
2. Cost of goods, services, or products provided, along with any applicable taxes or fees.
3. Except for the number 2, which is the only even prime number, all prime numbers are odd.
4. I for 1, V for 5, X for 10, L for 50, C for 100, D for 500, M for 1000
5. No, the Roman numeral system does not use the principle of place value in the same way as the decimal system.

6. $\frac{x}{5}$, 0.2 x, $\frac{x}{100} \times 20\%$

7. $\frac{35}{16}$

8. -18

9. $\frac{3}{4}$

10. 1

11. 2

12. 185

13. 39951

14. 93940

Assignments

1. a. Prove that $(x^3 + y^3) = (x + y)(x^2 - xy + y^2)$

   b. Factorize $x^6 - y^6$

   c. Factorize $50x^2 - 20xy$

2. A sales person receives 5x + 12 y dollars, where x is the number of hours worked and y is the number of miles of automobile usage. Find the amount due to the sale person if he worked 4 hours and travels 6 miles.

3. A stereo shop has the following history during the first three years of operation. It lost x dollars during the first three years and made a profit y dollar during the second year. If the profit of the third year is 3 times the net profit of first and second year. Find the final year profit when x = 10 and y = 2.

4. A person buys x_1, x_2, x_3 units of three goods whose prices per unit are respectively p_1, p_2, p_3. What is the total expenditure?

5. If we have the relation $C = \frac{5}{9}(F-32)$ where C and F are the units of temperature in Celsius and Fahrenheit. Find C when F = 32 and F when C = 100.Find a general expression for F in terms of C.

Suggested Readings

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4. G S Monga, Mathematics and Statistics for Economics, Second revised edition,Vikas Publishing House PVT LTD, 2002

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1. Taro Yamane, Mathematics for Economists: An Elementary Survey, Prentice Hall of India
2. Mike Rosser, Basic Mathematics for Economist, 2nd Edition, Routledge, London and New York, 2003
3. Sydsaeter K and Hammond P, Essential Mathematics for Economic Analysis, Prentice Hall
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3

UNIT

Algebraic Operations

Learning Outcomes

After completing this unit, the learner will be able to

- apply algebraic techniques to model and analyse economic scenarios
- have a deep understanding of fundamental algebraic concepts
- possess strong problem-solving skills, using algebraic methods to address economic challenges
- communicate their algebraic solutions clearly and concisely

Prerequisites

Think of algebra as a helpful friend who can make your math journey easier. Before we jump into Addition and Subtraction, imagine that these are like the starting point. Just like when you learn to walk before you run, these are the basic moves in algebra. Now, let us talk about Multiplication and Division. Multiplication is like making groups of things, while Division is like sharing things equally. They help you work with numbers in a more organized way.

Well, beyond math class, these skills are like superpowers. They help you solve all kinds of problems, not just math ones. Imagine you want to figure out how much you will save on a sale item or understand how prices change over time in economics. These algebra basics will be your secret weapon.

So, remember, algebra is not just about numbers and letters; it is about making life's puzzles easier to solve. And as you grow, you will see how these simple concepts become incredibly useful in your studies and future career in economics. So, let us start this exciting journey together.

Keywords

Addition, Subtraction, Multiplication, Division, Algebraic Expressions

Discussion

1.3.1 Algebraic Expression

An algebraic expression is a mathematical phrase that can contain numbers, variables, arithmetic operations (addition, subtraction, multiplication, division), and exponentiation, but usually not equality or inequality symbols. It is a combination of symbols and terms that can be evaluated, simplified, or manipulated using algebraic rules.

For example: $3x + 3y, 4x - 10$ are algebraic expressions

Example. 1. Let P be the basic pay of an employee (monthly) with annual increase of 8% of the basic during that year. Then he receives P + $\frac{8}{100}$ P during the second year. During the third year he receives (P + $\frac{8P}{100}$) + $\frac{8}{100}$ (P + $\frac{8P}{100}$) which is an expression involved variable and constant 8%.

Such an expression involving variables and constants is called an algebraic expression.
An expression in which integer constants, variables and algebraic operation '+' or '-' or '×' or '÷' is called an algebraic expression.

Illustration 1.3.1

Evaluate (1) $3x - 5$ when $x = 8$

(2) $(7x - 2)(5x + 7)$ when $x = 3$

Solution

(1) $3x - 5 = (3)(8) - 5$

$= 24 - 5 = 19$

(2) $(7x - 2)(5x + 7) = (7 \times 3 - 2)(5 \times 3 + 7)$

$= (21 - 2)(15 + 7)$
$= 19 \times 22 = 418$

Illustration 1.3.2

Find $\frac{3m + n}{m - n}$, when m = 3, n = 5

Solution

$$\frac{3m + n}{m - n} = \frac{3 \times 3 + 5}{3 - 5}$$
$$= \frac{9 + 5}{-2}$$
$$= \frac{14}{-2}$$
$$= -7$$

Illustration 1.3.3

Simplify $8x(x + y) - 3(x + 2)$

Solution

$$8x(x + y) - 3(x + 2) = 8x^2 + 8xy - 3x - 6$$

1.3.2 Algebraic Operations

The combination of operations such as addition, subtraction, multiplication, and division between two or more algebraic expressions is called the algebraic operations.

Order of Operations

It is very important to note the order of performing the operations.

(1) Perform all operations within parenthesis.
(2) Compute any square or powers.
(3) Find multiplication and division working left to right.
(4) Find addition and subtraction working from left to right.

For evaluation put the values first and then apply the hierarchy operation rules.

Illustration 1.3.4

Evaluate $\frac{3(x-1)}{y+2} + (x - y)(x + y)$, when $x = 5, y = 2$

Solution

Ans. $\frac{3(x-1)}{y+2} + (x - y)(x + 2) = \frac{3(5-1)}{2+2} + (5-2)(5+2)$

$= \frac{3 \times 4}{4} + 3 \times 7$

$= 3 + 21 = 24$

Illustration 1.3.5

Evaluate $\frac{2(x-1)}{(x+1)+(x+3)}$ when x = 2

Solution

Ans. $\frac{2(x-1)}{(x+1)+(x+3)} = \frac{2(2-1)}{(2+1)+(2+3)}$

$= \frac{2 \times 1}{3+5}$

$= \frac{2}{8}$

$= \frac{1}{4}$

Illustration 1.3.6

Find $1 + (2 - x) + \frac{y}{y-1}$, when $x = 1, y = 2$

Solution

$1 + (2 - x) + \frac{y}{y-1} = 1 + (2 - 1) + \frac{2}{2-1}$

$= 1 + 1 + 2$

$= 4$

Illustration 1.3.7

Evaluate the following when $a = 3, b = 4$

1. 2a + b (2) 3a – b (3) ab (4) $\frac{2a-b}{b}$

Solution

Given a = 3, b = 4

(1) $2a + b = 2(3) + 4 = 6 + 4 = 10$

(2) $3a - b = 3(3) - 4 = 9 - 4 = 5$

(3) $ab = (3)(4) = 12$

(4) $\frac{2a-b}{b} = \frac{2(3)-4}{4} = \frac{6-4}{4} = \frac{2}{4} = \frac{1}{2}$

Illustration 1.3.8

Evaluate algebraic expression $3x - \frac{2x-y}{y+2}$, when $x = 3$, y = 2

Solution

$3x - \frac{2x-2}{y+2} = 3(3) - \frac{2(3)-2}{2+2}$

$= 9 - \frac{6-2}{4}$

$= 9 - \frac{4}{4}$

= 9 - 1 = 8

Illustration 1.3.9

Find 10 (y − 2x) + 7x (x + 2), when x = 1, y = −1.

Solution

10 (y − 2x) + 7x (x + 2) = 10 (−1 −2 × 1) + 7 × 1 (1 + 2)

= 10 (−3) +7 (3)

= −30 + 21 = −9

Illustration 1.3.10

At 1 PM the temperature is 5° above 0° and at 10 PM it is 3° below zero. How many degrees has the temperature dropped?

Solution

Temperature at 1 PM $t_1 = 5°$

Temperature at 10 PM $t_2 = -3°$

Then the temperature dropped at 10 pm = $t_1 - t_2$

= 5 − (−3)

= 8°C

Illustration 1.3.11

A stationary store had loss of Rs. 500 during 2020 and a loss of Rs. 800 during 2021. How much amount did the store lose during these years?

Solution

Loss during 2020, $L_1 = 500$

Loss during 2021, L_2 = 800.
Total loss = $L_1 + L_2$ = 500 + 800 = 1300
Total loss is Rs.1300

Illustration 1.3.12

A bicycle repair shop had a profit of Rs. 600 for the month of June 2020 and a loss of Rs.300 for the months of June 2021. What is the profit of the shop from June 2020 to 2021?

Solution

Profit of the shop for the year June 2020, $P_1 = 600$
Profit of the shop for the June 2021, $P_2 = -300$
Net profit = $P_1 + P_2$
$= 600 + (-300) = 300$
Total Profit is Rs. 300

Illustration 1.3.13

Vimal Fabrics has a loss of x rupees during the financial year 2019 - 2020 and three times of it during the financial year 2020 - 2021. Write down to an expression of the amount of profit during these periods?

Solution

Profit during the year 2019 - 2020 (P_1) = $-x$
Profit on the year 2020 – 2021$(P_2) = -y = -3x$.
Total profit = $P_1 + P_2$
$= -x + (-3x)$
$= -4x$

Illustration 1.3.14

A general store has an annual loss of Rs. 32000 during 2019. During 2020 the annal loss is Rs. 5000. On the year 2021, the annual profit is Rs. 12000. Find the net profit during

(1) 2019 and 2020
(2) 2020 and 2021
(3) During these three years.

Solution

Ans. Profit during 2019 (P_1) = -32000
Profit during 2020 (P_2) = -5000
Profit during 2021 (P_3) = +12000

(i) Profit during 2019 and 2020 = $P_1 + P_2$

= (-32000) + (-5000)

= -37000

i.e., Loss for the general store during 2019, 2020 is Rs 37000

(ii) Net profit during the year 2020, 2021 = $P_2 + P_3$

= -5000 + 12000

= 7000

(iii) Net profit during 2019, 2020, 2021 = $P_1 + P_2 + P_3$

= (-32000) + (-5000) + 12000

= -25000

i.e., Loss of the three years = Rs. 25000

Illustration 1.3.15

The stereo shop has a loss of Rs. 2500 during its first year of business, a profit of Rs. 1000 during the second year and a profit of Rs. 4500 during third year. What was the average loss or profit for over the three years period?

Solution

First year profit(loss) P_1 = -2500

Second year profit P_2 = 1000

Third year profit P_3 = 4500

Average income = $P_1 + P_2 + P_3$

= (-2500) + 1000 + 4500

= (-2500) + 5500

= 3000

Hence the net profit for the three years = Rs. 3000

1.3.2.1 Addition or subtraction of Algebraic Expression

To add (subtract) algebraic expressions, combine like terms, which are terms with the same variables and exponents. Here is a step-by-step process for adding algebraic expressions:

Step 1: Identify Like Terms.

Step 2: Add (subtract) the Coefficients of Like Terms.

Step 3: Write the resulting expression with the added/subtracted coefficients.

For example,

1) The sum of $2x^2 + 5x$ and $3x^2 - 2x$ is $2x^2 + 5x + 3x^2 - 2x$

$= (2x^2 + 3x^2) + (5x - 2x)$

2) Subtract $2x^2 + 5x$ from $3x^2 - 2x$ is $x^2(2+3) + x(5-2)$

= $5x^2$- $3x$

1.3.2.2 Multiplication of Algebraic Expression

To multiply algebraic expressions, distribute each term in one expression by each term in the other expression, and then combine like terms.

Step 1: Distribute each term in the first expression by each term in the second expression.

Step 2: Multiply the coefficients (numeric parts) and add the exponents of the variables according to the rules of exponents.

Step 3: Combine the resulting terms that have the same variables and exponents.

For example, multiply the f algebraic expression $(2x + 3)(x + 2)$

$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6 = 2x^2 + 7x + 6$

1.3.2.3 Division of Algebraic Expression

To divide algebraic expressions, we follow a process similar to dividing numbers, but we use the rules of exponents and algebra.

Step1. Take the first digit of the dividend.

Stet 2. Divide the dividend by the divisor and write the corresponding quotient at the top.

Step 3. Subtract the product of the quotient and result and write below the remaining expression.

Step 4. Bring down the next digit of the dividend (if present).

This method of finding division is called method for dividing large numbers.

Usually, dividend is higher degree and divisor is of lower degree.

Illustration 1.3.16

Add $-3x^2 + 5x - 7$, $2x^2 - 3x + 10$

Solution

$(-3x^2 + 5x - 7) + (2x^2 - 3x + 10) = (-3x^2 + 2x^2) + (5x - 3x) + (-7 + 10)$

$= -x^2 + 2x + 3$

Illustration 1.3.17

Find $-3x^2 + x - 4) + (3x + 8 - x^2)$.

Solution

$(-3x^2 + x - 4) + (3x + 8 - x^2)$

$= (-3x^2 - x^2) + (x + 3x) + (-4 + 8)$

$= -4x^2 + 4x + 4$

Illustration 1.3.18

Subtract $2x^2 + 4x - 8$ from $4x - 3$.

Solution

$(4x - 3) - (2x^2 + 4x - 8) = (0x^2 + 4x - 3) - (2x^2 + 4x - 8)$

$= (0x^2 - 2x^2) + (4x - 4x) + (-3 + 8)$

$= -2x^2 + 0 + 5$

$= -2x^2 + 5$

Illustration 1.3.19

Subtract $2x^2 + 3x + 2$ from $3x^2 - 5x - 1$.

Solution

$(3x^2 - 5x - 1) - (2x^2 + 3x + 2) = x^2 - 8x - 3$

Illustration 1.3.20

Find $(6x^2 - 3x + 4) - (-2x^2 + 2x - 2)$

Solution

$(6x^2 - 3x + 4) - (-2x^2 + 2x - 2) = (6x^2 - 3x + 4) + (2x^2 - 2x + 2)$

$= 8x^2 - 5x + 6$

Illustration 1.3.21

Find $(6x - 3y)(2x + 5y)$

Solution

$(6x - 3y)(2x + 5y) = 6x(2x + 5y) - 3y(2x + 5y)$

$= (12x^2 + 30xy) - (6yx + 15y^2)$

$= 12x^2 + 24xy - 15y^2$

Illustration 1.3.22

Find $(5y + 3z)(7y - 15z)$

Solution

$(5y + 3z)\,(7y - 15z) = 5y(7y - 15z) + 3z(7y - 15z)$

$= 35y^2 - 75yz + 21zy - 45z^2$

$= 35y^2 - 75yz + 21yz - 45z^2$

$= 35y^2 - 54yz - 45z^2$

Illustration 1.3.23

Find $(a^2 - 2a)(a + 2b - 3c)$

Solution

$(a^2 - 2a)\ (a + 2b - 3c) = a^2 \times a + a^2 \times 2b + a^2\ (-8)\ c - 2a \times a$

$= a^3 + 2a^2b - 3a^2c - 2a^2 - 4ab + 6ac$

Illustration 1.3.24

Find $(2x + 3y)(x - y)$

Solution

$(2x + 3y)(x - y)$

$= 2x^2 - 2xy + 3xy - 3y^2$

$= 2x^2 + xy - 3y^2$

Illustration 1.3.25

Divide $x^3 - 2x^2 - 4$ by $x - 3$

Solution

$$
\begin{array}{r|l}
 & x^2 + x + 3 \\
\hline
x-3 & x^3 - 2x^2 - 4 \\
 & x^3 - 3x^2 \\
\hline
 & \quad x^2 + 0x \\
 & \quad x^2 - 3x \\
\hline
 & \qquad 3x - 4 \\
 & \qquad 3x - 9 \\
\hline
 & \qquad\quad 5
\end{array}
$$

$$\frac{x^3 - 2x^2 - 4}{x - 3} = x^2 + x + 3 + \frac{5}{x - 3}$$

Illustration 1.3.26

Divide $6x^2 + 10x - 24$ by $2x + 6$

Solution

$$
\begin{array}{r|l}
 & \quad 3x - 4 \\ \hline
2x+6 & 6x^2+10\,x-24 \\
 & 6x^2+18x \\ \hline
 & \quad -8x - 24 \\
 & \quad -8x - 24 \\ \hline
 & \qquad 0
\end{array}
$$

$$\frac{6x^2 + 10x - 24}{2x+6} = 3x - 4$$

Illustration 1.3.27

Divide $4x^3 - 5x - 21$ by $x - 3$

Solution

$$
\begin{array}{r|l}
 & \quad 4x + 7 \\ \hline
x - 3 & 4x^2 - 5x - 21 \\
 & 4x^2 - 12x \\ \hline
 & \quad 7x - 21 \\
 & \quad 7x - 21 \\ \hline
 & \qquad 0
\end{array}
$$

$$\frac{4x^2 - 5x - 21}{x-3} = 4x + 7$$

1.3.2.4 Properties of Algebraic Expression

1. Closure Property in Algebraic Expressions.

Let $a = a_1x + a_2y$, $b = b_1x + b_2y$

Then $a + b = (a_1 + a_2)\,x + (b_1 + b_2)\,y$

$a.b = (a_1x + a_2y).(b_1x + b_2y)$

The sum and product of two algebraic expression is an algebraic expression.
Hence algebraic expressions and closed with respect to addition and multiplication

Example. 1. $(2x + 7) + (7y - 2)$ is an algebraic expression.
2. $(7y - 2).(3x + 5)$ is an algebraic expression.

2. Commutative Properties

If a, b, c are algebraic expressions then $a + b = b + a$, $a \cdot b = b \cdot a$
Example. 1. Let $a = 5 + x$, $b = 9 - x$ then

$a + b = (5 + x) + (9 - x)$

$= (9 - x) + (5 + x)$

$= b + a$

Since real numbers addition is commutative

$a \cdot b = (5 + x)(9 - x)$

$= (9 - x)(5 + x)$

3. Associative Property in Algebraic Expressions.

Let a, b, c are algebraic expressions, then

$a + (b + c) = (a + b) + c$

$a(b \cdot c) = (a \cdot b) c$

Assume $a = a_1x + a_2$, $b = b_1x + b_2y$, $c = c_1x + c_2y$

Now $a + (b + c) = (a_1x + a_2y) + [(b_1x + b_2y) + (c_1x + c_2y)]$

$= [(a_1x + a_2y) + (b_1x + b_2y)] + (c_1x + c_2y)$

$= (a + b) + c$

$a \cdot (b \cdot c) = (a_1x + a_2y)\,[(b_1x + b_2y)(c_1x + c_2y)]$

$= [(a_1x + a_2y)(b_1x + b_2y)]\,(c_1x + c_2y)$

$= (a \cdot b) \cdot c$

Remark. On addition and multiplication of algebraic expressions (real number) translate them in to algebraic symbols. By associative property we may group them in any order.

Illustration 1.3.28

Evaluate $(-2)\,[(-5)(-x)]$

Solution

Ans. $(-2)[(-5)(-x)] = [(-2)(-5)]\,(-x)$

$= 10(-x)$

$= -10x$

Illustration 1.3.29

Find $10 \cdot (20x)$

Solution

$10 \cdot 20x = (10)(20)x$

$= 200\ x$

Illustration 1.3.30

Simplify the following using commutative property and associative property.

(1) $(7 - y) + 3$

(2) $\left(\frac{-3}{2}\right)\left(\frac{8}{5}x\right)$

(3) $\left(\frac{1}{2}x - \frac{3}{2}\right) \times \left(\frac{7}{2} \times \frac{8}{5}\right)$

Solution

(1) $(7 - y) + 3 = 3 + (7 - y)$ by commutative property

$= (3 + 7) - y$ by associative property

$= 10 - y$

(2) $\left(\frac{-3}{2}\right)\left(\frac{8}{5}x\right) = \left(\frac{8}{5}x\right)\left(\frac{-3}{2}\right)$

$= x \cdot \frac{8}{5}\left(\frac{-3}{2}\right)$

$= x \cdot \left(\frac{-24}{10}\right)$

$= x \cdot \left(\frac{-12}{5}\right)$

$= \frac{-12}{5}x$

(3) $\left(\frac{1}{2}x - \frac{3}{2}\right) \times \left(\frac{7}{2} \times \frac{8}{5}\right)$

$= \left(\frac{1}{2}x - \frac{3}{2}\right) \times \frac{56}{10}$

$= \left(\frac{1}{2}x - \frac{3}{2}\right) \times \frac{28}{5}$

$= \frac{28}{5}\left(\frac{1}{2}x - \frac{3}{2}\right)$

$= \frac{28}{5}\left(\frac{x-3}{2}\right)$

$= \frac{14}{5}(x-3)$

Illustration 1.3.31

Simplify

1. $8 + (5 - 6x)$
2. $7 + (3x - 6)$
3. $(3x - 2) + 8x$
4. $(8x - 5) + 5$

Solution

1. $8 + (5 - 6x) = (8 + 5) - 6x$

$= 13 - 6x$

2. $7 + (3x - 6) = 7 + [(-6) + 3x]$

$= [7 + (-6)] + 3x$

$= 1 + 3x$

3. $(3x - 2) + 8x = [(-2) + (3x)] + 8x$

$= (-2) + [(3x) + (8x)]$

$= -2 + 11x$

4. $(8x - 5) + 5 = 8x + [(-5) + 5]$

$= 8x$

4 Distributive Law in Algebraic Expressions

Distributive law of multiplication over addition can be stated as follows

$a.(b + c) = a.b + a.c$

$(a + b).c = a.c + b.c$

For example (1) $7(x + 2y) = 7x + (7)(2)y$

$= 7x + 14y$

Example (2) $7(3 + 2) = (7)(3) + (7)(2)$

$= 21 + 14$

$= 35$

Illustration 1.3.32

Expand (1) $8(7x - 2y + 5z)$ (2) $(-8x - 20y + x)5$

Solution

(1) $8\,(7x - 2y + 5z) = (8)(7)\,x + (8)(-2)\,y + (8)(5)z$

$= 56x - 16y + 40z$

(2) $(-8x - 20y + x)\,5 = [(-8)\,x + (-20)\,\mathrm{y} + x]\,5$

$= (-8)(x)(5) + (-20)(\mathrm{y})(5) + x(5)$

$= (-8)(5)x + (-20)(5)\mathrm{y} + 5x$

$= -40x - 100\mathrm{y} + 5x$

Illustration 1.3.33

Expand using the distributive properties

(1) 4 (2x + 3)   (2) (9x − y) 8   (3) – (3x + 2y)

Solution

1) 4 (2x + 3) = 4 (2x) + 4 (3)

= 8x + 12

2) (9x − y) 8 = [9x + (−y)] 8

= (9)(x)(8) + 8(−y)

= (9)(8)(x) + 8(−y)

= 72x − 8y

(3) – (3x + 2y) = (−1)(3x + 2y)

= (−1)(3)x + (−1)(2)y

= −3x − 2y

Illustration 1.3.34

Find (8x − 3y) − (3x − 2y)

Solution

(8x − 3y) − (3x − 2y)

= (8x − 3y) + 1(−3x + 2y)

= 8x − 3x − 3y + 2y

= 5x – y

Illustration 1.3.35

Simplify using distributive properties.

(1) $(7x - 2)\,5$

(2) $-10\,(8x - 3y)$

(3) $5\,(a - b + c)$

(4) $(3a + 5b - 2c)\,6$

Solution

(1) $(7x - 2)\,5 = (7x)(5) + (-2)\,5$

$= (5)(7)x + (-2)\,5$

$= 35x - 10$

(2) $-10\,(8x - 3y) = (-10)(8)x + (-10)(-3)\,y$

$= -80x + 30y$

(3) $5\,(a - b + c) = (5)(a) + 5\,(-b) + (5)(c)$

$= 5a - 5b + 5c$

(4) $(3a + 5b - 2c)6 = (3a)(6) + 5b\,(6) + (-2c)\,6$

$= (6)(3a) + (6)(5b) - 6\,(-2c)$

$= 18a + 30b - 12c$

Recap

- Algebraic expression - combination of symbols and terms that can be evaluated, simplified, or manipulated using algebraic rules
- Algebraic expression - expression in which integer constants, variables and algebraic operation '+' or '-' or 'x' or '÷'
- Algebraic operations - combination of operations
- Order of performing the operations is very important
- Sum and product of two algebraic expression is an algebraic expression
- If a, b, c are algebraic expressions then $a + b = b + a,\ a \cdot b = b \cdot a$

Objective Questions

1. Evaluate $7x - 5$ when $x = 4$
2. Find $\frac{r + 2s}{s - r}$, when $r = 1, s = 3$
3. Find $\frac{2(x - 1)}{(x + 1) + (x + 3)}$ when $x = 2$
4. Evaluate $(-15 + 6) + 9$
5. Evaluate the expression. $x - 3x - y$ when $x = 2, y = 1$
6. Find $(-4)^4$
7. Using commutative property write the expression $\left(\frac{3}{2}x\right)\left(\frac{1}{2}y\right)$ in another form.
8. Use associate property to simplify
   a. $3 + (2 + x)$
   b. $6 \cdot 2\ y\ x$
9. Find the value of $\left|\frac{1}{7} - \frac{13}{7}\right|$
10. Solve
    a. $\frac{8}{3}x \cdot \frac{y}{2}$
    b. $(\frac{3x}{5})(-\frac{1}{2}y)$

Answers

1. 23
2. $\frac{7}{2}$
3. $\frac{1}{4}$
4. 0
5. -5
6. 256

7. $\left(\frac{1}{2}y\right)\left(\frac{3}{2}x\right)$
8. a. 5+ x

   b. 12xy
9. $\frac{12}{7}$
10. a. $\frac{4}{3}xy$

    b. $\frac{3}{10}xy$

Assignments

1. Explain in detail about addition, subtraction, multiplication, and division of algebraic expressions with examples.
2. Explain in detail about closure property, commutative property, associative property, distributive property of algebraic expressions with examples.
3. Find
   a. (-3)(-2)
   b. (-12)(6)
   c. (25)(-4)
   d. 8×5
   e. $(5)\frac{3}{4} \times -\frac{3}{2}$
   f. $-\frac{3}{5} \times \frac{-5}{-4}$
   g. $6\left(\frac{7-9}{3}\right)$
   h. $\frac{1}{5}\left(\frac{5-6}{4}\right)$
   i. $\left(-\frac{3}{7}\right)\left(\frac{11-15}{8}\right)$

4. Solve

   a. $(2x - 1)(3x + 1)$, when x = 1

   b. $\frac{(7x - 4)(3x - 2)}{(2x + 1)}$, when x = 2

   c. $\frac{(20x - 5y)(x + 2)}{(2x + 3)}$, when x = 5, y = 3

   d. $\frac{|x + y||2x - y|}{|3x + 2|}$, when x = 1, y = 1

   e. 7x.5y, when x = 3, y = −2

   f. (7 - 2x)(5x - 2), when x = 3

5. Simplify

   a. Using commutative and associative property

      (i) $4 + (2x + 2)$

      (ii) $\left(\frac{4}{5}x\right)\left(\frac{10}{2}\right)$

      (iii) $(5 - 3x) + 6$

   b. Using distributive property

      (i) $5(3x - y)$

      (ii) $7x(2y)[3 + 2z]$

      (iii) $-10(5a + b - c)$

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UNIT 4

Linear Equations - Solution to a Pair of Linear Simultaneous Equations

Learning Outcomes

After completing this unit, the learner will be able to:

- solve linear equations
- gain an understanding of solving pairs of linear simultaneous equations
- apply linear equations and simultaneous equation techniques to model and analyze economic scenarios

Prerequisites

Suppose you have a bread manufacturing unit. It is important to know about the demand for the commodity you produce, in order to decide on the quantity of commodity to be produced. The demand for bread is the sum of individual demand for bread. If there is an established price for the commodity, then the demand for the commodity depends on the price of the commodity. The demand for the commodity can be represented as a linear equation. Suppose Q is the demand for bread, P is the price of bread, 'a' is the amount of bread that consumer will consume without any dependence on the price of bread, and 'x'is the quantity depended on price, then demand can be represented as Q = a - Px. If you know the demand for the commodity from the past consumption habit and price of the commodity, it is possible to get the demand for the commodity. Linear equation is widely used in representing economic concepts mathematically. It is possible to represent supply function, budget function, and so on in linear equation where the unknowns in the equation can be find out mathematically. Let us discuss the mathematical operation of linear equations.

Keywords

Linear equations, Unknown variables, Solutions, Simultaneous equations

Discussion

In this unit, we are discussing about the linear equations. Let us consider an economic concept to introduce linear equation, such as the cost of a commodity. If the cost of a commodity, represented by x, increases daily it can be expressed as 5x + 3. Therefore, to find the root of the equation 5x + 3=0, we can solve for x as $x = \frac{(-3)}{5}$. Hence $x = \frac{-3}{5}$ which is the root of the equation. In this example, we have one variable, but there are cases where an equation contains two variables.

The profit of a commodity depends on the supply y. so we can write it as 5x= 3y
Hence an equation in which two variables are also linear. Similarly, there are equations of 3 first degree unknowns x, y and z say y = 2x - 3z

1.4.1 Linear Equation

An equation of the form $ax + b = 0$ is called a linear equation. It is also called a simple equation. A linear equation has one or two variables. No variable in a linear equation is raised to a power greater than 1 or used as a denominator of a fraction.

An equation which contains two variables of first degree is a linear equation in x and y. Two or more such equations of first degree in x and y are called simultaneous simple equations. The values of x and y that satisfies these equations are called solutions of the equations.

If x, y and z are three unknowns, the values of x, y, and z are the solutions and we will get the values from three simultaneous linear equations.

To find the value of x, y, and z, there will be three simultaneous equations with three unknowns x, y and y. Using the two equations, eliminate one unknown and then we have two equations in two unknowns. Solving the remaining two and find out one more unknown and substitute the value in these equations, we have the value of the third variable. Let us see how the values of unknown are calculated under linear equations.

Illustration 1.4.1

A batsman finds that by scoring a century in the 11^{th} innings of his test games, he has bettered his average of the previous ten innings by 5. What is the average after the 11^{th} innings?

Solution

Let's denote the batsman's average for the first ten innings as "x". The total runs scored in these ten innings would be 10x, as the average is calculated by dividing the total runs by the number of innings.

The average after the 11th innings is better by 5 runs, so the average for the 11 innings would be x + 5. The total runs scored in eleven inning is 11(x+5).

In the 11th inning, the batsman scores a century (100 runs), so the total runs after the 11th inning would also be given as $10x + 100$.

So, 10x + 100 =11(x + 5)

10x + 100 = 11x + 55

x = 45

So, the average of the first ten innings is 45.

The average after the 11th inning = (x + 5)

= 45 + 5 = 50

So, the average after the 11th inning is 50.

Illustration 1.4.2

The average monthly income of a person for the first eight month of a year exceeds that of the first seven months of the year by Rs 25. If he earns Rs 1275 in the eighth month of the year. Find his average monthly income for the first eight months of the year?

Solution

Assume the average income for the first eight months = Rs x per month

If average monthly income of eight months exceeds the average monthly income of seven months by Rs. 25, the average monthly income of first seven months is x-25, the total income for seven months will be 7(x-25).

The average income of eight months can be written as:

$$\frac{7(x-25)+1275}{8} = x$$

i.e. 7x-175+1275 = 8x

⟹ x = 1275-175 = Rs. 1100

Illustration 1.4.3

Suppose the ratio of the prices of two houses was 16:23. After two years, first house had risen by 10 % and second house had risen Rs 477. Then, the ratio of the prices of two houses becomes 11:20. Find the original prices of the two houses.

Solution

Let the prices of two houses be 16x and 23x,

Increasing in price of the first house is $16x + 16x\,\frac{10}{100}$ and the price of second house is 23x + 477. Since the ratio of price of two houses after two years is given as 11:20, we can write

$$\frac{16x+16x.\frac{10}{100}}{23x+477} = \frac{11}{20}$$

Doing the operation of LCM

$$\frac{\frac{160x+16x}{10}}{23x+477} = \frac{11}{20}$$

Rearranging the equation,

$$\frac{160x+16x}{10(23x+477)} = \frac{11}{20}$$

$(160x + 16x)20 = 11[10(23x + 477)]$

3200x + 320x = 2530x + 52470

3520x = 2530x + 52470

3520x – 2530x = 52470

990x = 52470

x = 53

The original price of first house = 16x

= 16 × 53

= 848

The original price of second house is 23x

= 23 × 53

= 1219

∴ The original prices of the two houses are Rs 848 and Rs 1219

Illustration 1.4.4

Solve the simultaneous equations, $3x + 5y + 7 = 0$; $4x + 7y + 3 = 0$

Solution

Given equations $3x + 5y + 7 = 0$ – – – – – – – – – – (1)

$4x + 7y + 3 = 0 ------------ (2)$

Equation $(1) \times 4$ is $\quad 12x + 20y = -28$

Equation $(2) \times 3$ is $\quad 12x + 21y = -9$

Subtracting the above two equations, -1 y = -19

So, $y = 19$

Substituting the value of y in equation (1)

$$3x = -7 - 5y$$

$$3x = -7 - 95$$

$$3x = -102$$

Hence $x = \frac{-102}{3}$

= -34, when $y = 19$

$x = -34$ and $y = 19$

Illustration 1.4.5

Solve the simultaneous equations, $5x - 2y + 25 = 0; \quad 4y - 3x - 29 = 0$

Solution

Given equations $\quad 5x - 2y + 25 = 0 ---------- (1)$

$$-3x + 4y - 29 = 0 ------------ (2)$$

Equation $(1) \times 3$ is $\quad 15x - 6y = -75$

Equation $(2) \times 5$ is $\quad -15x + 20y = 145$

Adding the above two equations,

$$14y = 70$$

$y = \frac{70}{14} = 5$ Substituting the value of y in equation (1)

$$5x = -25 + 2y$$

$$5x = -25 + 10$$

$5x = -15, \ x = \frac{-15}{5} = -3,$ when $y = 5$

$x = -3$ and $y = 5$

Illustration 1.4.6

Solve the simultaneous equations,

$x + y + z = 6, \quad -x + 2y + 3z = 14, \quad -x + y - z = -2$

Solution

Given equations $\quad x + y + z = 6 ---------- (1)$

$$-x + 2y + 3z = 14 ----------- (2)$$

$$-x + y - z = -2 ---------- (3)$$

Adding equation (1) and (2) is $3y + 4z = 20 ------ (4)$

Subtracting equation (2) - (3) is $y + 4z = 16 - - - - - - (5)$

Subtracting the equation (4) and (5) is $2y = 4$

$$y = 2$$

Substitute the value of y in the equation (5)

$$4z = 14$$

$$z = \frac{14}{4} = \frac{7}{2}$$

Substitute the value of y and z in the equation (1)

$$x = 6 - y - z$$

$$x = 6 - 2 - 7/2$$

$$= 4 - 7/2$$

$$x = \frac{8-7}{2}$$

$$x = \frac{1}{2}$$

So, $x = \frac{1}{2}$, $y = 2$, $z = \frac{7}{2}$

Illustration 1.4.7

Solve $\frac{x}{4} = \frac{y}{3} = \frac{z}{2}$, $7x + 8y + 5z = 62$

Solution

Lets say given equations $\frac{x}{4} = \frac{y}{3} = \frac{z}{2} = k$ ------------------(1)

$7x + 8y + 5z = 62 - - - - - - - - (2)$

From equation (1), the value of the variables can be written as $x = 4k, y = 3k, z = 2k$

Substitute the values of variables in equation (2),

$7 \times 4k + 8 \times 3k + 5 \times 2k = 62$

$28k + 24k + 10k = 62$

$62k = 62$

$k = 1$

Substitute the value of k,

$x = 4 \times 1 = 4$, $y = 3 \times 1 = 3$, $z = 2 \times 1 = 2$

$x = 4, y = 3$ and $z = 2$

Recap

- Linear equation - An equation of the form ax + b = 0
- A linear equation may have more than one variable
- Variables are not raised to a power greater than 1
- Equation which contains two variables of first degree is a linear equation in x and y
- Simultaneous simple equations - two or more equations of first degree in x and y
- Solutions – values of unknown variables
- Eliminating one unknown and later solve for other unknowns
- Substitute value of solution to equation for the unknown variables

Objective Questions

1. What is linear equation?
2. What does x and y in linear equations represents?
3. What are solutions?
4. What are simultaneous equations?
5. What is the power of variables under linear equations?
6. Give a normal form of linear function.

Answers

1. Linear equation is an equation of the form $ax + b = 0$
2. Unknown variables
3. Solutions are values of unknown variables
4. Simultaneous simple equations are two or more equations of first degree in x and y
5. 1
6. ax+b

Assignments

1. Solve

   $x + y + z = 6$ ………. (1)

   $x - y + z = 2$ ……….. (2)

   $2x + y - z = 1$ ……….. (3)

2. Solve

   $x+y = 5$ …………… (1)

   $x-y = -1$ ……………. (2)

3. Solve

   $3x + y = -7$ ………….. (1)

   $x-4y = 2$ ……………… (2)

4. Solve

   $3x + 4y = 2$ …………. (1)

   $6x + 8y = 24$ ………… (2)

5. Determine the price when demand, $d = 75 - 4P$ and supply, $s = 20 + 3P$

Suggested Readings

1. Carl P. Simon and Lawrence Blume, Mathematics for Economist, W. W. Norton & Company, 1994
2. Dowling Edward T, Mathematical Methods for Business and Economics, Schaums Outline Series, McGraw Hill, 1993
3. Mike Rosser, Basic Mathematics for Economist, 2^{nd} Edition, Routledge, London and New York, 2003
4. Sydsaeter K and Hammond P, Essential Mathematics for Economic Analysis, Prentice Hall

References

1. Dowling Edward.T, Introduction to Mathematical Economics, 2nd/3rd Edition, Schaum‘s Outline Series, McGraw-Hill, New York, 2003
2. G S Monga, Mathematics and Statistics for Economics, Second revised edition,Vikas Publishing House PVT LTD, 2002
3. Taro Yamane, Mathematics for Economists: An Elementary Survey, Prentice Hall of India

5

UNIT

Quadratic Equations

Learning Outcomes

After completing this unit, the learner will be able to:

- have a comprehensive understanding of quadratic equations
- solve quadratic equations using various techniques
- apply quadratic equations to model economic situations
- know the economic implications of quadratic equation solutions

Prerequisites

Alice was a young math enthusiast. She had always been curious about solving puzzles, but she faced challenges while solving the puzzles. She had set her heart on solving the mysterious Quadratic Riddle. However, she knew that to tackle the Quadratic Riddle, she needed to prepare herself. Her journey to find a solution comes to a head at the doorstep of the wise mathematician. He revealed the use of a Quadratic formula for unlocking the quadratic riddles. Then she realized that quadratic equations were more than just mathematical enigmas. They were the keys to understanding economic optimization, modeling, profit and loss, and predicting market fluctuations.

The world of quadratic equations awaits, it allows informed decision making Her adventure was just beginning, but she knew that she wanted to know about algebra, numbers, and related concepts to use the quadratic equations for solving quadratic riddles. They are the stepping stones that help students successfully navigate toward the study of quadratic equations.

Keywords

Quadratic equation, Polynomial, Linear equation, Discriminant, Identity

Discussion

1.5.1 Quadratic Equation

A polynomial of second degree is called a quadratic polynomial. Any equation f(x) = 0, where f(x) is a quadratic polynomial is called a quadratic equation.

Example (i) $x^2 - 5x + 6 = 0$

(ii) $3x^2 + 2x - 9 = 0$

These equations can be factorised $x^2 - 5x + 6 = 0$ can be factorised as $(x-2)(x-3) = 0$. Then 2 and 3 are the roots of the equation.

An algebraic expression equated to another expression or zero is called an equation. In an equation only specified values are satisfied. That is, in an equation only specified values for unknowns satisfies the equation.

But if all the unknowns are satisfied by an equation, it is called an identity. Hence identity is a statement of equality between two expressions which is true for all values of the variables involved.

Examples (i) $x^2 - 2xy + y^2 = (x - y)^2$

(ii) $(x-y)(x+y) = x^2 - y^2$

(i)and (ii) are example of identity.

1.5.2 Solutions of Quadratic Equations

Formula for solving $ax^2 + bx + c = 0$

Given $ax^2 + bx + c = 0$

Multiplying by 4a; $4a^2x^2 + 4abx + 4ac = 0$

ie $4a^2x^2 + 4abx = -4ac$

adding b^2 both sides

$b^2 + 4a^2x^2 + 4abx = b^2 - 4ac$

$(2ax + b)^2 = b^2 - 4ac$

$2ax + b = \pm \sqrt{b^2 - 4ac}$

$2ax = -b \pm \sqrt{b^2 - 4ac}$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ -------------(1)

Hence $\Delta = b^2 - 4ac$ is called the discriminant.

From (1) we will get two values for x say α and β where

$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Remark: If α and β are the roots of the equation $ax^2 + bx + c = 0$ then

$\alpha + \beta = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

$\alpha\beta = \frac{\text{constant term}}{\text{coefficient of } x^2}$

We have $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$,

$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

$\alpha + \beta = \frac{-2b}{2a} = \frac{-b}{a}$

$\alpha\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$

$= \frac{b^2 - (b^2-4ac)}{4a^2} = \frac{c}{a}$

1.5.2.1 Condition for Nature of Roots of a Quadratic Equation

If a, b, c are real numbers determine the condition that the roots of the equation $ax^2 + bx + c = 0$ are

(i) real and unequal
(ii) real and equal
(iii) imaginary

we have the roots of the equation $ax^2 + bx + c = 0$ is $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

i.e. $x = \frac{-b\pm\sqrt{\Delta}}{2a}$ where Δ is the discriminant $b^2 - 4ac$.

Since the roots are $x = \frac{-b\pm\sqrt{\Delta}}{2a}$

when $\Delta > 0$, the values of x (roots) are real and unequal.
When $\Delta = 0$, roots are real and equal.

When $\Delta < 0$, roots are imaginary and unequal.

Illustration 1.5.1

Solve the equation

$21x^2 - 41x + 10 = 0$

Solution

$a = 21, b = -41, c = 10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{41 \pm \sqrt{(-41)^2 - 4 \times 21 \times 10}}{42}$$

$$= \frac{5}{3} \text{ or } \frac{2}{7}$$

Illustration 1.5.2

Examine the nature of the roots of the equation $3x^2 - 15x + 8 = 0$

Solution

Given equation $3x^2 - 15x + 8 = 0$

Discriminant $\Delta = b^2 - 4ac$

$= (-15)^2 - 4 \times 3 \times 8$

$= 225 - 96$

$= 129$

Roots are real and unequal.

Illustration 1.5.3

Show that the roots of the equation $3x^2 + 4x + 4 = 0$ is imaginary.

Solution

$a = 3, b = 4, c = 4$

$b^2 - 4ac = 16 - 4 \times 3 \times 4$

$= 16 - 48 = -32 < 0$

So the roots are imaginary.

Illustration 1.5.4

If $b = a + c$ prove that the roots of the equation $ax^2 + bx + c = 0$ are real.

Solution

$\Delta = b^2 - 4ac$

$= (a + c)^2 - 4ac$

$= (a - c)^2 > 0$

$\therefore$ roots are real.

Illustration 1.5.5

If the roots of $a^2x^2 + 2bx + c^2 = 0$ are imaginary, then show that the roots of $b(x^2 + 1) + 2acx = 0$ are real and different.

Solution

Given two equations

$a^2x^2 + 2bx + c^2 = 0$ -------------------(1)

and $bx^2 + 2acx+b = 0$ -------------(2) in which discriminant of (1) is less than 0.

Let Δ_1 be the discriminant of (1) and Δ_2 be the discriminant of (2)

Also given $\Delta_1 < 0$

i.e. $(2b^2)^2 - 4a^2c^2 < 0$

$4(b^2 - a^2c^2) < 0 \Longrightarrow b^2 - a^2c^2 < 0$

$\Longrightarrow a^2c^2 - b^2$ -----------------(3)

Δ_2 = discriminant of (2)

$= (2ac)^2 - 4b^2$

$= 4\, a^2c^2 - 4b^2$

$= 4\,(a^2c^2 - b^2) > 0$

$\therefore$ Roots of (2) are real and unequal.

Illustration 1.5.6

Find the value of m for which $x^2 - 2mx + 8m - 15 = 0$ has equal roots.

Solution

Given the equation $x^2 - 2mx + 8m - 15 = 0$----------(1)

Its discriminant = $(-2m)^2 - 4 \times 1 \times (8m - 15)$

$= 4m^2 - 32m + 60$

$= 4(m^2 - 8m + 15)$

When roots of (1) are equal its discriminant = 0

i.e. $4(m^2 - 8m + 15) = 0$

i.e. $m^2 - 8m + 15 = 0$

i.e. $(m-3)(m-5) = 0$, $m = 3,5$

Illustration 1.5.7

A piece of cloth costs Rs 35. If the piece were 4 metre longer and each metre costs Rs 1.00 less, the cost would remains unchanged. How long is the piece?

Solution

Let x be the length of the cloth.

Then cost of cloth piece = Rs 35

$\therefore$ cost per metre = $\frac{35}{x}$

Given a length of cloth piece 4 metre.

Longer cost Rs 1 less

$\therefore$ cost per metre = $\frac{35}{x} - 1$

$\therefore$ cost of cloth with 4+x metre length = $(4 + x)\left(\frac{35}{x} - 1\right)$

Since costs remains unchanged,

$(4 + x)\left(\frac{35}{x} - 1\right) = 35$

i.e. $(4 + x)(35 - x) = 35x$

$x^2 + 4x - 140 = 0$

$x = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times -140}}{2x1}$

$= \frac{-4 \pm \sqrt{16 + 560}}{2}$

$= \frac{-4 \pm \sqrt{576}}{2}$

= 10 or -14

Here length is non-negative, take x = 10

Illustration 1.5.8

The sum of the squares of two numbers is 233 and one of the number is 3 less than twice the other. Find the numbers.

Solution

Let x and y be the numbers where y = 2x - 3

Also given that $x^2 + y^2 = 233$

$\Longrightarrow x^2 + 4x^2 - 12x + 9 = 233$

$\Longrightarrow 5x^2 - 12x - 224 = 0$

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4\times5\times-224}}{2x5}$$

$$= \frac{12 \pm \sqrt{144+4480}}{10}$$

$$= \frac{12 \pm 68}{10} = 8 \text{ or } -\frac{28}{5}$$

When $x = 8, y = 2 \times 8 - 3 = 13$

$$x = \frac{-28}{5}, y = 2 \times \frac{-28}{5} - 3$$

$$= \frac{-71}{5}$$

1.5.2.2 Equations of the Form

$a.[f(x)]^2 + b\, f(x) + c = 0$

Illustration 1.5.9

Solve $(x^2 - 5x)^2 - 30(x^2 - 5x) - 216 = 0$

Solution

Let $u = x^2 - 5x$

Then given equation becomes

$u^2 - 30u - 216 = 0$

$(u - 36)(u + 6) = 0$

$\Longrightarrow u = 36, u = -6$

$u=36 \Longrightarrow x^2 - 5x = 36$

$\Rightarrow x^2 - 5x - 36 = 0$

$(x + 4)(x - 9) = 0$ $\quad\quad\quad\quad$ $p + q = -5,\ pq = -36$

$x = -4,\ x = 9$

$u = -6 \Rightarrow x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0,\ x = 2,3$

Illustration 1.5.10

Solve $(x^2 - 6x)^2 - 2(x^2 - 6x) = 35$

Solution

Put $u = x^2 - 6x$. then given equation becomes $u^2 - 2x - 35 = 0$

ie $(u - 7)(u + 5) = 0$, $u = 7$ or -5

when $u = 7$, $x^2 - 6x = 7$

ie $x^2 - 6x - 7 = 0$,

$(x - 7)(x + 1) = 0$

$x = 7$, or -1

$u = -5 \Rightarrow x^2 - 6x = -5$

$\Rightarrow x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0,\ x = 5,1$

Illustration 1.5.11

Solve $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1$

Solution

$(x^2 - 5x + 7)^2 - (x^2 - 5x + 6) - 1 = 0$

Put $u = x^2 - 5x + 7$

Then given equation becomes $u^2 - (u + 1) - 1 = 0$

i.e. $u^2 - u = 0$

$u(u-1) = 0$

$u = 0$ or $u = 1$

when $u = 0$, $x^2 - 5x + 7 = 0$

$\Rightarrow x = 2,3$

When u = 1 $\Longrightarrow x^2 - 5x + 7 - 1 = 0$, i.e., $x^2 - 5x + 6 = 0$

x = 3, 2

Illustration 1.5.12

$(x + \frac{1}{x})^2 - \frac{7}{2}(x + \frac{1}{x}) = 2$

Solution

Given equation $(x + \frac{1}{x})^2 - \frac{7}{2}(x + \frac{1}{x}) = 2$

Put $x + \frac{1}{x} = u$ then (1) becomes

$u^2 - \frac{7}{2}u = 2$

Then both sides multiply with 2, we get $2u^2 - 2\,\frac{7}{2}\,u = 2 \times 2$

$2u^2 - 7u - 4 = 0$

p + q = -7, pq = -8

$2u^2 + u - 8u - 4 = 0$

$u(2u + 1) - 4(2u+1) = 0$

$(2u + 1)(u-4) = 0$

$2u + 1 = 0$ or $u-4 = 0$

$u = \frac{-1}{2}$, $u = 4$

when $u = \frac{-1}{2}$, $x + \frac{1}{x} = \frac{-1}{2}$

$2x^2 + 2 = -x$

$2x^2 + x + 2 = 0$

$x = \frac{-1 \pm \sqrt{15}\, i}{4}$

when u = 4; $x + \frac{1}{x} = 4$

$x^2 - 4x + 1 = 0$

$x = 2 \pm \sqrt{3}$

Hence $x = 2 \pm \sqrt{3}, \frac{-1 \pm i\sqrt{15}}{4}$

Illustration 1.5.13

Solve $2\left(x + \frac{1}{x}\right)^2 - 7\left(x + \frac{1}{x}\right) + 5 = 0$

Solution

Given equation $2\left(x + \frac{1}{x}\right)^2 - 7\left(x + \frac{1}{x}\right) + 5 = 0$

Put $u = \left(x + \frac{1}{x}\right)$

Then $2u^2 - 7u + 5 = 0$, $u = 1, \frac{5}{2}$

$u=1$, $x + \frac{1}{x} = 1$ or $x^2 - x + 1 = 0$

$x = \frac{1 \pm \sqrt{3}\, i}{2}$

For $u = \frac{5}{2}$, $x + \frac{1}{x} = \frac{5}{2}$

Or $2x^2 - 5x + 2 = 0$ then $x = 2, \frac{1}{2}$

Hence the four roots are $2, \frac{1}{2}, \frac{1 \pm \sqrt{3}\, i}{2}$

Illustration 1.5.14

Solve $10\left(\frac{x}{2x-1}\right) - 2\left(\frac{2x-1}{x}\right) + 1 = 0$

Solution

Put $u = \frac{x}{2x-1}$

$10u - 2.\frac{1}{u} + 1 = 0$

ie $10u^2 + u - 2 = 0$

$u = \frac{2}{5}, -\frac{1}{2}$

$u = \frac{2}{5} \Longrightarrow \frac{x}{2x-1} = \frac{2}{5}$

$5x = 4x-2$

$x = -2$

$u = -\frac{1}{2}, \frac{x}{2x-1} = -\frac{1}{2}$

$\Longrightarrow 2x = -2x - 1$

$\Longrightarrow 4x = 1, x = \frac{1}{4}$

Illustration 1.5.15

Solve $x^2 - 4x - 3 = \frac{14}{x^2 - 4x+2}$

Solution

Put $x^2 - 4x + 2 = u$

Given equation becomes

$x^2 - 4x + 2 - 5 = \frac{14}{x^2 - 4x+2}$

$u - 5 = \frac{14}{4}$

$u^2 - 5u - 14 = 0$, $u = 7$ or -2

when $u = 7$; $x^2 - 4x + 2 = 7$

$x^2 - 4x - 5 = 0 \Longrightarrow x = 5$ or -1

$u = -2$; $x^2 - 4x + 2 = -2$

$x^2 - 4x + 4 = 0$, $x = 2, 2$

Illustration 1.5.16

Solve $10\sqrt{\frac{x}{x+3}} - \sqrt{\frac{3+x}{x}} = 3$

Solution

Put $\sqrt{\frac{x}{x+3}} = u$

Given equation becomes

$10.u - \frac{1}{u} = 3$

$10u^2 - 3u - 1 = 0$

$u = -\frac{1}{2}$ or $\frac{1}{5}$

when $u = -\frac{1}{2} \Rightarrow \sqrt{\frac{x}{x+3}} = -\frac{1}{2}$

$\Rightarrow \frac{x}{x+3} = \frac{1}{4}$

$4x = x + 3 \Rightarrow x = 1$

$u = \frac{1}{5} = \sqrt{\frac{x}{x+3}} = \frac{1}{5} \Rightarrow \frac{x}{x+3} = \frac{1}{25}$

$25x = x + 3$, $x = 1, \frac{1}{8}$

Illustration 1.5.17

Solve $7^{x+1} + 7^{1-x} = 50$

Solution

Given equation is $7^x.7 + 7^1.7^{-x} = 50$

i.e. $7.7^x + 7.\frac{1}{7^x} = 50$

put $7^x = u$, then equation becomes

$7u + 7.\frac{1}{u} = 50$

ie $7u^2 - 50u + 7 = 0$

$u = 7$ or $\frac{1}{7}$

When $u = 7$, $7^x = 7 \Rightarrow x = 1$

$u = \frac{1}{7} = 7^{-1}$, $7^x = 7^{-1} \Rightarrow x = -1$

1.5.2.3 Solutions of Radical Equation

Illustration 1.5.18

Solve $\sqrt{x} = x-2$

Solution

Given equation $\sqrt{x} = x-2$

Squaring, $x = (x-2)^2$

Then $x = x^2 - 4x + 4$

i.e. $x^2 - 4x + 4 – x$

$= x^2 - 5x + 4$

$x = 4, 1$

$x = 1$

Illustration 1.5.19

Find the limits for m that ensure the roots of the equation

$x^2 - 4(m-1)x + (m-1)(m-2)$ are be real.

Solution

Given equation $x^2 - 4(m-1)x + (m-1)(m-2)$

Roots are real discriminant $\Delta > 0$

ie when $[4(m-1)]^2 – 4\times1\times (m-1)(m-2) > 0$

ie $4(m-1)^2 – (m-1)(m-2) > 0$

$4(m^2 - 2m + 1) - (m^2 - 3m + 2) > 0$

$3m^2 - 5m + 2 > 0$

$(m-1)(3m-2) > 0$

i.e., either $m-1 > 0$ and $3m-2 > 0$
or $m-1 < 0$ and $3m - 2 < 0$

$\Longrightarrow$ either $m > 1$ and $m > \frac{2}{3}$

Or $m < 1$ and $m < \frac{2}{3}$

$\Longrightarrow$ either $m > 1$ and $m < \frac{2}{3}$

$\Longrightarrow$ m should not lie between $\frac{2}{3}$ and 1

Illustration 1.5.20

If one of the roots of the equation $x^2 + px + q = 0$ is square of the other, show that $p^2 + q^2 + q = 3pq$

Solution

Given equation $x^2 + px + q = 0$ ----------------(1)

Let α and α^2 be the roots of (1)

Then $\alpha + \alpha^2 = -p$ $\quad\quad$ $\alpha . \alpha^2 = q$

$(\alpha + \alpha^2)^3 = (-p)^3$

$\alpha^3 + \alpha^6 + 3\alpha^3(\alpha + \alpha^2) = -p^3$

$q + q^2 + 3q(-p) = -p^3$

$p^3 + q^2 + q = 3pq$

Illustration 1.5.21

If α and β are the roots of the equation $x^2 - px + q = 0$, find the values of

(i) $\alpha^2 + \beta^2$ $\quad$ (ii) $\alpha^3 + \beta^3$ $\quad$ (iii) $\alpha^2 \beta + \alpha \beta^2$

Solution

Given equation $x^2 - px + q = 0$--------------(1)

Since α and β are the roots of (1)

$\alpha + \beta = p, \quad \alpha\beta = q$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= p^2 - 2q$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= p^3 - 3qp$

(iii) $\alpha^2 \beta + \alpha \beta^2 = \alpha\beta(\alpha + \beta)$

$= q.p$

Illustration 1.5.22

Obtain a quadratic equation when roots α and β are 2 and 3?

Solution

Given the roots $\alpha = 2; \beta = 3$

Sum of the roots = 2 + 3 = 5

Product of the roots = 2.3 = 6

$\therefore$ equation of the required equation is $x^2 - 5x + 6 = 0$

Illustration 1.5.23

If α and β are the roots of the equation $ax^2 + bx + c = 0$ form an equation whose roots are α^3 and β^3?

Solution

α and β are given roots of $ax^2 + bx + c = 0$

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

The equation with α^3 and β^3 are having sum of roots $\alpha^3 + \beta^3$

$$= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right) \times -\frac{b}{a}$$

$$= -\frac{b^3}{a^3} + \frac{3bc}{a^2}$$

$$= -\frac{b^3}{a^3} + \frac{3bc}{a^2} \times \frac{a}{a}$$

$$= -\frac{b^3}{a^3} + \frac{3abc}{a^3}$$

$$= \frac{3abc - b^3}{a^3}$$

Products of the roots $= \alpha^3 \beta^3 = \frac{c^3}{a^3}$

$\therefore$ required equation is $x^2 - \frac{3abc - b^3}{a^3}x + \frac{c^3}{a^3} = 0$

ie $a^3x^2 + (b^3 - 3abc)x + c^3 = 0$

Recap

- Quadratic polynomial- polynomial of second degree
- Quadratic equation - $f(x) = 0$, where $f(x)$ is a quadratic polynomial
- Equation - algebraic expression equated to another expression or zero
- Identity- if all the unknown values are satisfied by an equation
- Discriminant $\Delta = b^2 - 4ac$
- $\Delta > 0$, the values of x (roots) are real and unequal
- $\Delta = 0$, roots are real and equal
- $\Delta < 0$, roots are imaginary and unequal

Objective Questions

1. Define equation.
2. What is quadratic polynomial?
3. What is quadratic equation?
4. What is identity?
5. What is discriminant?
6. Solve $\sqrt{x} = x-2$
7. Condition for two roots is to be equal.
8. What is the condition for two roots are imaginary and unequal?
9. Comment the formula for quadratic equations.
10. Give an example for identity.

Answers

1. An algebraic expression equated to another expression or zero
2. A polynomial of second degree is quadratic polynomial
3. Any equation f(x) = 0, where f(x) is a quadratic polynomial
4. If all the unknown values are satisfied by an equation- identity
5. $\Delta = b^2 - 4ac$
6. Squaring, $x = (x-2)^2$

   i.e. $x^2 - 5x + 4 = 0$

   $x = 4, 1$
7. When $\Delta = 0$, roots are equal.
8. When $\Delta < 0$, roots are imaginary and unequal
9. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
10. $(x-y)(x+y) = x^2 - y^2$

Assignments

1. Solve for x
   a. $5 - 3x = -1$
   b. $2x^2 - 5x + 3=0$

2. Solve the following

   a. $\frac{x+2}{3} - \frac{x+3}{5} = \frac{x+1}{2} + \frac{x+5}{4}$

   b. Solve $\frac{x}{x-3} + \frac{x-9}{x-7} = \frac{x+1}{x-1} + \frac{x-8}{x-5}$

   c. Solve $8x + \frac{x+2}{3} = 2x - 3\frac{(x-5)}{2}$

3. Prove that the roots of the equations $x^2 - 2ax + a^2 + b^2 + c^2 = 0$

Suggested Readings

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2

BLOCK

Fundamentals of Matrix Algebra I

UNIT 1

Introduction to Matrices

Learning Outcomes

After completing this unit, the learner will be able to:

- grasp the meaning and types of matrixes
- understand the meaning and calculation of vector
- apply the matrix and vectors to solve economic problems

Prerequisites

A small harbour where fishing boats set sails each day. The villagers relied on the sea for their livelihood, and the ocean was both their ally and their challenge. They had heard whispers of a mystical map, one that could guide them through the vast sea with precision and accuracy. Before embarking on their journey, these villagers needed to understand two remarkable tools: vectors and matrices. Vectors were like the compass, pointing the way with direction and magnitude. They allowed the villagers to chart their course, whether it be towards the open sea or back to the safety of the harbour. But vectors alone were not enough. The explorers also needed matrices, which were like the map itself. Matrices could organise vast amounts of information and reveal hidden patterns, just as the sea held its secrets beneath the waves. They could help the villagers plan their voyages, allocate resources, and predict weather patterns.

With the knowledge of vectors and matrices, the villagers boarded on their explorations. They used vectors to navigate the false currents and matrices to analyse the changing tides. They discovered new lands, optimised their fishing routes, and adapted to the ever-shifting sea. Just like those villagers, you are about to begin on a voyage of mathematical navigation. The world of vectors and matrices is your map and compass, guiding you through the complexities of economics. As you familiarise into these tools, remember the seaside village, the open sea, and the mysteries that lay beneath. Embrace the challenge, for within the realm of vectors and matrices lies the power to navigate the economic landscape, make informed decisions, and discover the treasures of knowledge that await your exploration. The voyage begins, and you are the navigators, poised to chart the course to economic understanding.

Keywords

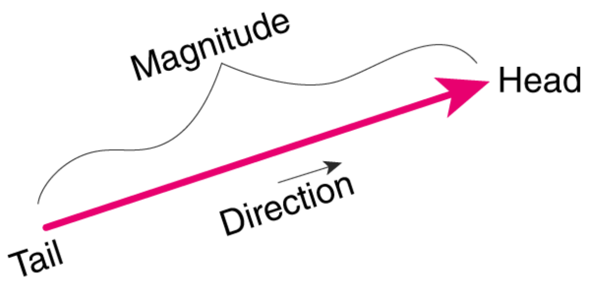
Vector, Matrix, Order, Transpose, Equality

Discussion

2.1.1 Vectors

Suppose a warehouse has n different commodities denoted by $X_1, X_2, \ldots\ldots X_n$. Each month the units $a_1, a_2 \ldots\ldots a_n$ of each good in stock is recorded. It is convenient to represent these stock quantities by either a row: $(a_1, a_2 \ldots\ldots a_n)$ or a column $\begin{pmatrix} a1 \\ a2 \\ \vdots \\ an \end{pmatrix}$. Such an ordered set of numbers, which is distinguished not only by the elements it contains, but also by the order in which they appear, is called a vector. In particular, $(a_1, a_2 \ldots\ldots a_n)$ is called row vector where as $\begin{pmatrix} a1 \\ a2 \\ \vdots \\ an \end{pmatrix}$ is column vector. It is clear that the row vector and the column vector contain exactly the same information - the numbers and their order are the same, only the arrangement of the numbers is different.

In other words, a geographical entity that has magnitude and direction is called a vector. In a line, the length of the line is its magnitude and arrow is its direction. The starting point of the vector is known as tail and end point is head.



2.1.2 Types of Vectors

1.Zero Vector: When the magnitude of the vector is zero, it is called zero vector.
2.Unit Vector: When the magnitude of the vector is one unit, it is said to be a unit vector.
3.Coinitial Vectors: Two or more vectors with same initial point is called a coinitial vector.
4.Collinear Vectors: If two or more vectors lying on the same or parallel lines, then it is collinear vectors.
5. Equal Vectors: Equal vectors are two more vectors with same magnitude and direction.

2.1.3 Operations on Vectors

Two n- vectors x and y are said to be equal if all their corresponding components are equal. Then we can write x = y. If two vectors are not equal, we use the symbol ≠. The equality is possible only between vectors of the same dimension.
For example, (x,y,z) = (2,-1,3) only if x = 2,y = -1, and z = 3

2.1.3.1 Addition of Vectors

If a and b are two n vectors, the sum of a and b, denoted by a + b, is the n vector obtained by adding each component of 'a' to the corresponding component of 'b'.

For example, for row vectors,
$(a_1, a_2,\ldots,a_n)+ (b_1,b_2,\ldots..,b_n) =(a_1+b_1,a_2+b_2,\ldots..a_n+b_n)$

Illustration 2.1.1

Find a +_b if a = (3,-2,5) and b = (-2,10,-3)

Solution

a = (3,-2,5)

b = (-2,10,-3)

Then a + b = (3 +-2,-2 + 10,5 + -3)

= (1,8,2)

2.1.3.2 Subtraction of Vectors

If a and b are both n vectors, then the difference between a and b is defined by
a - b = a + (-1)b

This implies that

$(a_1, a_2,\ldots,a_n)-(b_1,b_2,\ldots,b_n) = (a_1-b_1,a_2-b_2,\ldots,a_n-b_n)$

Illustration 2.1.2

Find a - b if a = (3,-2,5) and b = (-2,10,-3)

Solution

a = (3,-2,5)

b = (-2,10,-3)

Then a-b = 3-(-2), -2-10, 5-(-3)

= (5, -12, 8)

2.1.3.3 Multiplication by a Scalar

If 'a' is an n vector and t is a real number, we define 'ta' as the n vector whose components are t times the corresponding components in 'a'.
ie, $t(a_1, a_2,\ldots,a_n)= (ta_1, ta_2,\ldots,ta_n)$

Illustration 2.1.3

Find 3a, if a = (3,-2,5)

Solution

3a = 3(3,-2,5)

= (3×3, 3×-2, 3×5)

= (9,-6,15)

2.1.3 Meaning and Types of Matrices

Suppose Aleena went for a shopping with Jaya and Neha. Aleena bought 2 sarees. We can write it as [2], with the understanding that the number inside [ ] represents the number of sarees Aleena purchased. Aleena also bought 4 churidars and 1 gown. We may express all of them now as [2 4 1] with the understanding that first number inside [ ] is the number of sarees, second one is churidars and the last one is the number of gown purchased by Aleena.

Let us now suppose that we wish to express the information about the purchase of sarees, churidars and gowns by Aleena and her two friends Jaya and Neha which is as follows:

Aleena purchased	2 sarees	4 churidars	1 gown
Jaya purchased	1 saree	3 churidars	2 gowns
Neha purchased	5 sarees	1 churidar	0 gown

This could be arranged in the tabular form as follows:

Name/ Item	sarees	churidars	gowns
Aleena	2	4	1
Jaya	1	3	2
Neha	5	1	0

This can be expressed as,

$$\begin{array}{cl} \begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \\ 5 & 1 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow \text{First Row} \\ \leftarrow \text{Second Row} \\ \leftarrow \text{Third Row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} & \text{Third Column} \end{array} & \end{array}$$

An arrangement or display of the above kind is called a matrix.

Note: The horizontal arrays of a matrix are called its rows and the vertical arrays are called its columns.

Or

Item/ Name	Aleena	Jaya	Neha
sarees	2	1	5
churidars	4	3	1
gowns	1	2	0

which can be expressed as:

$$\begin{array}{cl} \left(\begin{array}{ccc}2&1&5\\4&3&1\\1&2&0\end{array}\right) & \begin{array}{l}\leftarrow \text{First Row}\\ \leftarrow \text{Second Row}\\ \leftarrow \text{Third Row}\end{array} \\ \begin{array}{ccc}\uparrow & \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} & \text{Third Column}\end{array} & \end{array}$$

In the first arrangement the entries in the first column represent the number of sarees purchased by Aleena, Jaya and Neha, respectively and the entries in the second column represent the number of churidars purchased by Aleena, Jaya and Neha, respectively. Similarly, the entries in the third column represent the number of gowns purchased by Aleena, Jaya and Neha, respectively.

In the second arrangement, the entries in the first row represent the number of sarees purchased by Aleena, Jaya and Neha, respectively and the entries in the second row represent the number of churidars purchased by Aleena, Jaya and Neha, respectively. Similarly, the entries in the third row represent the number of gowns purchased by Aleena, Jaya and Neha, respectively.

Definition: *Matrix is an array of numbers, symbols or parameters that are arranged in rows and columns.*

Origin and meaning of term Matrix

- According to research, the term matrix dates back to the second century BC, with traces dating back to the fourth century BC. It is a Latin word for "mother," and it was later discovered that it was also used to represent the "womb."
- English mathematician James Joseph Sylsevter and British mathematician Arthur Cayley were among the first to contribute to matrix theory. In 1848, James Joseph Sylsevter coined the term matrix, which is derived from the Latin word for womb. Later that year, in 1856, Arthur Cayley introduced matrix multiplication and matrix inverse.

The term matrix is widely used in a variety of fields of study. A matrix is a system or environment in which something emerges and grows. It is an arrangement or a series of information that has been ordered. The term matrix is used as a noun, and the plural term is "matrices."

The term "matrix" is commonly used in different subjects. It is used to compactly represent large systems. It is a type of short hand. In economics, a matrix can be used to study a market, the interaction between industries in an economy, or the relationship between inputs and outputs of a production system.

Generally, it is written within a pair of second brackets, “[ ]”, parentheses, "( )”, or, on occasion, double lines, “| |”. It is a system for storing information. Each matrix member is referred to as an element of the matrix, and the elements are separated by spaces. There may or may not be a connection between the elements.

Matrices can have either a rectangular or a square shape. The number of rows and columns in a rectangular matrix varies. A square matrix has the same number of rows and columns as a rectangular matrix.

A matrix is generally modelled by a letter of the alphabet. Thus, a matrix Z as in our above example can be written as:

$$Z = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \\ 5 & 1 & 0 \end{bmatrix}$$

The elements of a matrix are represented as Z_{ij}, where i represents the element in the i^{th} row and j is the element in the j^{th} column.

Thus, Z_{11} is the element in the first row and the first column and that is the number 2.
Similarly,

Z_{12} is the element in the first row and the second column and that is the number 4.
Z_{13} is the element in the first row and the third column and that is the number 1.
Z_{21} is the element in the second row and the first column and that is the number 1.
Z_{22} is the element in the second row and the second column and that is the number 3.
Z_{23} is the element in the second row and the third column and that is the number 2.
Z_{31} is the element in the third row and the first column and that is the number 5.
Z_{32} is the element in the third row and the second column and that is the number 1.
Z_{33} is the element in the third row and the third column and that is the number 0.

Note: Z_{34} is the element in the third row and fourth column. Since there is no fourth column in the matrix Z, hence, this is undefined.

2.1.3.1 Order of a matrix

The order or dimension of a matrix is the total number of elements in the matrix. The product of the rows and columns represents the order, which is written in the matrix's bottom right corner.

Note: There is a rule for writing the order of a matrix, and that is, the number of rows is multiplied by the number of columns. Thus, if "m" is the number of rows and "n" is the number of columns, then the order of the matrix with "m" rows and "n" columns may be written as

"$m \times n$".

For example,

$$Z = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \\ 5 & 1 & 0 \end{bmatrix}_{3 \times 3}$$

The above matrix is an example of an "3 × 3" (read as 3 by 3) matrix; that is, there are 3 horizontal rows and 3 vertical columns. There are (3 × 3) = 9 elements in this matrix. Since the number of rows and columns are equal, this is an example of a square matrix.

Illustration 2.1.4

Suppose three students purchased books and pens. How can we represent it with a matrix of order 3 × 2 and 2 × 3.

Solution

$$\text{Matrix A} = \begin{bmatrix} 7 & 4 \\ 1 & 3 \\ 5 & 1 \end{bmatrix}$$

Since there are 3 horizontal rows (in our case three students) and 2 vertical columns (in our case books and pens respectively), it is a matrix of order 3 × 2.

$$\text{Matrix B} = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Since there are 2 horizontal rows (in our case books and pens respectively) and 3 vertical columns (in our case three students), it is a matrix of order 2 × 3.

2.1.3.2 Types of matrices

We can arrange the items we discussed so far in different ways. In this section, we shall discuss different types of matrices.

1. **Square matrix:** A matrix with equal number of rows and columns is called a square matrix. Since the number of rows is equal to number of columns we can define the matrix as a square matrix of order m or n, where m = n.
For example,

$$Z = \underset{2\times2}{\begin{bmatrix} 6 & 9 \\ 4 & 7 \end{bmatrix}} \text{ is a square matrix of order 2.}$$

2. **Rectangular matrix:** Any matrix with 'm' rows and 'n' columns is called a rectangular matrix of m x n. In a rectangular matrix, the number of rows and number of columns are not equal that is m ≠ n.
For example,

$$Z = \underset{2\times3}{\begin{bmatrix} 6 & 9 & 3 \\ 4 & 7 & 2 \end{bmatrix}} \text{ is a rectangular matrix of order } 2 \times 3.$$

3. **Row matrix:** A matrix having only one row is called a row matrix. There may be many columns in such matrices.
For example,

$$Z = [\,4 \quad 7 \quad 2 \quad 6 \quad 1\,] \text{ is a row matrix.}$$

4. **Column matrix:** A matrix having only one column is called a column matrix. There may be many rows in such matrices.
For example,

$$Z = \begin{bmatrix} 6 \\ 4 \\ 1 \\ 7 \\ 3 \end{bmatrix} \text{ is a column matrix.}$$

5. **Diagonal matrix:** Only square matrices with the same number of rows and columns can make a diagonal matrix. A diagonal matrix is one in which all of the elements that make up the diagonal are non-zero and all of the other elements are zero. That is, a square matrix, in which all the elements except the leading diagonal are zero, is called a diagonal matrix.

Leading diagonals: Elements of a matrix a_{11}, a_{22}, a_{33}, a_{44}, a_{55} …etc form the leading diagonal.

For example,

$$Z = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ is a diagonal matrix.}$$

Note: A square matrix a_{ij} is a diagonal matrix when

$$aij = 0 \, if \, i \neq j \, and \, aij \neq 0 \, when \, i = j.$$

In the above example $a_{12} = 0$, $a_{13} = 0$, $a_{21} = 0$, $a_{23} = 0$, $a_{31} = 0$ and $a_{32} = 0$.

Whereas, $a_{11} \neq 0$, $a_{22} \neq 0$ and $a_{33} \neq 0$.

6. **Scalar matrix:** A diagonal matrix in which all the diagonal elements are same number is called a scalar matrix.
For example,

$$Z = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ is a scalar matrix.}$$

Note: A matrix of order n is a scalar matrix, if $a_{ij} = x$ when $i = j$ and
$aij = 0$ *when* $i \neq j$, *where x is an element.*

7. **Unit matrix:** A diagonal matrix in which all the diagonal element is unity, is said to be a unit matrix. It is also known as identity matrix.

For example,

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is a unit matrix.}$$

Note: A matrix of order n is a unit matrix, if $a_{ij} = 1$ when $i = j$ and
$a_{ij} = 0$ *when* $i \neq j$.

8. **Null matrix:** A matrix in which all the elements are zero, is said to be null matrix or zero matrix.

For example,

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a null or zero matrix.}$$

Note: A matrix of order m x n is a null or zero matrix, if $a_{ij} = 0$ for all i and j.

9. **Triangular Matrix:** A matrix may be a lower triangular matrix or an upper triangular matrix.
If all the elements above the diagonal elements of the matrix are zero and the elements below the diagonal matrix are non-zero, then it is referred as a lower triangular matrix.

For example,

$$Z = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 2 & 9 \end{bmatrix} \text{ is a lower triangular matrix.}$$

On the other hand, if all the elements below the diagonal elements are zero and the elements above the diagonal elements are non-zero, it is termed as an upper triangular matrix.

For example,

$$Z = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 3 & 7 \\ 0 & 0 & 9 \end{bmatrix} \text{ is an upper triangular matrix.}$$

10. **Transpose of a matrix:** If the rows and columns of a matrix are interchanged, the new matrix formed is called the transpose of the original matrix. It is generally denoted by A^t or A'.

For example,

If $Z = \begin{bmatrix} 5 & 2 & 9 \\ 7 & 1 & 4 \\ 3 & 6 & 8 \end{bmatrix}$ is a matrix of order 3×3 then the transpose of Z or Z' will be

$$Z^t = \begin{bmatrix} 5 & 7 & 3 \\ 2 & 1 & 6 \\ 9 & 4 & 8 \end{bmatrix}$$

Here, the elements of the first row that is 5, 2 and 9 become the elements of the first column. The elements of the second row that is 7, 1 and 4 become the elements of the second column. Similarly, the elements of the third row that is 3, 6 and 8 become the elements of the third column.

Illustration 2.1.5

Find the transpose of the given matrix $A = \begin{bmatrix} 7 & -8 \\ 3 & 2 \\ -4 & 1 \\ -1 & 5 \end{bmatrix}$

Solution

We can find the transpose of matrix A by interchanging the rows and columns of matrix A. Thus the new matrix A^t will be as follows.

$$A^t = \begin{bmatrix} 7 & 3 & -4 & -1 \\ -8 & 2 & 1 & 5 \end{bmatrix}$$

11. **Symmetric matrix:** If the transpose of a matrix is equal to the original matrix, then that matrix is called a symmetric matrix.
For example,

If $B = \begin{bmatrix} 5 & 2 & 9 \\ 2 & 1 & 6 \\ 9 & 6 & 8 \end{bmatrix}$ is a matrix of order 3 x 3 then the transpose of B or B' will be

$$B^t = \begin{bmatrix} 5 & 2 & 9 \\ 2 & 1 & 6 \\ 9 & 6 & 8 \end{bmatrix}$$

That is, B is symmetric if $B = B^t$

Note: Symmetry occurs only in square matrices.

12. Skew symmetric matrix: Any square matrix is said to be skew symmetric if it is equal to its negative transpose.

For example,

If $M = \begin{bmatrix} 0 & 4 & -7 \\ -4 & 0 & -6 \\ 7 & 6 & 0 \end{bmatrix}$ is a matrix of order 3 x 3 then the negative transpose of M or M^t will be

$$M^t = \begin{bmatrix} 0 & -4 & 7 \\ 4 & 0 & 6 \\ -7 & -6 & 0 \end{bmatrix}, \; -M^t = \begin{bmatrix} 0 & 4 & -7 \\ -4 & 0 & -6 \\ 7 & 6 & 0 \end{bmatrix}$$

In short, M is skew symmetric, if $M = -M^t$

2.1.3.3 Equality of matrices

Two matrices $R = [r_{ij}]$ and $S = [s_{ij}]$ are said to be equal, if

a) they are of the same order

b) each element of R is equal to the corresponding element of S, that is $r_{ij} = s_{ij}$ for all i and j.

For example,

$$R = \begin{bmatrix} 4 & 9 & 8 \\ 3 & 7 & 2 \\ 5 & 1 & 6 \end{bmatrix} \text{ and } S = \begin{bmatrix} 4 & 9 & 8 \\ 3 & 7 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

Then R = S, since all the elements in matrix R is equal to all the elements of matrix S. That is they are equal matrix.

Let us consider another matrix P.

$$P = \begin{bmatrix} 4 & 9 & 8 \\ 3 & 1 & 2 \\ 5 & 0 & 7 \end{bmatrix}$$

Here R and P are not equal matrix since all the elements of matrix R is not equal to all the elements of matrix P.

Note:

If $A = \begin{bmatrix} Apple & Mango \\ Banana & Orange \\ Grapes & Pineapple \end{bmatrix}$, $B = \begin{bmatrix} 4 & 7 \\ 8 & 6 \\ 2 & 5 \end{bmatrix}$ and A = B, then the above matrices show that $Apple$ = 4 kg, $Banana$ = 8 kg, $Grapes$ = 2 kg, $Mango$ = 7 kg, $Orange$ = 6 kg and $Pineapple$ = 5 kg.

The above technique of equality between two matrices can be used in mathematics to solve certain problems. Now let us try to solve the given problem.

Illustration 2.1.6

Let $\begin{bmatrix} x-2 & a-1 \\ y+4 & b-4 \\ 0 & c+6 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 2 & 8 \\ z-9 & 11 \end{bmatrix}$. Find the values of x, y, z, a, b and c.

Solution

It is given that,

$$\begin{bmatrix} x-2 & a-1 \\ y+4 & b-4 \\ 0 & c+6 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 2 & 8 \\ z-9 & 11 \end{bmatrix}$$

Since the given matrices are equal, their corresponding elements are also equal. Thus,

x – 2 = 6	a – 1 = - 4
y + 4 = 2	b – 4 = 8
0 = z - 9	c + 6 = 11

Simplifying the above equations, we get,

x = 6 + 2 = 8
y = 2 – 4 = - 2
z = 9
a = - 4 + 1 = - 3
b = 8 + 4 = 12
c = 11 – 6 = 5

Therefore, x = 8, y = - 2, z = 9, a = -3, b = 12 and c = 5

Hope you have understood the equality of matrices. Now try to solve the following problem as we discussed.

Recap

- Matrix - An array of numbers and symbols that are arranged in rows and columns
- Dimension of a matrix - Total number of elements in the matrix
- Square matrix - Matrix having an equal number of rows and columns
- Rectangular matrix - Any matrix with 'm' rows and 'n' columns
- Row matrix - Matrix having only one row
- Column matrix - Matrix having only one column
- Diagonal matrix - All diagonal elements are equal
- Unity matrix - A diagonal element in which all the diagonal elements are unity
- Unity matrix is also known as identity matrix
- Null matrix - Matrix in which all the elements are zero
- Null matrix also known as zero matrix
- Transpose of a matrix- A new matrix is formed by interchanging its rows and columns
- Transpose of a matrix is denoted by A^t or A'
- Lower triangular matrix - Matrix has non-zero elements only below its main diagonal
- Upper triangular matrix - Matrix has non-zero elements only above its main diagonal

Objective Questions

1. What is defined as an array of numbers, symbols, or parameters arranged in rows and columns?
2. What does the order or dimensions of a matrix represent?
3. How many elements are there in a 2×3 matrix?
4. What is the type of matrix when it has an equal number of rows and columns?

5. Define rectangular matrix.
6. What is row matrix?
7. What is column matrix?
8. What is a diagonal matrix where all the diagonal elements are same?
9. What is a diagonal matrix in which all the diagonal elements are unity?
10. What is a matrix in which all the elements are zero?
11. What is a matrix called when the rows and columns are interchanged?
12. What is a matrix called if its transpose is equal to the original matrix?
13. What is a square matrix called if it is equal to its negative transpose?
14. What is the condition for the equality of two matrices?

Answers

1. Matrix
2. The total number of elements in the matrix
3. 6
4. Square matrix
5. Matrix having m rows and n columns
6. Matrix with only one row
7. Matrix with only one column
8. Scalar matrix
9. Identity matrix/unit matrix
10. Null matrix
11. Transpose of matrix
12. Symmetric matrix
13. Skew-symmetric matrix
14. Both matrices have the same order

Assignments

1. Explain the meaning and different types of Matrix
2. Explain about meaning, types, and operations of vectors
3. Find

   a) a+b

   b) a-b

   c) 2a+3b

   d) -5a+2b

   when vectors a = (2,-1) and b = (3,4)

Suggested Readings

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UNIT 2 Elementary Operations of a Matrix

Learning Outcomes

After completing this unit, the learner will be able to:

- understand the elementary operations of a matrix
- know the properties of matrix calculation
- use the matrix calculations to solve the economic problem

Prerequisites

In a village, there lived a community of skilled craftsmen and artisans. They were known far and wide for their ability to create intricate patterns and designs on their crafts. The secret to their artistry lay in an ancient tool - the canvas of matrices.

Imagine a bustling marketplace where these craftsmen displayed their goods. Each item was a work of art, a testament to the creativity and precision of its creator. Yet, beneath the surface of this creativity lay a mathematical language, a canvas of matrices that held the key to their craft.

Before the villagers could create these masterpieces, they needed to understand the essence of matrices. Matrices were like the canvas upon which their art was painted. These grids held the structure, the form, and the potential for transformation.

To grasp this world of matrices, the villagers needed to learn the elementary operations - addition, subtraction, and multiplication (up to 3×3 matrices).

These operations were like the brushes and paints, allowing them to blend colours, reshape forms, and create intricate patterns.

With the knowledge of matrix operations, the craftsmen could craft more than just beautiful artifacts; they could optimize their resources, analyze market trends, and adapt to changing customer preferences. The canvas of matrices became a versatile tool for economic transformation.

Keywords

Matrix, Addition, Subtraction, Multiplication, Commutative, Associative

Discussion

In this section, we shall discuss certain operations on matrices, namely, addition of matrices, difference of matrices, multiplication of a matrix by a scalar and multiplication of matrices.

2.2.1 Matrix Addition and Subtraction

Suppose an electronic company makes two items, X and Y, and sells them in two different markets, one on Kerala and the other in Tamil Nadu. The following are the weekly sales for each market:

Kerala Markets – Weekly sales in thousand				
Products	**Week 1**	**Week 2**	**Week 3**	**Week 4**
X	**8**	**9**	**7**	**6**
Y	**9**	**5**	**8**	**7**

Tamil Nadu Markets – Weekly sales in thousand				
Products	**Week 1**	**Week 2**	**Week 3**	**Week 4**
X	11	6	5	9
Y	7	9	10	8

Let the markets in Kerala be called A and market in Tamil Nadu be called B. Then, the given information may be arranged in matrix form as follows:

$$A=\begin{bmatrix}8 & 9 & 7 & 6\\ 9 & 5 & 8 & 7\end{bmatrix} \text{ and } B=\begin{bmatrix}11 & 6 & 5 & 9\\ 7 & 9 & 10 & 8\end{bmatrix}$$

We can find the weekly total sales of the two products of the company using matrix algebra. For this we can use matrix addition.

If we want to find the weekly sale of product X in the first week in both in Kerala and Tamil Nadu markets, we have to add 8 and 11. Likewise, we can simply find weekly sales of product X and product Y in both markets for each weeks. That is,

$$A + B = \begin{bmatrix}8 & 9 & 7 & 6\\ 9 & 5 & 8 & 7\end{bmatrix} + \begin{bmatrix}11 & 6 & 5 & 9\\ 7 & 9 & 10 & 8\end{bmatrix}$$

$$A + B = \begin{bmatrix}19 & 15 & 12 & 15\\ 16 & 14 & 18 & 15\end{bmatrix}$$

The resultant matrix gives the total sales of product X and Y in both Kerala and Tamil Nadu markets can be tabulated as follows. This is how we do matrix addition and subtraction.

Kerala & Tamil Nadu Markets – Weekly sales in thousands				
Products	**Week 1**	**Week 2**	**Week 3**	**Week 4**
X	19	15	12	15
Y	16	14	18	15

As we discussed above, it is possible to add or subtract two or more matrices. The rule to follow is that the matrices must be in the same order. Because the number of rows in the first matrix is not equal to the number of rows in the second matrix, a matrix of order cannot add or subtract from another matrix of order.

While we do addition or subtraction, each the element we add or subtract has to be the same in both matrices. For example, consider two matrices Matrix A and Matrix B. If we add both matrices, then a_{11} of matrix A has to be added with b_{11} of matrix B. Likewise, a_{12} of matrix A has to be added with b_{12} of matrix B.

The order of the matrices added or subtracted will be the same in the resultant matrix (matrix formed after performing the mathematical operation). As a result, if two order matrices are combined, the resultant matrix's order will be the same.

For example:

Let matrix A = $\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}$ and matrix B = $\begin{bmatrix} 5 & 3 \\ 6 & -7 \end{bmatrix}$

Then A + B = $\begin{bmatrix} 4+5 & 1+3 \\ -5+6 & 2-7 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 1 & -5 \end{bmatrix}$

Let us consider another example.

Given matrix X = $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 8 & -1 \end{bmatrix}$ and matrix Y = $\begin{bmatrix} 3 & 5 & 6 \\ -4 & -2 & 7 \end{bmatrix}$

Then X + Y = $\begin{bmatrix} 1+3 & 4+5 & 2+6 \\ 3-4 & 8-2 & -1+7 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 8 \\ -1 & 6 & 6 \end{bmatrix}$

2.2.1.1 Properties of Matrix Addition

The basic properties of matrix addition can be discussed as follows.

1. Commutative Law: Matrix addition is commutative i.e., A + B = B + A.

For example, let the two outlets, say A and B, of a company sell three commodities. Let us find the total sales of three months of the two outlets.

A= $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 7 & 6 \\ 5 & 2 & 8 \end{bmatrix}$ and B= $\begin{bmatrix} 8 & 6 & 7 \\ 2 & 4 & 9 \\ 6 & 3 & 3 \end{bmatrix}$

Let the three columns of the given matrices be three months and three rows be the three commodities. If we add the total sales of the three commodities of the two outlets in the three months, we get,

$$A + B = \begin{bmatrix} 2+8 & 1+6 & 4+7 \\ 3+2 & 7+4 & 6+9 \\ 5+6 & 2+3 & 8+3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 7 & 11 \\ 5 & 11 & 15 \\ 11 & 5 & 11 \end{bmatrix}$$

If we do the same by taking the second outlet first and the first outlet second, we get,

$$B + A = \begin{bmatrix} 8+2 & 6+1 & 7+4 \\ 2+3 & 4+7 & 9+6 \\ 6+5 & 3+2 & 3+8 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 7 & 11 \\ 5 & 11 & 15 \\ 11 & 5 & 11 \end{bmatrix}$$

Since we get the same matrix in both cases we can say that matrices,

A + B = B + A.

2. Associative Law: Matrix addition is associative i.e., (A + B) + C = A + (B + C). Suppose we add the total sales of another outlet, say outlet C, in the above example,

A= $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 7 & 6 \\ 5 & 2 & 8 \end{bmatrix}$, B = $\begin{bmatrix} 8 & 6 & 7 \\ 2 & 4 & 9 \\ 6 & 3 & 3 \end{bmatrix}$ and C= $\begin{bmatrix} 5 & 8 & 5 \\ 3 & 6 & 7 \\ 9 & 4 & 9 \end{bmatrix}$ then,

$$(A + B) + C = \begin{bmatrix} 10 & 7 & 11 \\ 5 & 11 & 15 \\ 11 & 5 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 8 & 5 \\ 3 & 6 & 7 \\ 9 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 15 & 15 & 16 \\ 8 & 17 & 22 \\ 20 & 9 & 20 \end{bmatrix}$$

(For A+B refer to the above example).

$$B + C = \begin{bmatrix} 8 & 6 & 7 \\ 2 & 4 & 9 \\ 6 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 8 & 5 \\ 3 & 6 & 7 \\ 9 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 13 & 14 & 12 \\ 5 & 10 & 16 \\ 15 & 7 & 12 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 7 & 6 \\ 5 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 13 & 14 & 12 \\ 5 & 10 & 16 \\ 15 & 7 & 12 \end{bmatrix} = \begin{bmatrix} 15 & 15 & 16 \\ 8 & 17 & 22 \\ 20 & 9 & 20 \end{bmatrix}$$

Therefore, (A + B) + C = A + (B + C).

3. Existence of Additive Identity: Zero matrix O is the additive identity of a matrix because adding a matrix with zero matrix leaves it unchanged i.e., X + O = O + X = X. For example,

Let X = $\begin{bmatrix} 4 & 9 & 8 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ and take a null matrix or zero matrix, a matrix in which all the elements are zero, O = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us add the null matrix O with the given matrix X. Then,

$$X + O = \begin{bmatrix} 4 & 9 & 8 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0 & 9+0 & 8+0 \\ 3+0 & 1+0 & 2+0 \\ 5+0 & 6+0 & 7+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 8 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$$

The new matrix is same as matrix X. Therefore, X + O = X.

If we calculate O + X also, we get the same matrix. That is, as per the matrix property of Commutative Law, X + O = O + X. Thus, X + O = O + X = X.

Note: The method we followed in matrix addition can also be used in matrix subtraction.

Illustration 2.2.1

Let matrix A= $\begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 9 \\ 6 & 3 \end{bmatrix}$ and C= $\begin{bmatrix} 2 & 7 \\ 3 & 6 \end{bmatrix}$. Prove that (A + B) + C = A + (B + C).

Solution

It is given that, A= $\begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 9 \\ 6 & 3 \end{bmatrix}$ and C= $\begin{bmatrix} 2 & 7 \\ 3 & 6 \end{bmatrix}$.

Then,

$$A + B = \begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 9 \\ 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7+2 & 6+9 \\ 2+6 & 8+3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 15 \\ 8 & 11 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 9 & 15 \\ 8 & 11 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2 & 15+7 \\ 8+3 & 11+6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 22 \\ 11 & 17 \end{bmatrix}$$

That is, $(A + B) + C = \begin{bmatrix} 11 & 22 \\ 11 & 17 \end{bmatrix}$

$$A= \begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2 & 9 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 9+7 \\ 6+3 & 3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 16 \\ 9 & 9 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 16 \\ 9 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 22 \\ 11 & 17 \end{bmatrix}$$

That is, $A + (B + C) = \begin{bmatrix} 11 & 22 \\ 11 & 17 \end{bmatrix}$

Therefore, (A + B) + C = A + (B + C).

Illustration 2.2.2

Two farmers Tarun and Varun cultivates only three varieties of rice namely Basmati, Permal and Naura. The sales in rupees of these varieties of rice by both farmers in the month of August and September are given by the following matrices A and B.

August Sales in Rupees

$$A = \begin{array}{c} \text{Basmati} \quad \text{Permal} \quad \text{Naura} \\ \begin{bmatrix} 20{,}000 & 10{,}000 & 40{,}000 \\ 30{,}000 & 20{,}000 & 50{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ Tarun \\ Varun \end{array}$$

September Sales in Rupees

$$B = \begin{array}{c} \text{Basmati} \quad \text{Permal} \quad \text{Naura} \\ \begin{bmatrix} 10{,}000 & 5{,}000 & 20{,}000 \\ 20{,}000 & 10{,}000 & 30{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ Tarun \\ Varun \end{array}$$

a) Find the combined sales in August and September for each farmer in each variety.
b) Find the change in sales from August to September.

Solution

a) Combined sales in August and September for each farmer in each variety is given by,

$$A + B = \begin{array}{c} \text{Basmati} \qquad\qquad \text{Permal} \qquad\qquad \text{Naura} \\ \begin{bmatrix} 20{,}000 + 10{,}000 & 10{,}000 + 5{,}000 & 40{,}000 + 20{,}000 \\ 30{,}000 + 20{,}000 & 20{,}000 + 10{,}000 & 50{,}000 + 30{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ Tarun \\ Varun \end{array}$$

$$A + B = \begin{array}{c} \text{Basmati} \quad \text{Permal} \quad \text{Naura} \\ \begin{bmatrix} 30{,}000 & 15{,}000 & 60{,}000 \\ 50{,}000 & 30{,}000 & 80{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ Tarun \\ Varun \end{array}$$

b) Change in sales from August to September.

$$A - B = \begin{array}{c c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} & \\ \begin{bmatrix} 20{,}000 - 10{,}000 & 10{,}000 - 5{,}000 & 40{,}000 - 20{,}000 \\ 30{,}000 - 20{,}000 & 20{,}000 - 10{,}000 & 50{,}000 - 30{,}000 \end{bmatrix} & \begin{array}{l} Tarun \\ Varun \end{array} \end{array}$$

$$A - B = \begin{array}{c c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} & \\ \begin{bmatrix} 10{,}000 & 5{,}000 & 20{,}000 \\ 10{,}000 & 10{,}000 & 20{,}000 \end{bmatrix} & \begin{array}{l} Tarun \\ Varun \end{array} \end{array}$$

Illustration 2.2.3

Given the matrices,

$$X = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 8 & 1 \\ 5 & -4 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -5 & 2 & 5 \\ 1 & 4 & 6 \\ 3 & 7 & 2 \end{bmatrix}$$

Find X – Y

Solution

It is given that,

$$X = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 8 & 1 \\ 5 & -4 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -5 & 2 & 5 \\ 1 & 4 & 6 \\ 3 & 7 & 2 \end{bmatrix}$$

Let us solve the given matrices mathematically.

$$X - Y = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 8 & 1 \\ 5 & -4 & 0 \end{bmatrix} - \begin{bmatrix} -5 & 2 & 5 \\ 1 & 4 & 6 \\ 3 & 7 & 2 \end{bmatrix}$$

$$X - Y = \begin{bmatrix} 2+5 & 1-2 & -4-5 \\ 3-1 & 8-4 & 1-6 \\ 5-3 & -4-7 & 0-2 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -9 \\ 2 & 4 & -5 \\ 2 & -11 & -2 \end{bmatrix}$$

Illustration 2.2.4

Given three matrices,

$$X = \begin{bmatrix} -2 & 1 \\ -5 & 2 \\ 3 & -8 \\ 2 & 4 \end{bmatrix}, Y = \begin{bmatrix} 2 & 4 \\ -6 & 5 \\ 2 & -2 \\ 7 & -1 \end{bmatrix} \text{ and } Z = \begin{bmatrix} 1 & 11 \\ 4 & 9 \\ -2 & 7 \\ 3 & 0 \end{bmatrix}$$

Find X - Y + Z.

Solution

$$X - Y + Z = \begin{bmatrix} -2 & 1 \\ -5 & 2 \\ 3 & -8 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -6 & 5 \\ 2 & -2 \\ 7 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 11 \\ 4 & 9 \\ -2 & 7 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2-2+1 & 1-4+11 \\ -5+6+4 & 2-5+9 \\ 3-2-2 & -8+2+7 \\ 2-7+3 & 4+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 8 \\ 5 & 6 \\ -1 & 1 \\ -2 & 5 \end{bmatrix}$$

Illustration 2.2.5

If, $X = \begin{bmatrix} 4 & -3 \\ 1 & 8 \\ -5 & 2 \\ 3 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 6 & -4 \\ 2 & 7 \\ -6 & 4 \\ 1 & 3 \end{bmatrix}$. Find the matrix such that $X + Y - Z = 0$.

Solution

It is given that $X + Y - Z = 0$.

Therefore, $X + Y = Z$

It is also given that $X = \begin{bmatrix} 4 & -3 \\ 1 & 8 \\ -5 & 2 \\ 3 & 6 \end{bmatrix}$ and $Y = \begin{bmatrix} 6 & -4 \\ 2 & 7 \\ -6 & 4 \\ 1 & 3 \end{bmatrix}$

Thus, $X + Y = \begin{bmatrix} 4 & -3 \\ 1 & 8 \\ -5 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 6 & -4 \\ 2 & 7 \\ -6 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4+6 & -3-4 \\ 1+2 & 8+7 \\ -5-6 & 2+4 \\ 3+1 & 6+3 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -7 \\ 3 & 15 \\ -11 & 6 \\ 4 & 9 \end{bmatrix}$$

Therefore, $Z = \begin{bmatrix} 10 & -7 \\ 3 & 15 \\ -11 & 6 \\ 4 & 9 \end{bmatrix}$

Then $X + Y - Z = 0$

Illustration 2.2.6

Let $X + \begin{bmatrix} 7 & 5 \\ 3 & 8 \\ -8 & -4 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ 12 & -4 \\ 3 & 7 \\ 4 & 9 \end{bmatrix}$. Solve the equation.

Solution

It is given that $X + \begin{bmatrix} 7 & 5 \\ 3 & 8 \\ -8 & -4 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ 12 & -4 \\ 3 & 7 \\ 4 & 9 \end{bmatrix}$. Since the given matrices are in addition from, we can write it as,

$$X = \begin{bmatrix} 11 & 9 \\ 12 & -4 \\ 3 & 7 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} 7 & 5 \\ 3 & 8 \\ -8 & -4 \\ -7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11-7 & 9-5 \\ 12-3 & -4-8 \\ 3+8 & 7+4 \\ 4+7 & 9-2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 9 & -12 \\ 11 & 11 \\ 11 & 7 \end{bmatrix}$$

Therefore, $X = \begin{bmatrix} 4 & 4 \\ 9 & -12 \\ 11 & 11 \\ 11 & 7 \end{bmatrix}$

Illustration 2.2.7

Let matrix $A = \begin{bmatrix} 7 & 9 & -8 \\ 3 & 1 & 12 \\ 6 & 0 & 7 \end{bmatrix}$ and matrix B = A + Unit Matrix of 3×3. Find matrix B.

Solution

It is given that matrix $A = \begin{bmatrix} 7 & 9 & -8 \\ 3 & 1 & 12 \\ 6 & 0 & 7 \end{bmatrix}$

B = A + Unit Matrix of 3×3

A diagonal matrix in which all the diagonal element is unity, is said to be a unit matrix. Let Z be the unit matrix of 3×3. That is,

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, B = A + Z

$$B = \begin{bmatrix} 7 & 9 & -8 \\ 3 & 1 & 12 \\ 6 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7+1 & 9+0 & -8+0 \\ 3+0 & 1+1 & 12+0 \\ 6+0 & 0+0 & 7+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 9 & -8 \\ 3 & 2 & 12 \\ 6 & 0 & 8 \end{bmatrix}$$

$$\text{Therefore, } B = \begin{bmatrix} 8 & 9 & -8 \\ 3 & 2 & 12 \\ 6 & 0 & 8 \end{bmatrix}$$

2.2.2 Multiplication of a Matrix by a Scalar

In a shop the customer gets a purchase coupon for each commodity he purchases. Suppose Ayush purchases 5 books, 2 pens and 3 pencils. Then how many purchase coupons will he get?

Let K be the coupon and the items he purchased be as follows in matrix form.

$$\text{Let } A = \begin{bmatrix} Books \\ Pens \\ Pencils \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

Then he gets, $\begin{bmatrix} 5K \\ 2K \\ 3K \end{bmatrix}$. That is 10 coupons.

How did we get this?

We simply multiplied the number of commodities he purchased with the coupon. That is,

$$K \times A = K \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5K \\ 2K \\ 3K \end{bmatrix}.$$

This is how we multiply a matrix using a scalar. When a matrix is multiplied by a scalar, the scalar amount is multiplied by each element of the matrix, yielding a new matrix.

For example,

Let matrix A = $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 6 & 2 \\ 2 & 1 & 5 \end{bmatrix}$. If we multiply the given matrix by a scalar, say 2. Then a new matrix will be formed as follows.

$$2 \times A = 2\begin{bmatrix} 4 & 5 & 3 \\ 1 & 6 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 & 2 \times 5 & 2 \times 3 \\ 2 \times 1 & 2 \times 6 & 2 \times 2 \\ 2 \times 2 & 2 \times 1 & 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 & 6 \\ 2 & 12 & 4 \\ 4 & 2 & 10 \end{bmatrix}$$

Illustration 2.2.8

Let matrix A = $\begin{bmatrix} 2 & -5 \\ 3 & 4 \\ -5 & 6 \\ 7 & -1 \end{bmatrix}$. Multiply the given matrix by a scalar 3 and find the new matrix.

Solution

It is given that matrix A = $\begin{bmatrix} 2 & -5 \\ 3 & 4 \\ -5 & 6 \\ 7 & -1 \end{bmatrix}$

Let the new matrix be B.

That is,

$$B = 3\begin{bmatrix} 2 & -5 \\ 3 & 4 \\ -5 & 6 \\ 7 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 2 & 3 \times -5 \\ 3 \times 3 & 3 \times 4 \\ 3 \times -5 & 3 \times 6 \\ 3 \times 7 & 3 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -15 \\ 9 & 12 \\ -15 & 18 \\ 21 & -3 \end{bmatrix}$$

Thus the new matrix B $= \begin{bmatrix} 6 & -15 \\ 9 & 12 \\ -15 & 18 \\ 21 & -3 \end{bmatrix}$

Illustration 2.2.9

If $A = \begin{bmatrix} 2 & 9 & 4 \\ 4 & 5 & 2 \\ 3 & 2 & 1 \\ 8 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 & 6 \\ 3 & 3 & 4 \\ 7 & 1 & 8 \\ 5 & 8 & 3 \end{bmatrix}$ Find 3A – 2B.

Solution

It is given that $A = \begin{bmatrix} 2 & 9 & 4 \\ 4 & 5 & 2 \\ 3 & 2 & 1 \\ 8 & 1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 & 6 \\ 3 & 3 & 4 \\ 7 & 1 & 8 \\ 5 & 8 & 3 \end{bmatrix}$

$$3A = 3 \begin{bmatrix} 2 & 9 & 4 \\ 4 & 5 & 2 \\ 3 & 2 & 1 \\ 8 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 2 & 3 \times 9 & 3 \times 4 \\ 3 \times 4 & 3 \times 5 & 3 \times 2 \\ 3 \times 3 & 3 \times 2 & 3 \times 1 \\ 3 \times 8 & 3 \times 1 & 3 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 27 & 12 \\ 12 & 15 & 6 \\ 9 & 6 & 3 \\ 24 & 3 & 18 \end{bmatrix}$$

Therefore, $3A = \begin{bmatrix} 6 & 27 & 12 \\ 12 & 15 & 6 \\ 9 & 6 & 3 \\ 24 & 3 & 18 \end{bmatrix}$

$$2B = 2 \begin{bmatrix} 9 & 2 & 6 \\ 3 & 3 & 4 \\ 7 & 1 & 8 \\ 5 & 8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 9 & 2\times 2 & 2\times 6 \\ 2\times 3 & 2\times 3 & 2\times 4 \\ 2\times 7 & 2\times 1 & 2\times 8 \\ 2\times 5 & 2\times 8 & 2\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 4 & 12 \\ 6 & 6 & 8 \\ 14 & 2 & 16 \\ 10 & 16 & 6 \end{bmatrix}$$

Therefore, $2B = \begin{bmatrix} 18 & 4 & 12 \\ 6 & 6 & 8 \\ 14 & 2 & 16 \\ 10 & 16 & 6 \end{bmatrix}$

Thus, $3A - 2B = \begin{bmatrix} 6 & 27 & 12 \\ 12 & 15 & 6 \\ 9 & 6 & 3 \\ 24 & 3 & 18 \end{bmatrix} - \begin{bmatrix} 18 & 4 & 12 \\ 6 & 6 & 8 \\ 14 & 2 & 16 \\ 10 & 16 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 6-18 & 27-4 & 12-12 \\ 12-6 & 15-6 & 6-8 \\ 9-14 & 6-2 & 3-16 \\ 24-10 & 3-16 & 18-6 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 23 & 0 \\ 6 & 9 & -2 \\ -5 & 4 & -13 \\ 14 & -13 & 12 \end{bmatrix}$$

Therefore, $3A - 2B = \begin{bmatrix} -12 & 23 & 0 \\ 6 & 9 & -2 \\ -5 & 4 & -13 \\ 14 & -13 & 12 \end{bmatrix}$

2.2.2.1 Multiplication of Matrices

Suppose, Manju and Rani are two friends. Manju wants to buy 2 LED bulbs and 3 LED tubes, while Rani needs 4 LED bulbs and 2 LED tubes. They both go to a shop to enquire about the rates which are quoted as follows:

LED bulbs - Rs. 90 each and LED tubes - Rs. 160 each.
How much money does each need to spend?

Manju needs Rs. $(2 \times 90 + 3 \times 160)$. That is $180 + 480 = 660$.
While, Rani needs Rs. $(4 \times 90 + 2 \times 160)$. That is $360 + 320 = 680$.

In terms of matrix representation, we can write the above information as follows,

Requirements	**Price per item**	**Money required**
$\begin{array}{cc} \begin{array}{cc} \textit{LED bulb} & \textit{LED tube} \end{array} & \\ \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} & \begin{array}{c} \textit{Manju} \\ \textit{Rani} \end{array} \end{array}$	$\begin{bmatrix} 90 \\ 160 \end{bmatrix}$	$\begin{bmatrix} 2 \times 90 + 3 \times 160 \\ 4 \times 90 + 2 \times 160 \end{bmatrix} = \begin{bmatrix} 660 \\ 680 \end{bmatrix}$

The rule that is followed while multiplying two matrices is that the number of columns in the first matrix must equal the number of rows in the second matrix. The product of the number of rows in the first matrix and the number of columns in the second matrix determines the order of the resultant matrix.

Note: Two matrices A and B can be multiplied only if the number of columns of matrix A is equal to the number of rows of matrix B.

For example,

Let us consider the following two matrices A and B.

$$A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix} \text{ and } B = \begin{bmatrix} b11 & b12 \\ b21 & b22 \\ b31 & b32 \end{bmatrix}$$

Then the product of AB =

$$\begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{13} & a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32} \\ a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{13} & a_{21} b_{12} + a_{22} b_{22} + a_{23} b_{32} \end{pmatrix}$$

Note: Matrix A is of order 2×3 and Matrix B is of order 3×2. The number of columns of matrix A, i.e., 3, is equal to the number of rows of matrix B, i.e., 3. Thus the matrix multiplication is possible in the above case. The newly formed matrix will be of order 2×2, i.e., the number of rows of matrix A and the number of columns of matrix B.

Illustration 2.2.10

Let matrix A = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 4 \\ -1 & 3 & 3 \end{bmatrix}$. Find AB.

Solution

Given, A = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 4 \\ -1 & 3 & 3 \end{bmatrix}$

Matrix A is of order 3×3 and matrix B is of order 3×3. The number of columns of matrix A is equal to the number of rows of matrix B. Therefore, matrix multiplication is possible among the given matrices.

AB =

$$\begin{bmatrix} 1\times 2+2\times 1+(-1\times -1) & 1\times 1+2\times -2+(-1\times 3) & 1\times 2+2\times 4+(-1\times 3) \\ 2\times 2+3\times 1+0\times -1 & 2\times 1+3\times -2+0\times 3 & 2\times 2+3\times 4+0\times 3 \\ 4\times 2+2\times 1+(-3\times -1) & 4\times 1+2\times -2+(-3\times 3) & 4\times 2+2\times 4+(-3\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+1 & 1-4-3 & 2+8-3 \\ 4+3+0 & 2-6+0 & 4+12+0 \\ 8+2+3 & 4-4-9 & 8+8-9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -6 & 7 \\ 7 & -4 & 16 \\ 13 & -9 & 7 \end{bmatrix}$$

Therefore, the product of AB = $\begin{bmatrix} 5 & -6 & 7 \\ 7 & -4 & 16 \\ 13 & -9 & 7 \end{bmatrix}$

Illustration 2.2.11

If X = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and Y = $\begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$. Find the products XY and YX. Show that XY ≠ YX.

Solution

Given that,

X = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and Y = $\begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix}$

Matrix X is of order 2×2 and matrix Y is of order 2×2. The number of columns of matrix X is equal to the number of rows of matrix Y. Therefore, matrix multiplication is possible among the given matrices. Thus,

$$XY = \begin{bmatrix} 2\times 4+3\times 3 & 2\times 1+3\times 0 \\ 1\times 4+2\times 3 & 1\times 1+2\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8+9 & 2+0 \\ 4+6 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 2 \\ 10 & 1 \end{bmatrix}$$

Therefore, XY = $\begin{bmatrix} 17 & 2 \\ 10 & 1 \end{bmatrix}$

$$YX = \begin{bmatrix} 4\times2+1\times1 & 4\times3+1\times2 \\ 3\times2+0\times1 & 3\times3+0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+1 & 12+2 \\ 6+0 & 9+0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 14 \\ 6 & 9 \end{bmatrix}$$

Therefore, YX = $\begin{bmatrix} 9 & 14 \\ 6 & 9 \end{bmatrix}$

We found that XY = $\begin{bmatrix} 17 & 2 \\ 10 & 1 \end{bmatrix}$ and YX = $\begin{bmatrix} 9 & 14 \\ 6 & 9 \end{bmatrix}$

Therefore, the product XY and YX are not equal. That is, XY ≠ YX.
Note: If X and Y are two matrices, it is not necessary that the product XY ≠ YX

Illustration 2.2.12

Let Q = $\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ and P = $\begin{bmatrix} 2 & 3 \\ -2 & 0 \end{bmatrix}$. Prove that QP ≠ PQ.

Solution

It is given that Q = $\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ and P = $\begin{bmatrix} 2 & 3 \\ -2 & 0 \end{bmatrix}$

Matrix Q is of order 2×2 and matrix P is of order 2×2. The number of columns of matrix Q is equal to the number of rows of matrix P. Therefore, matrix multiplication is possible among the given matrices. Thus,

$$QP = \begin{bmatrix} 1\times2+(-1\times-2) & 1\times3+(-1\times0) \\ 3\times2+(2\times-2) & 3\times3+2\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 3+0 \\ 6-4 & 9+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$$

Therefore, QP = $\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$

$$PQ = \begin{bmatrix} 2\times1+3\times3 & (2\times-1)+3\times2 \\ -2\times1+0\times3 & (-2\times-1)+2\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9 & -2+6 \\ -2+0 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 4 \\ -2 & 2 \end{bmatrix}$$

Therefore, PQ = $\begin{bmatrix} 11 & 4 \\ -2 & 2 \end{bmatrix}$

Since QP = $\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$ and PQ = $\begin{bmatrix} 11 & 4 \\ -2 & 2 \end{bmatrix}$ it is evident that QP ≠ PQ.

Illustration 2.2.13

If X = $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 3 \\ -1 & 2 & -2 \end{bmatrix}$ Y = $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -2 & 2 \end{bmatrix}$ and Z = $\begin{bmatrix} 3 & 1 & 3 & 2 \\ -1 & 2 & -2 & 3 \end{bmatrix}$.

Find X(YZ), (XY)Z and show that X(YZ) = (XY)Z.

Solution

It is given that X = $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 3 \\ -1 & 2 & -2 \end{bmatrix}$ Y = $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -2 & 2 \end{bmatrix}$ and Z = $\begin{bmatrix} 3 & 1 & 3 & 2 \\ -1 & 2 & -2 & 3 \end{bmatrix}$.

Matrix Y is of order 3×2 and matrix Z is of order 2×4. The number of columns of matrix Y is equal to the number of rows of matrix Z. Therefore, matrix multiplication is possible among the given matrices. Thus,

YZ=

$$\begin{bmatrix} 1\times 3+(2\times -1) & 1\times 1+2\times 2 & 1\times 3+(2\times -2) & 1\times 2+2\times 3 \\ 2\times 3+(3\times -1) & 2\times 1+3\times 2 & 2\times 3+(3\times -2) & 2\times 2+3\times 3 \\ (-2\times 3)+(2\times -1) & (-2\times 1)+(2\times 2) & (-2\times 3)+(2\times -2) & (-2\times 2)+2\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 1+4 & 3-4 & 2+6 \\ 6-3 & 2+6 & 6-6 & 4+9 \\ -6-2 & -2+4 & -6-4 & -4+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & -1 & 8 \\ 3 & 8 & 0 & 13 \\ -8 & 2 & -10 & 2 \end{bmatrix}$$

Therefore, YZ = $\begin{bmatrix} 1 & 5 & -1 & 8 \\ 3 & 8 & 0 & 13 \\ -8 & 2 & -10 & 2 \end{bmatrix}$

Let us find the product of X and the new matrix YZ.

$$X = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 3 \\ -1 & 2 & -2 \end{bmatrix} \text{ and } YZ = \begin{bmatrix} 1 & 5 & -1 & 8 \\ 3 & 8 & 0 & 13 \\ -8 & 2 & -10 & 2 \end{bmatrix}$$

Matrix X is of order 3x3 and the new matrix YZ is of order 3x4. The number of columns of matrix X is equal to the number of rows of matrix YZ. Therefore, matrix multiplication is possible among the given matrices. Thus,

X(YZ) =

$$\begin{bmatrix} 1.1 + 2.3 + (4.-8) & 1.5 + 2.8 + 4.2 & (1.-1) + 2.0 + (4.-10) & 1.8 + 2.13 + 4.2 \\ 2.1 + 5.3 + (3.-8) & 2.5 + 5.8 + 3.2 & (2.-1) + 5.0 + (3.-10) & 2.8 + 5.13 + 3.2 \\ -1.1 + 2.3 + (-2.-8) & -1.5 + 2.8 + (-2.2) & -1.-1 + 2.0 + (-2.-10) & -1.8 + 2.13 + (-2.2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 6 - 32 & 5 + 16 + 8 & -1 + 0 - 40 & 8 + 26 + 8 \\ 2 + 15 - 24 & 10 + 40 + 6 & -2 + 0 - 30 & 16 + 65 + 6 \\ -1 + 6 + 16 & -5 + 16 - 4 & 1 + 0 + 20 & -8 + 26 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 29 & -41 & 42 \\ -7 & 56 & -32 & 87 \\ 21 & 7 & 21 & 14 \end{bmatrix}$$

Therefore, $X(YZ) = \begin{bmatrix} -25 & 29 & -41 & 42 \\ -7 & 56 & -32 & 87 \\ 21 & 7 & 21 & 14 \end{bmatrix}$

Now let us find out the product of the two matrices X and Y.

It is given that $X = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 3 \\ -1 & 2 & -2 \end{bmatrix}$ $Y = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -2 & 2 \end{bmatrix}$

Matrix X is of order 3×3 and matrix Y is of order 3×2. The number of columns of matrix X is equal to the number of rows of matrix Y. Therefore, matrix multiplication is possible among the given matrices. Thus,

$$XY = \begin{bmatrix} 1 \times 1 + 2 \times 2 + (4 \times -2) & 1 \times 2 + 2 \times 3 + 4 \times 2 \\ 2 \times 1 + 5 \times 2 + (3 \times -2) & 2 \times 2 + 5 \times 3 + 3 \times 2 \\ (-1 \times 1) + 2 \times 2 + (-2 \times -2) & (-1 \times 2) + 2 \times 3 + (-2 \times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 4 - 8 & 2 + 6 + 8 \\ 2 + 10 - 6 & 4 + 15 + 6 \\ -1 + 4 + 4 & -2 + 6 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 16 \\ 6 & 25 \\ 7 & 0 \end{bmatrix}$$

Therefore, $XY = \begin{bmatrix} -3 & 16 \\ 6 & 25 \\ 7 & 0 \end{bmatrix}$

Let us find the product of new matrix XY and given matrix Z.

$$XY = \begin{bmatrix} -3 & 16 \\ 6 & 25 \\ 7 & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 3 & 1 & 3 & 2 \\ -1 & 2 & -2 & 3 \end{bmatrix}.$$

Matrix new matrix XY is of order 3×2 and the given matrix Z is of order 2×4. The number of columns of matrix XY is equal to the number of rows of matrix Z. Therefore, matrix multiplication is possible among the given matrices. Thus,

$$XY(Z) = \begin{bmatrix} -3.3 + (16.-1) & -3.1 + 16.2 & -3.3 + (16.-2) & -3.2 + 16.3 \\ 6.3 + (25.-1) & 6.1 + 25.2 & 6.3 + (25.-2) & 6.2 + 25.3 \\ 7.3 + 0.-1 & 7.1 + 0.2 & 7.3 + 0.-2 & 7.2 + 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 - 16 & -3 + 32 & -9 - 32 & -6 + 48 \\ 18 - 25 & 6 + 50 & 18 - 50 & 12 + 75 \\ 21 & 7 & 21 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 29 & -41 & 42 \\ -7 & 56 & -32 & 87 \\ 21 & 7 & 21 & 14 \end{bmatrix}$$

Therefore, $XY(Z) = \begin{bmatrix} -25 & 29 & -41 & 42 \\ -7 & 56 & -32 & 87 \\ 21 & 7 & 21 & 14 \end{bmatrix}$

Since, $X(YZ) = \begin{bmatrix} -25 & 29 & -41 & 42 \\ -7 & 56 & -32 & 87 \\ 21 & 7 & 21 & 14 \end{bmatrix}$, it is proved that $XY(Z) = X(YZ)$.

Recap

- Matrix addition is commutative i.e., A + B = B + A
- Matrix addition is associative i.e., (A + B) + C = A + (B + C)
- Zero matrix O is the additive identity of a matrix because adding a matrix with zero matrix leaves it unchanged i.e., X + O = O + X = X
- When a matrix is multiplied by a scalar, the scalar amount is multiplied by each element of the matrix
- Two matrices A and B can be multiplied only if the number of columns of matrix A is equal to the number of rows of matrix B
- If two matrices are in the order mxn and nxq respectively, after multiplication the final matrix will be of the order mxq

Objective Questions

1. What is the commutative law of matric addition?
2. How do you define the associative law of matric addition?
3. How do you define the existence of additive identity?
4. What happens to a matrix, if you multiply it with a scalar, say 3?
5. What is the condition of matrix multiplication?
6. Give an alternative name for the null matrix.
7. What is the rule to follow when we add or subtract two matrices?

Answers

1. A + B = B + A
2. Matrix addition is associative i.e., (A + B) + C = A + (B + C).
3. When, X + O = O + X = X
4. When a matrix is multiplied by a scalar (3), the scalar amount is multiplied by each element of the matrix.
5. The number of columns of matrix A is equal to the number of rows of matrix B.
6. Zero matrix
7. Matrices must be in the same order.

Assignments

1. Let A = $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$.

Find A(B+C) and AB +AC. Prove that A(B+C) = AB +AC.

2. Given two matrices

$$A = \begin{bmatrix} 6 & 2 & 4 \\ -2 & 3 & 2 \\ 0 & 1 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 8 & 4 \\ 11 & 1 & 8 \\ 9 & -5 & 3 \end{bmatrix}$$

a. Find A + B and A – B.

b. A + O = O + A = A.

3. Evaluate A+B, A-B, and 5A-3B when

A= $\begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 7 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & 9 \end{bmatrix}$

Suggested Readings

1. Carl P. Simon and Lawrence Blume, Mathematics for Economist, W. W. Norton & Company, 1994
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UNIT 3

Trace of a Matrix

Learning Outcomes

After completing this unit, the learner will be able to:

- determine the trace and rank of a matrix
- interpret the economic significance of trace and rank
- apply the trace and rank of a matrix to solve the economic problems

Prerequisites

In a crowded town nestled by the river's edge, there lived a community of fishermen whose lives were intricately woven into the rhythms of nature. Each day, as they cast their nets into the river, they noticed the intricate patterns formed by the interplay of fish, water currents, and the changing seasons. Little did they know that these patterns held the key to understanding the economic web that sustained their lives.

One day fishermen saw not just a river but a complex matrix of relationships, like threads woven into a grand tapestry. The fish, the water, the seasons—all were interconnected, creating a vast web of dependencies.

Before the fishermen could unravel this economic web, they needed to learn about matrices, particularly the concepts of trace and rank. Matrices were like the riverbed, the foundation upon which their understanding was built. Trace and rank were like the guiding stars, allowing them to navigate this intricate economic universe.

To grasp these concepts, they embarked on a journey of learning. Trace was like tracing the path of a river, helping them understand the flow of resources within the economic web. Rank was like counting the different types of fish, revealing the diversity and interconnectedness of economic variables.

With the knowledge of trace and rank, the fishermen could not only predict the ebbs and flows of their catches but also optimize their strategies. They understood that trace could help them to assess the overall stability of their economic system, while rank could unveil hidden dependencies and inefficiencies in their operations.

Keywords

Matrix, Trace, Rank, Elements, Diagonals

Discussion

2.3.1 Trace of a matrix

Trace of a matrix is the sum of the elements of the leading diagonals.

For example,

$$Z = \begin{bmatrix} 5 & 2 & 9 \\ 7 & 1 & 4 \\ 3 & 6 & 8 \end{bmatrix}, \text{ then the trace of } Z = 5 + 1 + 8 = 14$$

Illustration 2.3.1

Find the trace of the given matrix A.

$$A = \begin{bmatrix} 1 & 6 & -5 \\ 4 & 0 & 4 \\ 5 & -2 & -3 \end{bmatrix}$$

Solution

The trace of a matrix is the sum of the elements of the leading diagonals.

$$A = \begin{bmatrix} 1 & 6 & -5 \\ 4 & 0 & 4 \\ 5 & -2 & -3 \end{bmatrix}$$

The sum of leading diagonals in the given matrix A = 1 + 0 – 3 = - 2

Illustration 2.3.2

Let the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Solution

The trace of a matrix is the sum of the elements of the leading diagonals.

The sum of leading diagonals in the given matrix A = 1 + 3 + 1 = 5

Illustration 2.3.3

Let the matrix $A = \begin{bmatrix} 5 & 3 & 7 \\ 1 & 4 & 2 \\ 0 & 6 & X \end{bmatrix}$ and the trace of matrix A is 6. Find the value of X.

Solution

It is given that matrix $A = \begin{bmatrix} 5 & 3 & 7 \\ 1 & 4 & 2 \\ 0 & 6 & X \end{bmatrix}$ and trace of matrix A = 6

Trace of a matrix is the sum of the elements of the leading diagonals.

The leading diagonals of the given matrix are 5, 4 and X.

Therefore,

5 + 4 + X = 6

9 + X = 6

∴ X = 6 – 9 = -3

Therefore, the value of X = -3

2.3.2 Rank of Matrix

The rank of a matrix is the maximum number of linearly independent rows or columns in the matrix. The rank of a matrix cannot exceed the number of its column or rows. The rank of a matrix A can be written r(A). For a null matrix, the rank is zero. This is because the null matrix has no non- zero columns or rows. Therefore, there will not be any independent columns or rows. Knowing the rank of a matrix gives very useful information. We will study the rank of matrix in detail in block 3 after studying Minor and determinant of matrix.

Recap

- Trace of a matrix is the sum of the elements of the leading diagonals
- The rank of a matrix -maximum number of linearly independent rows or columns in the matrix
- Rank of a matrix cannot exceed the number of its column or rows
- For a null matrix, the rank is zero
- The rank of a matrix A can be written r(A)

Objective Questions

1. What is trace of a matrix?
2. What is the rank of a matrix?
3. How we symbolically write the rank of matrix A
4. What is the trace for the Matrix $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$?
5. What is the rank of a null matrix?

Answers

1. The trace of a matrix is the sum of the elements of the leading diagonals
2. The rank of a matrix is the maximum number of linearly independent rows or columns in the matrix
3. r(A)
4. 7
5. 0

Assignments

1. Find trace of the following matrices

a. $Z = \begin{bmatrix} 3 & 5 & 9 \\ 2 & 1 & 4 \\ 9 & 3 & 4 \end{bmatrix}$

b. $Z = \begin{bmatrix} 7 & 2 & 1 \\ 5 & 9 & 4 \\ 3 & 4 & 8 \end{bmatrix}$

c. $Z = \begin{bmatrix} -5 & 2 & 2 \\ 7 & 3 & 4 \\ 6 & 7 & 4 \end{bmatrix}$

Suggested Readings

1. Carl P. Simon and Lawrence Blume, Mathematics for Economist, W. W. Norton & Company, 1994
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3

BLOCK

Fundamentals of Matrix Algebra II

UNIT 1

Determinants

Learning Outcomes

After completing this unit, the learner will be able to:

- have a comprehensive understanding of determinants
- be proficient in applying determinant properties
- apply determinants to solve economic problems and model economic relationships
- evaluate the economic implications of determinant solutions

Prerequisites

Matrix has a wide range of applications in various fields such as mathematics, physics, economics, computer science, and even everyday life. In mathematics, a matrix means a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. Linear equations are solved with the help of a matrix and it also uses geometry, data compression and encryptions robotics, and economics among others. In this context, we explore the concept of determinants, which is a mathematical concept closely related to the square matrix. The determinant of a square matrix is a scalar value that can be calculated using specific rules. Mainly in economics determinants are used to analyse the relationship between the different sectors of an economy and are also used for economic planning and policy making. Here we consider the application of determinants in the case of trade between two countries. For instance, India and Indonesia are engaged in trade, India has an abundance in the production of mangoes and Indonesia is well known for the production of cocoa.

They decided to trade their mangoes for cocoas to help both countries grow economically. Here we used a special chart called a matrix to monitor their trading. This chart consists of two columns: one for mango and one for cocoas. Each row in the chart displayed how many mangoes or cocoas one country was buying or selling. It was like a tally sheet, helping them keep a record of their trades accurately. In this system, the determinant of the matrix played a vital role in narrating the trade balance between the two countries. After calculating the trade matrix we get a determinant value that shows the result of the trade. If the determinant value is positive, both countries' trade was balanced. Imagine India selling more mangoes and cocoas than it bought. Indonesia, just like India, would also be selling and buying goods to make the trade beneficial for both nations. On the other hand, if the value of the determinant is negative, it shows a different picture. Indonesia imports more apples and bananas than it exports which means trade is imbalanced. It denotes a situation where one nation was consuming more than it produced. Contrary if the value of the determinant is zero, it means one country is completely depends on the other for goods or maybe they are trading exactly equal amount.

Keywords

Matrix, Determinant, Linear equation, Transpose

Discussion

3.1.1 Determinants

A determinant is a compact form that depicts a set of numbers arranged in rows and columns, with the number of rows and columns being equal. In fact, a matrix's determinant is a number that is only defined for square matrices. That is, a determinant is a certain sort of statement written in a compact manner. It is worth noting that the numbers are placed

A Japanese mathematician named Seki Kowa (1683) is credited with inventing the determinant while systematising the old Chinese method of solving simultaneous equations with coefficients represented by calculating bamboos or sticks. Later, determinants were formalised by German mathematician Gottfried Wilhelm Von Leibnitz. Arthur Cayley proposed the current vertical notation in 1841. Crammer, who published his well-known formula for solving simultaneous equations in 1750, was the first to invent Determinant.

in a square between two vertical lines. A determinant is the name for this arrangement. The numbers in a determinant are referred to as the elements of the determinant.

The mathematical operation to be performed on the elements is referred to as the determinant. As a result of the operation, a simplified value of the determinant is formed as the determinant value. In fact, the determinants are mathematical objects that can be used to analyse and solve systems of linear equations. Determinants have numerous applications in economics, engineering, science, and social science. Let us look at the determinant of a matrix now.

3.1.1.1 Order of a Determinant

As discussed above, every determinant is in a square matrix form. That is, it has an equal number of rows and columns. Order of a determinant is the number of rows and columns of the determinant.

For example,

$\begin{vmatrix} 3 \end{vmatrix}$ is a determinant of order 1×1

$\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix}$ is a determinant of order 2×2

$\begin{vmatrix} 2 & 4 & 1 \\ 1 & 7 & 2 \\ 5 & 9 & 5 \end{vmatrix}$ is a determinant of order 3×3

$\begin{vmatrix} 1 & 3 & 7 & 4 \\ 2 & 6 & 4 & 1 \\ 1 & 5 & 3 & 2 \\ 5 & 2 & 1 & 0 \end{vmatrix}$ is a determinant of order 4×4

3.1.1.2 Value of a Determinant

The value of a determinant is a special number that can be calculated from a matrix. The value of a determinant is usually denoted as $|A|$ if the matrix is A.

***Value of a determinant of order* 2×2**

Let us see how the value of a determinant is calculated in a square matrix of order 2×2.

Let $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ be a determinant of order 2×2.

Step 1

Find out the product of the elements in the leading diagonal 'a and d'.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{ad}$$

Step 2

Find out the product of the elements in the other leading diagonal 'b and c'.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{bc}$$

Step 3

Minus the second value obtained from the first value, i.e., 'ad - bc'.

For example,

Let A = $\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}$ be a determinant of order 2×2

Value of the determinant will be,

$|A| = \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}$ = (3×4) – (2×5) = 12 – 10 = 2

Illustration 3.1.1

Find the value of the determinant A = $\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix}$

Solution

$|A| = \begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix}$ = (-4×2) – (1×3) = -8 – 3 = -11

Illustration 3.1.2

Find the value of the determinant A = $\begin{vmatrix} 6 & 5 \\ -1 & 3 \end{vmatrix}$

Solution

$|A| = \begin{vmatrix} 6 & 5 \\ -1 & 3 \end{vmatrix}$ = (6×3) – (5×-1) = 18 – (-5) = 18 + 5 = 23

Illustration 3.1.3

Find the value of the determinant A = $\begin{vmatrix} -2 & -6 \\ -7 & -4 \end{vmatrix}$

Solution

$|A| = \begin{vmatrix} -2 & -6 \\ -7 & -4 \end{vmatrix}$ = (-2×-4) – (-6×-7) = 8 – 42 = -34

Illustration 3.1.4

Find the value of the determinant A = $\begin{vmatrix} -4 & 2 \\ 2 & -1 \end{vmatrix}$

Solution

$|A| = \begin{vmatrix} -4 & 2 \\ 2 & -1 \end{vmatrix}$ = (-4×-1) – (2×2) = 4 – 4 = 0.

Illustration 3.1.5

Find the value of the determinant A = $\begin{vmatrix} 4a & b \\ 3 & a \end{vmatrix}$

Solution

$|A| = \begin{vmatrix} 4a & b \\ 3 & a \end{vmatrix}$ = $(4\text{a} \times \text{a}) - (\text{b} \times 3) = 4\text{a}^2 - 3\text{b}$

Value of a determinant of order 3×3

Let us see how the value of a determinant is calculated in a square matrix of order 3×3.

Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ be a determinant of order 3×3, the determinant can be calculated by a × (determinant obtained by eliminating the row and column where 'a' is located) - b× (determinant obtained by eliminating the row and column where 'b' is located) + c× (determinant obtained by the eliminating the row and column where 'c' is located)

That is, a × (determinant obtained by ignoring 1st row and 1st column) - b × (determinant obtained by ignoring 1st row and 2nd column) + c × (determinant obtained by the ignoring 1st row and 3rd column)

Step 1

a × (determinant obtained by ignoring 1st row and 1st column), can be written as

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

- b × (determinant obtained by ignoring 1st row and 2nd column), can be written as

$$b\begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

+ c× (determinant obtained by the ignoring 1st row and 3rd column), can be written as

$$c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Step 2

Write them as,

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Step 3

Find out the values of all the three new determinants as we found the determinants of order 2x2. Then find the final value of the given determinant of order 3×3. That is,

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \text{a (ei – fh) – b (di – fg) + c (dh – eg)}$$

For example,

$$|A| = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 4 & 6 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 4 & 6 \end{vmatrix}$$

= 2 ((1×7) – (2×6)) – 5 ((3×7) – (2×4)) + 4 ((3×6) – (1×4))

= 2 (7 – 12) – 5 (21 – 8) + 4 (18 – 4)

= 2 (-5) – 5 (13) + 4 (14)

= -10 – 65 + 56

= -75 + 56

= - 19

Illustration 3.1.6

Find the value of the determinant A = $\begin{vmatrix} 1 & -4 & 3 \\ 5 & 2 & -2 \\ 2 & -1 & 1 \end{vmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & -4 & 3 \\ 5 & 2 & -2 \\ 2 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 5 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix}$$

= 1(2×1) – (-2×-1)) – (-4) (5×1) – (-2×2) + 3(5×-1) – (2×2)

= 1(2 – 2) – (-4) (5 – (-4)) + 3(-5 – 4)

= 1(0) – (-4 × 9) + (3 ×-9)

= 0 – (-36) + (-27)

= 36 - 27

= 9

Illustration 3.1.7

Find the value of the determinant A = $\begin{vmatrix} 2 & 1 & -2 \\ 4 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix}$

Solution

$|A| = \begin{vmatrix} 2 & 1 & -2 \\ 4 & 0 & 3 \\ 1 & 2 & -3 \end{vmatrix}$ $= 2\begin{vmatrix} 0 & 3 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} 4 & 3 \\ 1 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix}$

= 2 ((0×-3) – (2×3)) – 1 ((4×-3) – (1×3)) + (-2) ((4×2) – (1×0))

= 1 (0 – 6) – 1(-12 - 3) + (-2) (8 – 0)

= 1 (-6) – 1(-15) + (-2) 8

= – 6 + 15 - 16 = -7

3.1.1.3 Properties of Determinants

Let us discuss the basic six properties of determinants. This will help us to discover the linear dependence among the rows.

Property 1

The value of the determinant remains unchanged, if the determinant's rows are changed into columns and the columns into rows. That is, $|A| = |A'|$
or, determinant of a matrix and its transpose are same.
For example,

Let the determinant be, $\begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix}$ value of the determinant,

$\begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix}$= (-3×2) – (4×1) = -6 – 4 = -10.

If the rows and columns of the determinant are interchanged, the initial determinant becomes,

$\begin{vmatrix} -3 & 1 \\ 4 & 2 \end{vmatrix}$, then the value of the new determinant is,

$\begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix}$ = (-3×2) – (4×1) = -6 – 4 = -10.

Property 2

Sign of the determinant changes when any two rows or columns are interchanged.

Let the determinant be $\begin{vmatrix} 2 & 5 & 9 \\ 3 & 1 & 2 \\ 7 & 4 & 6 \end{vmatrix}$. The value of the given determinant is,

$\begin{vmatrix} 2 & 5 & 9 \\ 3 & 1 & 2 \\ 7 & 4 & 6 \end{vmatrix}$ = 2 (1×6 – 4×2) – 5 (3×6 – 7×2) + 9 (3×4 – 7×1)

$= 2\ (6 - 8) - 5\ (18 - 14) + 9\ (12 - 7)$

$= 2\ (-2) - 5\ (4) + 9\ (5)$

$= -4 - 20 + 45$

$= 21$

If we interchange the first row with the third row (R1 → R3), the new determinant will be,

$$\begin{vmatrix} 7 & 4 & 6 \\ 3 & 1 & 2 \\ 2 & 5 & 9 \end{vmatrix}.$$

Let us find the value of the new determinant.

$$\begin{vmatrix} 7 & 4 & 6 \\ 3 & 1 & 2 \\ 2 & 5 & 9 \end{vmatrix} = 7\ (1\times9 - 5\times2) - 4\ (3\times9 - 2\times2) + 6\ (3\times5 - 1\times2)$$

$= 7\ (9 - 10) - 4\ (27 - 4) + 6\ (15 - 2)$

$= 7\ (-1) - 4\ (23) + 6\ (13)$

$= -7 - 92 + 78$

$= -21$

Thus, the sign of the determinant changes when the two rows or columns are interchanged.

Property 3

The value of the determinant is zero if all elements of a row or a column are zero.

For example,

Let the determinant be $\begin{vmatrix} 7 & 4 & 6 \\ 0 & 0 & 0 \\ 2 & 5 & 9 \end{vmatrix}$, then the value of the determinant will be,

$$\begin{vmatrix} 7 & 4 & 6 \\ 0 & 0 & 0 \\ 2 & 5 & 9 \end{vmatrix} = 7\ (0\times9 - 0\times5) - 4\ (0\times9 - 0\times2) + 6\ (0\times5 - 0\times2)$$

$= 7\ (0) - 4\ (0) + 6\ (0) = 0$

Property 4

If any two rows or columns are identical the value of the determinant is zero.

For example,

Let the determinant be $\begin{vmatrix} 1 & 3 & 1 \\ 4 & 2 & 5 \\ 1 & 3 & 1 \end{vmatrix}$, then the value of the determinant will be,

$$\begin{vmatrix} 1 & 3 & 1 \\ 4 & 2 & 5 \\ 1 & 3 & 1 \end{vmatrix} = 1\ (2\times1 - 3\times5) - 3\ (4\times1 - 1\times5) + 1\ (4\times3 - 1\times2)$$

$= 1\ (2 - 15) - 3\ (4 - 5) + 1\ (12 - 2)$

$= 1\ (-13) - 3\ (-1) + 1(10)$

$= -13 + 3 + 10 = 0$

Property 5

If all the elements of a row or a column of a determinant are multiplied by a constant 'k' the value of the determinant gets multiplied by 'k'.

For example,

Let the given determinant be $\begin{vmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \\ 2 & 4 & 1 \end{vmatrix}$, then the value of the determinant will be,

$$\begin{vmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \\ 2 & 4 & 1 \end{vmatrix} = 2\,(1\times1 - 4\times5) - 1\,(3\times1 - 2\times5) + 4\,(3\times4 - 1\times2)$$

$$= 2\,(1 - 20) - 1\,(3 - 10) + 4\,(12 - 2)$$

$$= 2\,(-19) - 1\,(-7) + 4\,(10)$$

$$= -38 + 7 + 40$$

$$= 9$$

Let us multiply the first row of the given determinant by a constant 2. That is R1× 2.

$$\begin{vmatrix} 2\times2 & 2\times1 & 2\times4 \\ 3 & 1 & 5 \\ 2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 8 \\ 3 & 1 & 5 \\ 2 & 4 & 1 \end{vmatrix}$$

The value of the new determinant will be,

$$\begin{vmatrix} 4 & 2 & 8 \\ 3 & 1 & 5 \\ 2 & 4 & 1 \end{vmatrix} = 4\,(1\times1 - 4\times5) - 2\,(3\times1 - 2\times5) + 8\,(3\times4 - 1\times2)$$

$$= 4\,(1 - 20) - 2\,(3 - 10) + 8\,(12 - 2)$$

$$= 4\,(-19) - 2\,(-7) + 8\,(10)$$

$$= -76 + 14 + 80$$

$$= 18$$

The value of the initial determinant was 9. Multiplying the first row of the initial determinant with a constant 2 leads to a new determinant value 18 which is a constant times the value of the initial determinant. That is 2×9 = 18.

Property 6

For a triangular determinant, the value of the determinant will be the product of the leading diagonals.

For example,

The value of the given determinant $\begin{vmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 2 & 9 \end{vmatrix}$ will be,

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 2 & 9 \end{vmatrix} = 4\,(1\times9 - 2\times0) - 0\,(3\times9 - 7\times0) + 0\,(3\times2 - 1\times7)$$

$$= 4\,(9 - 0) - 0\,(27 - 0) + 0\,(6 - 7)$$

$$= 4\,(9) - 0\,(27) + 0\,(-1)$$

$$= 36$$

Let us find the product of the leading diagonal elements now.

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 2 & 9 \end{vmatrix} \rightarrow 4\times1\times9 = 36$$

Thus, for a triangular determinant, the value of the determinant will be the product of the leading diagonals.

Recap

- Determinant - a compact form that depicts a set of numbers arranged in rows and columns
- Only square matrices have determinants
- The numbers in a determinant are referred to as the elements of the determinant
- Order of a determinant - the number of rows and columns of the determinant
- Value of a determinant - special number that can be calculated from a matrix
- The value of the determinant remains unchanged, if the determinant's rows are changed into columns and the columns into rows
- Sign of the determinant changes when any two rows or columns are interchanged
- The value of the determinant is zero if all elements of a row or a column are zero
- If any two rows or columns are identical the value of the determinant is zero
- For a triangular determinant, the value of the determinant will be the product of the leading diagonals.

Objective Questions

1. What type of matrices is the determinant specifically defined for?
2. Who is credited with inventing the determinant?
3. What does the order of a determinant represent?
4. How is the value of a determinant typically denoted in mathematical notation?
5. What is the value of a determinant if all elements of a row or a column are zero?
6. What will happen to the value of a determinant if the determinant's rows are changed into columns and the columns into rows?
7. What is the value of a triangular determinant in relation to its leading diagonals?
8. If all the elements of a row or a column of a determinant are multiplied by a constant k, how does this operation affect the value of the determinant?

Answers

1. Square matrix
2. Seki Kowa
3. Number of rows and columns in the determinants
4. $|A|$
5. Zero
6. The value of determinants is unchanged
7. Product of the leading diagonal.
8. The determinant is multiplied by k

Assignments

1. Prove the following properties of determinants with the help of suitable examples.

Property1: The value of the determinant remains unchanged, if the determinant's rows are changed into columns and the columns into rows.

Property 2: The value of the determinant is zero if all elements of a row or column are zero.

2. Find the determinants A. $\begin{vmatrix} 2 & 5 & 6 \\ 4 & 7 & 6 \\ 9 & 5 & 2 \end{vmatrix}$ B. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{bmatrix}$ C. $\begin{bmatrix} 3 & 4 & 5 \\ 7 & 8 & 8 \\ 11 & 7 & -4 \end{bmatrix}$

D. $\begin{bmatrix} -7 & 3 & 9 \\ 8 & 2 & -1 \\ 13 & 5 & 7 \end{bmatrix}$ E. $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 5 & 3 \\ 4 & 5 & 9 \end{bmatrix}$

Suggested Readings

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2

UNIT

Minors and Cofactors, Adjoint and Rank of Matrix

Learning Outcomes

After completing this unit, the learner will be able to:

- have a comprehensive understanding of minors and cofactors in matrix theory
- recognise the role of the adjoint matrix in finding the inverse of a matrix

Prerequisites

In a quaint village nestled amidst rolling hills and lush vineyards, there lived a community of puzzle enthusiasts. Their days were spent deciphering riddles, solving intricate jigsaw puzzles, and seeking answers to the mysteries that surrounded them. Little did they know that their love for puzzles held the key to understanding the enigmatic world of matrices, minors, cofactors, and the adjoint matrix.

Imagine, if you will, the puzzle workshop where villagers gathered to explore the world of mysteries. The tables were strewn with puzzle pieces, each one representing a fragment of a larger picture, much like the elements of a matrix. These pieces were the clues to solving the riddles, the hints that unlocked the secrets of their world.

Before the puzzle enthusiasts could dive into the world of matrices, they needed to understand the language of numbers and arrangements. Matrices were like the blueprints of complex puzzles, a grid of numbers waiting to be decoded. Minors and cofactors were the individual puzzle pieces, the elements of matrices that held hidden information.

To comprehend matrices, they embarked on a journey of learning. They learned how to calculate minors, understand cofactors, and construct the adjoint matrix. These were like the strategies for piecing together intricate jigsaw puzzles, each step leading to a more complete picture.

With the knowledge of matrices, minors, cofactors, and the adjoint matrix, the puzzle enthusiasts could not only solve their riddles but also apply their skills to the economic world. They understood that just as they assembled puzzle pieces to reveal a bigger picture, economists used matrices, minors, and cofactors to decode economic data, optimize resource allocation, and solve complex economic equations.

Keywords

Minor, Cofactor, Element, Adjoint, Rank

Discussion

3.2.1 Minor of a Matrix

The determinant of the elements of the matrix, derived after removing the row and column containing the particular element in the matrix, is the minor of the element in the matrix. In fact, it is the value of the determinant formed with elements obtained when the row and the column in which the element lies, are deleted. A number termed the minor is assigned to each element of a determinant.

3.2.1.1 Minor of 2×2 Matrix

Let us consider a 2×2 matrix, for example, minor of 2 in the determinant $\begin{vmatrix} 2 & 8 \\ 5 & 3 \end{vmatrix}$ is 3.

That is, by eliminating column and row of the element 2, $\begin{vmatrix} 2 & 8 \\ 5 & 3 \end{vmatrix}$, the remaining number in this 2×2 determinant is 3.

Now you may find the minors of 8, 5 and 3 of the above determinants.

You may have found that the minor of 8 is 5, the minor of 5 is 8 and the minor of 3 is 2.

Illustration 3.2.1

Find the minor of 9 from the determinant X = $\begin{vmatrix} 1 & -3 \\ 9 & 5 \end{vmatrix}$.

Solution

Determinant X = $\begin{vmatrix} 1 & -3 \\ 9 & 5 \end{vmatrix}$.

By eliminating column and row of the element 9, the remaining number in this 2×2 determinant is -3. Thus -3 is the minor of the determinant X.

3.2.1.2 Minor of 3x3 matrices

Let's consider a 3×3 matrix,

If the matrix A = $\begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & -1 \\ 2 & 8 & 9 \end{bmatrix}$ we have to use the following technique to find the minor of the matrix. Consider element 3. It appears in first row and first column. From the given matrix delete the first row and the first column. The determinant formed with the remaining elements is minor of 3.

That is, the minor of 3 is $\begin{vmatrix} 6 & -1 \\ 8 & 9 \end{vmatrix}$ = 6×9 – (8×-1) = 54 – (-8) = 54 + 8 = 62

Illustration 3.2.2

You may find the minor of 8 from the above matrix A.

Solution

Matrix A = $\begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & -1 \\ 2 & 8 & 9 \end{bmatrix}$. By eliminating the row and column the element 8 lies, we get, $\begin{vmatrix} 3 & 4 \\ 5 & -1 \end{vmatrix}$. Thus, the minor of 8 is $\begin{vmatrix} 3 & 4 \\ 5 & -1 \end{vmatrix}$ = 3×-1 – (4×5) = −3 – 20 = −23

3.2.2 Cofactor of a Matrix

A cofactor is a numerical factor associated with each element in a matrix. Specifically, the cofactor of a matrix element is calculated by multiplying the element's minor by -1 and adding the exponent (power) of the sum of the i^{th} row and j^{th} column containing the element. That is, cofactor of the element α_{ij} is obtained by multiplying the minor of a_{ij} with $(-1)^{i+j}$. Cofactors have important applications in finding the determinant of a matrix and in the process of matrix inversion. In this section, we will explore how to calculate cofactors, their relationship with minors, and how they contribute to solving complex problems involving matrices.

For example, if A = $\begin{bmatrix} -1 & 4 & 3 \\ 5 & -3 & 2 \\ -2 & 1 & 6 \end{bmatrix}$

Cofactor a_{12} is $\alpha_{12} = (-1)^{1+2} \begin{vmatrix} 5 & 2 \\ -2 & 6 \end{vmatrix} = (-1)^3 (5\times6 - (2\times-2)) = -1 (30 - (-4))$

$= -1 (30+4) = -1 (34) = -34$

Likewise let us find the cofactor a_{23}

Cofactor a_{23} is $\alpha_{23} = (-1)^{2+3} \begin{vmatrix} -1 & 4 \\ -2 & 1 \end{vmatrix} = (-1)^5 ((-1\times1) - (4\times-2)) = -1 (-1 - (-8))$

$= -1 (-1+8) = -1 (7) = -7$

Using the same method, we can find all the cofactors of the elements.
Let us find the cofactors of the following problem.

Illustration 3.2.3

If A = $\begin{bmatrix} 2 & -2 & 0 \\ -1 & 4 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ Find the cofactors of elements 4 and 7

Solution

A = $\begin{bmatrix} 2 & -2 & 0 \\ -1 & 4 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

Element 4 is a_{22}. Cofactor of a_{22} is α_{22}.

That is $\alpha_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 1 & 7 \end{vmatrix} = (-1)^4 ((2\times7) - (0\times1)) = 1 (14 - 0) = 1 (14) = 14$

Element 7 is a_{33}. Cofactor of a_{33} is α_{33}.

That is $\alpha_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} = (-1)^4((2\times4) - (-1\times-2)) = 1 (8 - (2)) = 1 (8-2) = 1 (6) = 6$

3.2.3 Adjoint of a Matrix

The adjoint of a matrix, also known as the adjugate or classical adjoint, is a matrix obtained from the cofactors of the elements of the original matrix. Adjoint of a square matrix is the transpose of the matrix formed by cofactors of the elements of a given square matrix, taken in order.

For example,

Let the square matrix be A = $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$ then Adjoint of the matrix A will be,

Adj A = Transpose of $\begin{bmatrix} \alpha 11 & \alpha 12 & \alpha 13 \\ \alpha 21 & \alpha 22 & \alpha 23 \\ \alpha 31 & \alpha 32 & \alpha 33 \end{bmatrix}$

Now, let us discuss the same with an example.

Illustration 3.2.4

Let matrix A = $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$. Find the adjoint of matrix A.

Solution

It is given that matrix A = $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$

First of all, let us find the cofactors of all the nine elements of the given 3×3 square matrix.

$\boldsymbol{\alpha}_{11}$ = co-factor of 1= $(-1)^2 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$ = 1 (3 – 0) = 3

$\boldsymbol{\alpha}_{12}$ = co-factor of 3 = $(-1)^3 \begin{vmatrix} 4 & 0 \\ 5 & 3 \end{vmatrix}$ = -1 (12 – 0) = -12

$\boldsymbol{\alpha}_{13}$ = co-factor of 2 = $(-1)^4 \begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}$ = 1 (8 – 5) = 3

$\boldsymbol{\alpha}_{21}$ = co-factor of 4 = $(-1)^3 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$ = -1 (9 – 4) = -5

$\boldsymbol{\alpha}_{22}$ = co-factor of 1 = $(-1)^4 \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix}$ = 1 (3 – 10) = -7

$\boldsymbol{\alpha}_{23}$ = co-factor of 0 = $(-1)^5 \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix}$ = -1 (2 – 15) = 13

$\boldsymbol{\alpha}_{31}$ = co-factor of 5 = $(-1)^4 \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix}$ = 1 (0 – 2) = -2

$\boldsymbol{\alpha}_{32}$ = co-factor of 2 = $(-1)^5 \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix}$ = -1 (0 – 8) = 8

$\boldsymbol{\alpha}_{33}$ = co-factor of 3 = $(-1)^6 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}$ = 1 (1 – 12) = -11

Adj A = Transpose of $\begin{bmatrix} 3 & -12 & 3 \\ -5 & -7 & 13 \\ -2 & 8 & -11 \end{bmatrix} = \begin{bmatrix} 3 & -5 & -2 \\ -12 & -7 & 8 \\ 3 & 13 & -11 \end{bmatrix}$

Illustration 3.2.5

Find the adjoint of the matrix $\begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

Solution

Let it be matrix A = $\begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

First of all, let us find the cofactors of all the nine elements of the given 3×3 square matrix.

$\boldsymbol{\alpha}_{11}$ = co-factor of 3 = $(-1)^2 \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$ = 1 (-1 – 2) = -3

$\boldsymbol{\alpha}_{12}$ = co-factor of -1 = $(-1)^3 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$ = -1 (-1 – 4) = 5

$\boldsymbol{\alpha}_{13}$ = co-factor of 1 = $(-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$ = 1 (1 – 2) = -1

$\boldsymbol{\alpha}_{21}$ = co-factor of 1 = $(-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$ = -1 (1 – 1) = 0

$\boldsymbol{\alpha}_{22}$ = co-factor of 1 = $(-1)^4 \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix}$ = 1 (-3 – 2) = -5

$\boldsymbol{\alpha}_{23}$ = co-factor of 2 = $(-1)^5 \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$ = -1 (3 – (-2)) = -5

$\boldsymbol{\alpha}_{31}$ = co-factor of 2 = $(-1)^4 \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}$ = 1 (-2 – 1) = -3

$\boldsymbol{\alpha}_{32}$ = co-factor of 1 = $(-1)^5 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$ = -1 (6 – 1) = -5

$\boldsymbol{\alpha}_{33}$ = co-factor of -1 = $(-1)^6 \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$ = 1 (3 – (-1)) = 4

Adj A = Transpose of $\begin{bmatrix} -3 & 5 & -1 \\ 0 & -5 & -5 \\ -3 & -5 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -3 \\ 5 & -5 & -5 \\ -1 & -5 & 4 \end{bmatrix}$

3.2.4 Rank of Matrix

A matrix has rank 'k' if at least one of its minors is not zero and all minors of order greater than 'k', if any, are zero.

Thus, if all minors of order 3 or more are zero and at least one minor of order 2 is not zero, the rank of a matrix is said to be 2.

Note: If the given matrix is square, the highest minor is the determinant formed by all of the given matrix's elements in the same position.

Let us understand the concept of the rank of a matrix with the following example.

$A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix}$ is a square matrix of order 3×3. Thus, the highest minor that can be obtained is of order 3. That is,

$$|A| = \begin{vmatrix} 3 & 5 & 1 \\ 1 & 1 & 0 \\ 2 & 6 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 \\ 6 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix}$$

$$= 3\,(1 - 0) - 5\,(1 - 0) + 1\,(6 - 2)$$

$$= 3\,(1) - 5\,(1) + 1\,(4)$$

$$= 3 - 5 + 4 = 2$$

Since the value is not zero, the rank of this given matrix is 3.

Illustration 3.2.6

Find the rank of matrix $A = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$.

Solution

$A = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ is a square matrix of order 3×3. Thus, the highest minor that can be obtained is of order 3. That is,

$$|A| = \begin{vmatrix} 5 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 5\,(0 - 3) - 0\,(0 - 1) + 2\,(6 - 1)$$

$$= 5\,(-3) - 0\,(-1) + 2\,(5)$$

$$= -15 - 0 + 10 = -5$$

Since the value is not zero, the rank of this given matrix is 3.

Illustration 3.2.7

Now let us see how to find the rank of a matrix if the determinant value is zero.

Find the rank of the matrix $A = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$.

Solution

Matrix A is a square matrix of order 3×3. Thus, the highest minor that can be obtained is of order 3 as we have discussed in the previous cases. That is,

$$|A| = \begin{vmatrix} 3 & 5 & 1 \\ 1 & 1 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 3\,(1 - 0) - 5\,(1 - 0) + 1\,(4 - 2)$$

$$= 3\,(1) - 5\,(1) + 1\,(2)$$

$$= 3 - 5 + 2 = 0$$

Since the value is zero, the rank of this given matrix is not 3.
Now consider the minors of order 2. They are,

$$\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 5 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix}$$

Let us find the determinant values of the above minors.

$$\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = 3 - 5 = -2$$

Since the determinant value of the minor of matrix A is not zero, the rank of the matrix is 2.

Illustration 3.2.8

Find the rank of the matrix A = $\begin{bmatrix} 3 & 2 & 1 \\ 9 & 6 & 3 \\ 6 & 4 & 2 \end{bmatrix}$

Solution

Matrix A = $\begin{bmatrix} 3 & 2 & 1 \\ 9 & 6 & 3 \\ 6 & 4 & 2 \end{bmatrix}$ is a square matrix of order 3×3. Thus, the highest minor that can be obtained is of order 3. That is,

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 9 & 6 & 3 \\ 6 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 9 & 3 \\ 6 & 2 \end{vmatrix} + 1\begin{vmatrix} 9 & 6 \\ 6 & 4 \end{vmatrix}$$

$$= 3\,(12 - 12) - 2\,(18 - 18) + 1\,(36 - 36)$$

$$= 3\,(0) - 2\,(0) + 1\,(0) = 0$$

Since the value is zero, the rank of this given matrix is not 3.
Now consider the minors of order 2. They are,

$$\begin{vmatrix} 3 & 2 \\ 9 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix}, \begin{vmatrix} 9 & 6 \\ 6 & 4 \end{vmatrix}, \begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix}$$

Let us find the determinant values of the above minors.

$$\begin{vmatrix} 3 & 2 \\ 9 & 6 \end{vmatrix} = 18 - 18 = 0$$

$\begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 6 - 6 = 0$

$\begin{vmatrix} 9 & 6 \\ 6 & 4 \end{vmatrix} = 36 - 36 = 0$

$\begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 12 - 12 = 0$

Since the determinant values of the minors of the given matrix A is zero, the rank of the matrix is not 2. Thus, the rank of the matrix A is 1.

Note: Minimum rank of a non-zero matrix is 1.
Note: The highest minor of order equal to m or n, whichever is less, if the given matrix is a rectangular matrix of order mxn.

Illustration 3.2.9

Find the rank of $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 5 & 1 & 2 & 0 \\ 1 & 3 & 0 & 2 \end{bmatrix}$.

Solution

The given matrix $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 5 & 1 & 2 & 0 \\ 1 & 3 & 0 & 2 \end{bmatrix}$ is a rectangular matrix of order 3×4. So, the highest minors that can be obtained are of order 3 only. Thus the highest minors of the given matrix are, $\begin{vmatrix} 2 & 1 & 4 \\ 5 & 1 & 2 \\ 1 & 3 & 0 \end{vmatrix}$ and $\begin{vmatrix} 1 & 4 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 1 & 4 \\ 5 & 1 & 2 \\ 1 & 3 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 5 & 2 \\ 1 & 0 \end{vmatrix} + 4\begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 2\,(0 - 6) - 1\,(0 - 2) + 4\,(15 - 1)$$

$$= 2\,(-6) - 1\,(-2) + 4\,(14)$$

$$= -12 + 2 + 56 = -12 + 58 = 46$$

Since the value is not zero, the rank of this given matrix is 3.

Recap

- Determinant of the elements of the matrix, derived after removing the row and column containing the particular element in the matrix, is the minor of the element in the matrix
- A number termed the minor is assigned to each element of a determinant
- Cofactor of a matrix element is calculated by multiplying the element's minor by -1 and adding the exponent (power) of the sum of the i^{th} row and j^{th} column containing the element
- Adjoint of a square matrix - transpose of the matrix formed by co-factors of the elements of a given square matrix, taken in order
- A matrix has rank 'k' if at least one of its minors is not zero and all minors of order greater than 'k', if any, are zero
- If all minors of order 3 or more are zero and at least one minor of order 2 is not zero, the rank of a matrix is said to be 2
- If the given matrix is square, the highest minor is the determinant formed by all of the given matrix's elements in the same position
- Minimum rank of a non-zero matrix is 1
- Highest minor of order equal to m or n, whichever is less, if the given matrix is a rectangular matrix of order m×n

Objective Questions

1. What is a minor of a matrix?
2. How are cofactors useful in finding the determinant of a matrix?
3. What is the significance of the exponent (power) in the formula for calculating a matrix element's cofactor?
4. What is the relationship between the adjoint of a square matrix and its cofactors?
5. How is the rank of a matrix determined based on its minors?

Answers

A

1. Minor of an element is the matrix in the determinant obtained after removing the row and column containing the particular element.
2. The determinant is the sum of the products of matrix elements and their corresponding cofactors.
3. It determines the sign of the cofactor.
4. The adjoint is the transpose of the matrix formed by cofactors of the elements of the given square matrix.
5. The rank is the order of the highest non-zero minor in the matrix.

Assignments

1. Find the rank of each of the following matrices.

1. $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ 2. $\begin{bmatrix} 7 & 4 \\ 5 & 8 \end{bmatrix}$ 3. $\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ 4. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{bmatrix}$

2. Find the minor of following 3×3 matrices.

1. Matrix A = $\begin{bmatrix} 3 & 1 & 4 \\ 5 & 6 & -1 \\ 2 & 8 & 9 \end{bmatrix}$ Matrix B= $\begin{bmatrix} 6 & 9 & 1 \\ 7 & 8 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

Matrix C= $\begin{bmatrix} 10 & 11 & 12 \\ 8 & 9 & 5 \\ 2 & 1 & 1 \end{bmatrix}$

3. Find the Cofactor of each of the following matrix.

1. $\begin{bmatrix} -1 & 2 & 3 \\ 5 & -3 & 8 \\ -1 & 0 & 6 \end{bmatrix}$ 2. $\begin{bmatrix} 7 & 10 & 11 \\ 13 & 4 & 9 \\ -5 & 7 & 1 \end{bmatrix}$

4. Find the adjoint of each of the following matrices.

1. $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 1 \\ 3 & -2 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 11 & 4 & 5 \\ 5 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 11 & 1 & 21 \\ 4 & 2 & 9 \\ 3 & 5 & 1 \end{bmatrix}$ 4. $\begin{bmatrix} 7 & 9 & 2 \\ 4 & -6 & 10 \\ 4 & -2 & -3 \end{bmatrix}$

Suggested Readings

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3

UNIT

Solution of Equations

Learning Outcomes

After completing this unit, the learner will be able to:

- solve systems of linear equations using various methods
- apply Cramer's Rule to solve economic problem
- calculate eigenvalues and eigenvectors for matrices

Prerequisites

The solution of linear equations is a fundamental concept of mathematics and has a significant role in various fields including economics, physics engineering, and computer science, etc. In this unit, we learn about the three power tools used to solve the linear equations Cramer's rule, eigenvalue, and eigenvectors. Cramer's rule is a well-defined method by using determinants for solving linear equations. It provides unique solutions for complex relationships represented by linear equations. For instance, a government agency conducted a study about the employment changes in the technology sector based on investments in research and development (R&D), infrastructure, and education initiatives. For this purpose, the agency sets up a system of equations based on the cramer's rule representing the relationships between investments in specific areas (R&D, Infrastructure, and Education) and job creation in different technology fields (Software Development, Hardware Engineering, and Data Science). By solving this equation's, the agency can accurately determine the employment projections in each technology field, allowing for informed resource allocation, targeted investment strategies, and effective workforce planning. Eigenvalues and eigenvectors are the essential concepts in linear algebra. When we applied to them to linear equations it revealed the structures of the equations. For example, in real-life situations, these patterns might help us understand if something is stable or likely to change, how things behave, and even find the best solutions to problems. In simpler terms, eigenvalues and eigenvectors are like special tools that help us see, understand, and make sense of the hidden rules in complex problems. In this unit, we will learn about the application level of Cramer's rule, eigenvalues, and eigenvectors.

Keywords

Cramer's Rule, Eigenvalue, Eigenvectors, Eigen Decomposition

Discussion

So far, we have studied the basic concepts of matrices and determinants. In this unit we shall overview the solution of system of linear equations. A linear equation is defined in mathematics as an equation written in the form Ax + By = C. It is the result of the interaction of two variables and a constant value. When we solve the system of linear equations, we get the values of the variables, which is known as a linear equation solution. Now, we will learn how a linear equation is solved using Cramer's rule, Eigenvalues, Eigenvector.

3.3.1 Cramer's Rule

Let us discuss a determinant-based method for solving systems of equations. This technique, known as Cramer's Rule, dates back to the middle of the 18th century and is named after its inventor, Swiss mathematician Gabriel Cramer (1704 -1752), who introduced it in 1750 in "Introduction to the analysis of algebraic curved lines", a geometry book. Cramer's Rule is a viable and efficient method for solving systems with an arbitrary number of unknowns, provided that the number of equations equals the number of unknowns.

Let us see how the Cramer's Rule works using the following example.
Consider a set of simultaneous equations say,
$2x + 3y = 1$, $3x + y = 5$

Step 1

Write the equations in the matrix form as follows,

$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ that is AX = B.

Where,

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Step 2

Obtain A_1 and A_2.

A_1 is obtained by replacing first column of A by B. That is,

$A_1 = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}$

A_2 is obtained by replacing second column of A by B. That is,

$A_2 = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

Step 3

Find the values of the determinants A, A_1 and A_2.

That is,

$|A| = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2 - 9 = -7$

$|A1| = \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} = 1 - 15 = -14$

$|A2| = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 10 - 3 = 7$

Step 4

Find the values of the unknowns using the formula,

$x = \frac{|A1|}{|A|}$, $y = \frac{|A2|}{|A|}$

In our example,

$$x = \frac{|A1|}{|A|} = \frac{-14}{-7} = 2$$

$$y = \frac{|A2|}{|A|} = \frac{7}{-7} = -1$$

Therefore, the value of $x = 2$ and $y = -1$

Illustration 3.3.1

Solve the linear equations using Cramer's Rule.

12x + 3y = 15

2x – 3y = 13

Solution

First of all, let us write the equations in the matrix form as follows,

$\begin{bmatrix} 12 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \end{bmatrix}$ that is AX = B.

$A = \begin{bmatrix} 12 & 3 \\ 2 & -3 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$B = \begin{bmatrix} 15 \\ 13 \end{bmatrix}$

$|A| = \begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}$ = (12×-3) – (2×3) = -36 – 6 = - 42

$A_1 = \begin{bmatrix} 15 & 3 \\ 13 & -3 \end{bmatrix}$

$|A1| = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}$ = (15×-3) – (3×13) = -45 – 39 = -84

$A_2 = \begin{bmatrix} 12 & 15 \\ 2 & 13 \end{bmatrix}$

$|A2| = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}$ = (12×13) – (15×2) = 156 – 30 = 126

Therefore,

$x = \frac{|A1|}{|A|} = \frac{-84}{-42} = 2$

$y = \frac{|A2|}{|A|} = \frac{126}{-42} = -3$

Thus, x = 2 and y = -3

Illustration 3.3.2

Solve the linear equations using Cramer's Rule.

6x + 4y + 2z = 12, 4x - 6y + 6z = 4 and 2x + 2y + 2z = 6

Solution

The given linear equations are,

6x + 4y + 2z = 12, 4x - 6y + 6z = 4 and 2x + 2y + 2z = 6

Let us write the given equations in the matrix form as follows,

$$\begin{bmatrix} 6 & 4 & 2 \\ 4 & -6 & 6 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 6 \end{bmatrix} \text{ that is AX = B}$$

Where,

$$A = \begin{bmatrix} 6 & 4 & 2 \\ 4 & -6 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 12 \\ 4 \\ 6 \end{bmatrix}$$

Since $A = \begin{bmatrix} 6 & 4 & 2 \\ 4 & -6 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 6 & 4 & 2 \\ 4 & -6 & 6 \\ 2 & 2 & 2 \end{vmatrix} = 6 \begin{vmatrix} -6 & 6 \\ 2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 4 & 6 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -6 \\ 2 & 2 \end{vmatrix}$$

$= 6 (-12 – 12) – 4 (8 – 12) + 2 (8 – (-12))$

$= 6 (-24) – 4 (-4) + 2 (20)$

$= -144 + 16 + 40 = -88$

$$A_1 = \begin{bmatrix} 12 & 4 & 2 \\ 4 & -6 & 6 \\ 6 & 2 & 2 \end{bmatrix}$$

$$|A1| = \begin{vmatrix} 12 & 4 & 2 \\ 4 & -6 & 6 \\ 6 & 2 & 2 \end{vmatrix} = 12 \begin{vmatrix} -6 & 6 \\ 2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 4 & 6 \\ 6 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -6 \\ 6 & 2 \end{vmatrix}$$

$= 12 (-12 – 12) – 4 (8 – 36) + 2 (8 – (-36))$

$= 12 (-24) – 4 (-28) + 2 (44)$

$= -288 + 112 + 88 = -88$

$$A_2 = \begin{bmatrix} 6 & 12 & 2 \\ 4 & 4 & 6 \\ 2 & 6 & 2 \end{bmatrix}$$

$$|A2| = \begin{vmatrix} 6 & 12 & 2 \\ 4 & 4 & 6 \\ 2 & 6 & 2 \end{vmatrix} = 6 \begin{vmatrix} 4 & 6 \\ 6 & 2 \end{vmatrix} - 12 \begin{vmatrix} 4 & 6 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 4 \\ 2 & 6 \end{vmatrix}$$

$= 6 (8 – 36) – 12 (8 – 12) + 2 (24 – 8)$

$= 6 (-28) – 12 (-4) + 2 (16)$

$= -168 + 48 + 32 = -88$

$$A_3 = \begin{bmatrix} 6 & 4 & 12 \\ 4 & -6 & 4 \\ 2 & 2 & 6 \end{bmatrix}$$

$$|A3| = \begin{vmatrix} 6 & 4 & 12 \\ 4 & -6 & 4 \\ 2 & 2 & 6 \end{vmatrix} = 6\begin{vmatrix} -6 & 4 \\ 2 & 6 \end{vmatrix} - 4\begin{vmatrix} 4 & 4 \\ 2 & 6 \end{vmatrix} + 12\begin{vmatrix} 4 & -6 \\ 2 & 2 \end{vmatrix}$$

$$= 6\,(-36 - 8) - 4\,(24 - 8) + 12\,(8 - (-12))$$

$$= 6\,(-44) - 4\,(16) + 12\,(20)$$

$$= -264 - 64 + 240 = -88$$

$$x = \frac{|A1|}{|A|} = \frac{-88}{-88} = 1$$

$$y = \frac{|A2|}{|A|} = \frac{-88}{-88} = 1$$

$$z = \frac{|A3|}{|A|} = \frac{-88}{-88} = 1$$

Therefore, $x = 1, y = 1, z = 1$

3.3.2 Eigenvalues and Eigenvector

A linear system of equations, i.e., a matrix equation, has a specific collection of scalars called eigenvalues. Each eigenvalue has an eigenvector that corresponds to it (or, in general, a corresponding right eigenvector and a corresponding left eigenvector; there is no analogous distinction between left and right for eigenvalues). The decomposition of a square matrix A into eigenvalues and eigenvectors is known as eigen decomposition. The eigen decomposition theorem states that this decomposition is always possible as long as the matrix A containing eigenvectors is square.

The eigenvector of a matrix A is a vector represented by a matrix X such that when X is multiplied by matrix A, the resultant matrix has the same direction as vector X.

Mathematically,

$$AX = \lambda X$$

Where,

A = Any arbitrary matrix
λ = Eigenvalues
X = Eigenvector corresponding to each eigenvalue.

We can observe that AX is parallel to X in this diagram. As a result, X is an eigenvector. For example,

Let matrix $A = \begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix}$, Eigenvector $X = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ and Eigenvalue $\lambda = 12$

That is,

AX = λX

$$AX = \begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -12 \times 2 + 6 \times 8 \\ 8 \times 2 + 10 \times 8 \end{bmatrix} = \begin{bmatrix} -24 + 48 \\ 16 + 80 \end{bmatrix} = \begin{bmatrix} 24 \\ 96 \end{bmatrix}$$

$$\lambda X = 12 \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 12 \times 2 \\ 12 \times 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 96 \end{bmatrix}$$

Let us see how to find eigenvectors and eigenvalues of any square matrix A.

We have,

$$AX = \lambda X$$

$$AX - \lambda X = 0$$

$$(A - \lambda I)\, X = 0 \qquad \rightarrow \quad (1)$$

Above condition will be true only if (A – λI) is singular. That means,

$$|A - \lambda I| = 0 \qquad \rightarrow \quad (2)$$

Equation 2 is known as the characteristic equation of the matrix.

The eigen values of the matrix A are the roots of the characteristic equation.

To find the eigenvectors, simply plug each eigenvalue into (1) and solve it using Gaussian elimination, that is, convert the augmented matrix (A – λI) = 0 to row echelon form and solve the resulting linear system of equations.

Let us try the equation 2 in the above example.

|A – λI| = 0

$$\left| \begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -12 - \lambda & 6 \\ 8 & 10 - \lambda \end{vmatrix} = 0$$

Let us find the value of the determinant.

$$(-12 - \lambda)(10 - \lambda) - (6x8) = 0$$

$$(-12 \times 10) + (-12 \times -\lambda) + (-\lambda \times 10) + (-\lambda \times -\lambda) - 48$$

$$-120 + 12\lambda - 10\lambda + \lambda^2 - 48 = 0$$

$$\lambda^2 + 2\lambda - 168 = 0$$

Now, let us solve this quadratic equation.

We have a = 1, b = 2 and c = -168

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda = \frac{-2 \pm \sqrt{2^2-4\times1\times-168}}{2\times1}$$

$$\lambda = \frac{-2 \pm \sqrt{4+672}}{2}$$

$$\lambda = \frac{-2 \pm \sqrt{676}}{2} = \frac{-2 \pm 26}{2}$$

$$\lambda = \frac{-2 + 26}{2} \text{ or } \frac{-2-26}{2}$$

$$\lambda = \frac{24}{2} \text{ or } \frac{-28}{2} = 12 \text{ or } -14$$

That is λ = 12 or -14.

Here are two possible eigenvalues.

Now we know eigenvalues, let us find their matching eigenvectors.

We have, A = $\begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix}$ and let us consider λ = 12.

Thus,

$$AX = \lambda X$$

$$\begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 12 \begin{bmatrix} x \\ y \end{bmatrix}$$

Multiplying this equation, we get,

$$-12x + 6y = 12x$$

$$8x + 10y = 12y$$

Let us solve it.

$$-12x - 12x + 6y = 0$$

$$8x + 10y - 12y = 0$$

$$-24x + 6y = 0$$

$$8x - 2y = 0$$

$$y = \frac{24x}{6}$$

$$y = 4x$$

Either equation reveals that y = 4x, so the eigenvector is any non-zero multiple of this: $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

In our case it is $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$.

And we get the solution we have found at the beginning.

$$AX = \begin{bmatrix} -12 & 6 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -12 \times 2 + 6 \times 8 \\ 8 \times 2 + 10 \times 8 \end{bmatrix} = \begin{bmatrix} -24 + 48 \\ 16 + 80 \end{bmatrix} = \begin{bmatrix} 24 \\ 96 \end{bmatrix}$$

$$\lambda X = 12 \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 12 \times 2 \\ 12 \times 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 96 \end{bmatrix}$$

Thus, $AX = \lambda X$.

Now it is your turn to find the eigenvector for the other eigenvalue of -14, another value of λ we found.

3.3.2.1 Eigenvalues and Eigenvectors for 3×3 Matrix

So far, we have discussed the technique to find the eigenvalues and eigenvectors for a 2×2 matrix. Now let us see how to find out the eigenvalues and eigenvectors for a 3×3 matrix.

Let Matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 10 \\ 0 & 8 & 6 \end{bmatrix}$

Step 1

Calculate $A - \lambda I$,

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 10 \\ 0 & 8 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 10 \\ 0 & 8 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & 8-\lambda & 10 \\ 0 & 8 & 6-\lambda \end{bmatrix}$$

Step 2

Equalise the determinant value to zero.

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 8-\lambda & 10 \\ 0 & 8 & 6-\lambda \end{vmatrix} = 0$$

That is,

$$(4-\lambda)\,((8-\lambda)\,(6-\lambda) - (10x8)) = 0$$

$$(4 - \lambda)(48 - 8\lambda - 6\lambda + \lambda^2 - 80) = 0$$

$$(4 - \lambda)\,(\lambda^2 - 14\lambda - 32) = 0$$

$$4\lambda^2 - 56\lambda - 128 - \lambda^3 + 14\lambda^2 + 32\lambda = 0$$

$$-\lambda^3 + 18\lambda^2 - 24\lambda - 128 = 0$$

Step 3

Solve the cubic equation and find the eigenvalues.

Solving the cubic equation $-\lambda^3 + 18\lambda^2 - 24\lambda - 128 = 0$ we get the eigenvalues λ = -2, 4 and 16.

Step 4

Now let us find the eigenvectors.

Let the eigenvalue λ = -2. The corresponding eigenvector can be found as follows.

$$AX = \lambda X$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 10 \\ 0 & 8 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Multiplying the equations, we get,

$$4x = -2x$$

$$8y + 10z = -2y$$

$$8y + 6z = -2z$$

That is,

$$4x + 2x = 0$$
$$10y + 10z = 0$$
$$8y + 8z = 0$$

Thus,

$$8x = 0$$
$$x = 0$$
$$10y = -10z$$
$$y = -z$$

That is x = 0, y = -z and so the eigenvector will be any non-zero multiple of the matrix $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Step 5

Test AX = λX

$$AX = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 10 \\ 0 & 8 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8-10 \\ 8-6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\lambda X = -2\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}.$$

$\lambda X = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}$ is a multiple of -2. Since the eigenvector is any non-zero multiple of the matrix

$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$,

AX = λX

Recap

- Cramer's Rule - viable and efficient method for solving systems with an arbitrary number of unknowns, provided that the number of equations equals the number of unknowns
- Linear system of equations or matrix equations has a specific collection of scalars called eigenvalues
- Each eigenvalue has an eigenvector that corresponds to it
- Decomposition of a square matrix A into eigenvalues and eigenvectors is known as eigen decomposition
- Eigen decomposition theorem states that this decomposition is always possible as long as the matrix A containing eigenvectors is square
- Eigenvector of a matrix A is a vector represented by a matrix X such that when X is multiplied by matrix A, the resultant matrix has the same direction as vector X. Mathematically, $AX = \lambda X$

Objective Questions

1. What condition must be met for Cramer's Rule to be applicable in efficiently solving equations?
2. What are eigenvalues of a matrix?
3. What does the eigen decomposition theorem state about possibly decomposing a square matrix into eigenvalues and eigenvectors?
4. What mathematical relationship is represented by the equation $AX = \lambda X$, where A is a matrix, X is an eigenvector, and λ is the corresponding eigenvalue?

Answers

1. The number of equations must equal the number of unknowns.
2. Linear system of equations or matrix equations has a specific collection of scalars called eigen values.
3. It is possible as long as the matrix containing eigenvectors is square.
4. Matrix Multiplication

Assignments

1. Solve the linear equation using Cramer's Rule.
   a. $2x - y = 3$, $4x + y = 3$
   b. $3x + 2y = 1$, $2x + 3y = 5$
   c. $2x - y = 5$, $x + y = 4$
2. Let Matrix A= $\begin{bmatrix} -14 & 6 \\ 8 & 10 \end{bmatrix}$ and eigenvalue $\lambda = -12$. Find the eigenvector.
3. Let Matrix A = $\begin{bmatrix} 8 & 2 \\ 4 & 12 \end{bmatrix}$ and eigenvalue $\lambda = -10$. Find eigenvector.

Suggested Readings

1. Carl P. Simon and Lawrence Blume, Mathematics for Economist, W. W. Norton & Company, 1994
2. Dowling Edward T, Mathematical Methods for Business and Economics, Schaums Outline Series, McGraw Hill, 1993
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4

UNIT

Inverse of a Matrix

Learning Outcomes

After completing this unit, the learner will be able to

- apply appropriate methods to find the inverse of a given matrix
- recognise how matrix inverses are used in input-output analysis, solving linear systems of equations in economics
- understand the significance of matrix inverses in economic analysis

Prerequisites

In economics, determinant is tool is handy when you want to figure out how new policies or decisions affect various factors. For instance, if you are wondering about the impact of a tax change on the economy, the determinant can help you gauge the scale of that impact. It is like having a clear lens to analyze the consequences of your decisions.

Now, when we talk about finding the inverse of a square matrix, it has the power to reverse transformations, just like you might want to undo certain economic decisions in the real world. Learning how to find the inverse is like gaining the ability to explore what might have happened if different choices were made. It is a valuable skill for making more informed decisions, almost like having a 'do-over' button in economic scenarios. As you venture further into the world of economics, understanding these fundamental concepts will be your toolkit for analysing, predicting, and making smart choices.

Keywords

Inverse of a matrix, Square matrix, Co-factor, Adjoint

Discussion

3.4.1 Inverse of a Matrix

In this unit we will discuss another topic of matrix algebra, that is, inverse of a square matrix. In simple words "Inverse" means reciprocal. For a scalar quantity, X, the reciprocal of X may be denoted by $\frac{1}{x}$. Whereas in case of a matrix, the reciprocal of a matrix is denoted by X^{-1}.

If A is a square matrix of order m and B is a square matrix of the same order m, then B is termed the inverse matrix of A and is represented by A^{-1}. A is said to be invertible in this instance. That is if A^{-1} exists, A is said to be invertible.

Note:

- *Only if A is a square matrix and A is non-singular is there an inverse for a given matrix A.*
- *A is also the inverse of B if B is the inverse of A.*
- *A rectangular matrix does not possess an inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.*

Formula: $A^{-1} = \frac{I}{|A|}(\text{Adj } A)$

3.4.2 Inverse of a 2×2 Matrix

Let us discuss the inverse of a 2×2 matrix. Let matrix A = $\begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix}$. Follow the steps to find the inverse of A.

Step 1

Find the determinant of A. (Refer Unit 1).

$|A| = \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix}$ = (2×-2) – (4×4) = -4 – 16 = -20

Therefore, $|A|$ = -20

Step 2

Find the cofactors. (Refer Unit 2).

$\alpha_{11} = (-1)^2\ (-2) = 1 \times -2 = -2$

$\alpha_{12} = (-1)^3\ (4) = -1 \times 4 = -4$

$\alpha_{21} = (-1)^3\ (4) = -1 \times 4 = -4$

$\alpha_{22} = (-1)^4\ (2) = 1 \times 2 = 2$

Step 3

Find the Adjoint of A. (Refer Unit 2).

Adj A = $\begin{bmatrix} -2 & -4 \\ -4 & 2 \end{bmatrix}$

Step 4

Use the formula to find the inverse of A.

$A^{-I} = \frac{I}{|A|}$ (Adj A)

$$= \frac{I}{-20} \begin{bmatrix} -2 & -4 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{-20} & \frac{-4}{-20} \\ \frac{-4}{-20} & \frac{2}{-20} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{-1}{10} \end{bmatrix}$$

Therefore, $A^{-I} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{-1}{10} \end{bmatrix}$

Illustration 3.4.1

Find the inverse of $\begin{bmatrix} 8 & 4 \\ -6 & 2 \end{bmatrix}$.

Solution

It is given that matrix A = $\begin{bmatrix} 8 & 4 \\ -6 & 2 \end{bmatrix}$

Determinant of A = $|A|$

$|A| = \begin{vmatrix} 8 & 4 \\ -6 & 2 \end{vmatrix}$ = (8×2) – (4×-6) = 16 – (-24) = 16 + 24 = 40.

Therefore, $|A|$ = 40.

Let us find the cofactors.

$\alpha_{11} = (-1)^2\ (2) = 1 \times 2 = 2$

$\alpha_{12} = (-1)^3 (-6) = -1 \times -6 = 6$

$\alpha_{21} = (-1)^3 (4) = -1 \times 4 = -4$

$\alpha_{22} = (-1)^4 (8) = 1 \times 8 = 8$

Therefore, the Adjoint of A,

$$\text{Adj A} = \begin{bmatrix} 2 & -4 \\ 6 & 8 \end{bmatrix}$$

Let us find the inverse of A.

$$A^{-I} = \frac{I}{|A|} (\text{Adj A})$$

$$= \frac{I}{40} \begin{bmatrix} 2 & -4 \\ 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{40} & \frac{-4}{40} \\ \frac{6}{40} & \frac{8}{40} \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & \frac{-1}{10} \\ \frac{3}{20} & \frac{1}{5} \end{bmatrix}$$

Therefore, $A^{-I} = \begin{bmatrix} \frac{1}{20} & \frac{-1}{10} \\ \frac{3}{20} & \frac{1}{5} \end{bmatrix}$

3.4.3 Inverse of a 3×3 Matrix

Now let us discuss the inverse of a 3×3 matrix.

Illustration 3.4.2

Let matrix A = $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 0 & -3 \end{bmatrix}$ Find the inverse of A

Solution

First of all, find the determinant of A = $|A|$

$$|A| = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 0 & -3 \end{vmatrix} = 1\,(1\times-3) - (0\times0) - 2\,(3\times-3 - (-2\times0)) + 4\,(3\times0 - (-2\times1))$$

$$= 1\,(-3) - 2\,(-9) + 4\,(2) = -3 + 18 + 8 = 23$$

Therefore, $|A| = 23$

Let us find the cofactors.

$$\alpha_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = 1\,(-3 - 0) = 1 \times -3 = -3$$

$\boldsymbol{\alpha}_{12} = (-1)^3 \begin{vmatrix} 3 & 0 \\ -2 & -3 \end{vmatrix} = -1\ (-9 - 0) = 9$

$\boldsymbol{\alpha}_{13} = (-1)^4 \begin{vmatrix} 3 & 1 \\ -2 & 0 \end{vmatrix} = 1\ (0 - (-2)) = 2$

$\boldsymbol{\alpha}_{21} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 0 & -3 \end{vmatrix} = -1\ (-6 - 0) = 6$

$\boldsymbol{\alpha}_{22} = (-1)^4 \begin{vmatrix} 1 & 4 \\ -2 & -3 \end{vmatrix} = 1\ (-3 - (-8)) = 5$

$\boldsymbol{\alpha}_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = -1\ (0 - (-4)) = -4$

$\boldsymbol{\alpha}_{31} = (-1)^4 \begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 1\ (0 - 4) = -4$

$\boldsymbol{\alpha}_{32} = (-1)^5 \begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} = -1\ (0 - 12) = 12$

$\boldsymbol{\alpha}_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1\ (1 - 6) = -5$

Therefore, the Adjoint of A,

$$\text{Adj A} = \begin{bmatrix} -3 & 6 & -4 \\ 9 & 5 & 12 \\ 2 & -4 & -5 \end{bmatrix}$$

Let us find the inverse of A.

$$A^{-I} = \frac{I}{|A|}\ (\text{Adj A})$$

$$= \frac{I}{23} \begin{bmatrix} -3 & 6 & -4 \\ 9 & 5 & 12 \\ 2 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{23} & \frac{6}{23} & \frac{-4}{23} \\ \frac{9}{23} & \frac{5}{23} & \frac{12}{23} \\ \frac{2}{23} & \frac{-4}{23} & \frac{-5}{23} \end{bmatrix}$$

Therefore, $A^{-I} = \begin{bmatrix} \frac{-3}{23} & \frac{6}{23} & \frac{-4}{23} \\ \frac{9}{23} & \frac{5}{23} & \frac{12}{23} \\ \frac{2}{23} & \frac{-4}{23} & \frac{-5}{23} \end{bmatrix}$

Illustration 3.4.3

Let matrix A = $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -3 \\ 2 & 1 & 1 \end{bmatrix}$. Find the inverse of A.

Solution

It is given that Matrix A = $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -3 \\ 2 & 1 & 1 \end{bmatrix}$

First of all, find the determinant of A = $|A|$

$|A|$ = $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 3 & -3 \\ 2 & 1 & 1 \end{vmatrix}$ = 2 (1×3) – (1×-3) + 1 (1×1 – (-3×2)) + 2 (1×1 – (3×2))

= 2 (3 + 3) + 1(1 + 6) + 2 (1 – 6)

= 2 (6) + 1(7) + 2 (-5)

= 12 + 7 - 10 = 9

Therefore, $|A|$ = 9

Let us find the cofactors.

$\alpha_{11} = (-1)^2 \begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix}$ = 1 (3 + 3) = 1 × 6 = 6

$\alpha_{12} = (-1)^3 \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix}$ = -1 (1 + 6) = -1 × 7 = -7

$\alpha_{13} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$ = 1 (1 – 6) = 1 × -5 = -5

$\alpha_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$ = -1 (-1 - 2) = -1 × -3 = 3

$\alpha_{22} = (-1)^4 \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$ = 1 (2 – 4) = 1 ×-2 = -2

$\alpha_{23} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix}$ = -1 (2 + 2) = -1 × 4 = -4

$\alpha_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 3 & -3 \end{vmatrix}$ = 1 (3 - 6) = 1 × -3 = -3

$\alpha_{32} = (-1)^5 \begin{vmatrix} 2 & 2 \\ 1 & -3 \end{vmatrix}$ = -1 (-6 – 2) = -1 × -8 = 8

$\alpha_{33} = (-1)^6 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$ = 1 (6 + 1) = 1×7 = 7

Therefore, the Adjoint of A,

Adj A = $\begin{bmatrix} 6 & 3 & -3 \\ -7 & -2 & 8 \\ -5 & -4 & 7 \end{bmatrix}$

Let us find the inverse of A.

$$A^{-I} = \frac{I}{|A|}\,(\text{Adj } A)$$

$$= \frac{I}{9}\begin{bmatrix} 6 & 3 & -3 \\ -7 & -2 & 8 \\ -5 & -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{9} & \frac{3}{9} & \frac{-3}{9} \\ \frac{-7}{9} & \frac{-2}{9} & \frac{8}{9} \\ \frac{-5}{9} & \frac{-4}{9} & \frac{7}{9} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{-7}{9} & \frac{-2}{9} & \frac{8}{9} \\ \frac{-5}{9} & \frac{-4}{9} & \frac{7}{9} \end{bmatrix}$$

Therefore, $A^{-I} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{-7}{9} & \frac{-2}{9} & \frac{8}{9} \\ \frac{-5}{9} & \frac{-4}{9} & \frac{7}{9} \end{bmatrix}$

Recap

- Inverse means reciprocal. For a scalar quantity, X, the reciprocal of X may be denoted by $\frac{1}{X}$. Whereas in case of a matrix, the reciprocal of a matrix is denoted by X^{-1}
- If A is a square matrix of order m and B is a square matrix of the same order m, then B is termed the inverse matrix of A and is represented by A^{-1}. A is said to be invertible in this instance. That is if A^{-1} exists, A is said to be invertible
- Only if A is a square matrix and A is non-singular is there an inverse for a given matrix A
- A is also the inverse of B if B is the inverse of A
- A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order
- Formula to find the inverse of a matrix A, $A^{-I} = \frac{I}{|A|}(\text{Adj } A)$
- Steps to find the inverse of a matrix A,
  1. Find the determinant of A, i.e., $|A|$
  2. Find the co-factors of A
  3. Find the Adjoint A
  4. Use the formula to find the inverse of A

Objective Questions

1. What is the formula for finding the inverse of a matrix A?
2. What symbol is used to represent the reciprocal of a scalar quantity X?
3. Why do rectangular matrices not possess an inverse matrix?
4. Under what conditions does a matrix A have an inverse?
5. For a square matrix A of order m and another square matrix B of the same order m, which condition is satisfy the term "invertible"?

Answers

1. $A^{-1} = \frac{1}{|A|}$ (Adj A)
2. $\frac{1}{x}$
3. It is necessary that matrices A and B should be square matrices of the same order
4. A must be a square matrix and a non-singular matrix.
5. If B is the inverse matrix of A, denoted as A^{-1}, then A is invertible

Assignments

1. Find the inverse of the following matrices.

1. A= $\begin{bmatrix} 6 & -2 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{bmatrix}$ 2. A= $\begin{bmatrix} 7 & 8 & 11 \\ 3 & 5 & 2 \\ -1 & 6 & -9 \end{bmatrix}$ 3. A= $\begin{bmatrix} 10 & 8 & 2 \\ 6 & -4 & 5 \\ 7 & 4 & 3 \end{bmatrix}$

4. A= $\begin{bmatrix} 12 & -1 & -2 \\ -6 & 7 & 8 \\ 3 & 4 & -10 \end{bmatrix}$ 5. A= $\begin{bmatrix} -4 & 6 & 8 \\ 7 & 3 & -4 \\ -6 & 1 & 1 \end{bmatrix}$

Suggested Readings

1. Carl P. Simon and Lawrence Blume, Mathematics for Economist, W. W. Norton & Company, 1994
2. Dowling Edward T, Mathematical Methods for Business and Economics, Schaums Outline Series, McGraw Hill, 1993
3. Dowling Edward.T, Introduction to Mathematical Economics, 2nd/3rd Edition, Schaum's Outline Series, McGraw-Hill, New York, 2003
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5

UNIT

Uses of Matrices

Learning Outcomes

After completing this unit, the learner will be able to:

- recognise matrices as tools for representing economic data
- create matrix representations of economic models
- understand how matrices help in resource management and assist in decision-making processes

Prerequisites

Matrix is a fundamental concept in mathematics. A matrix is a rectangular arrangement of numbers, symbols, or expressions, organized in rows and columns and is fundamental in various mathematical operations, including addition, subtraction, multiplication, and solving systems of linear equations. They find extensive applications in fields such as physics, engineering, computer science, economics, and statistics etc, Matrices have numerous real-life applications. One common example is in the field of economics. In this context, matrices help us to illustrate how goods and services flow between different sectors within an economy. An economy consists of various sectors like agriculture, manufacturing, and services. Each sector produces a particular amount of goods and services that become inputs for other sectors. These complex relationships can be found with the help of matrices.

By applying various matrix operations, economists can assess how changes in one sector affect others. This analytical approach helps policymakers and businesses in making well-informed decisions regarding resource allocation, production strategies, and economic growth initiatives. In this unit we learn about the wide application of matrices.

Keywords

Matrix Algebra, Input-output model, Linear equation, Cramer's rule

Discussion

3.5.1 Applications of a Matrix

Matrix applications are widely used in mathematics, economics and other disciplines. Matrix algebra provides the theoretical foundation for many quantitative methods, including linear programming, game theory, input-output models, and various statistical models. All of these models are created by constructing a system of linear equations that represents the problem at hand. It makes it easier to solve linear equations. In economics, matrices are used in a wide range of areas like analysing the trends of a business, share prices, and constructing business models…etc.

Matrixes are used to represent a complex system of equations and massive amounts of data in a compact form in real-world business situations with more than three variables. When a system of equations is expressed as a matrix, it can be solved fast and easily with a computer. The limitation of matrix algebra is that it can only be used in situations where linearity can be assumed.

The economic method of "input-output analysis," first suggested by Wassily W. Leontief in the 1930s, can be solved simply by using matrices. This strategy is rooted in the concept of "economic interdependence," which states that every economic sector (or industry) is linked to every other sector. That is, they are all interlinked and interdependent on one another. This means that each change in one industry has a variable impact on the other industries. Input-output analysis is built on the foundation of input-output tables. Tables like these contain a succession of rows and columns of data that quantify the supply chain for various industries. Hawkins – Simon conditions, which are connected to input-output analysis and are likewise based on matrices, ensure the system's viability.

Matrix Cramer's Rule and determinants are useful tools for resolving a variety of problems in business and economics involving profit maximisation and loss minimisation. Variance and covariance are calculated using matrices. With the use of a matrix determinant, Matrix Cramer's Rule is utilised to find solutions to linear equations. The IS-LM model's market equilibrium is solved with determinants and Matrix Cramer's Rule.

In the quantitative analysis of managerial decisions, matrices play a crucial role. They also give highly convenient and compact ways for constructing and solving a system of linear simultaneous equations. These tools have also shown to be extremely beneficial in all areas of management. Matrices also have the advantage of being able to solve systems of equations fast using a computer once they have been set up in matrix form.

So far, we have studied different matrix concepts and their operations. In this unit we shall discuss the application of determinants and matrices for solving the system of linear equations in economics.

Illustration 3.5.1

Two farmers Abhiram and Gakilal cultivate two varieties of wheat namely Sonalika and Kalyanasona. The sales in rupees of these varieties of wheat by both farmers in the month of October and November are given by the following matrices A and B.

October Sales in Rupees

$$A \begin{array}{c} \text{Sonalika} \quad \text{Kalyanasona} \\ \begin{bmatrix} 5{,}000 & 7{,}000 \\ 8{,}000 & 8{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ Abhiram \\ Gakilal \end{array}$$

November Sales in Rupees

$$B \begin{array}{c} \text{Sonalika} \quad \text{Kalyanasona} \\ \begin{bmatrix} 9{,}000 & 8{,}000 \\ 10{,}000 & 9{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ Abhiram \\ Gakilal \end{array}$$

a) Find the combined sales in October and November for each farmer in each variety.
b) Find the change in sales from October to November.

Solution

a) Combined sales in October and November for each farmer in each variety is given by,

$$A + B \quad \begin{array}{cc} \text{Sonalika} & \text{Kalyanasona} \\ \end{array}$$
$$A + B \begin{bmatrix} 5{,}000 + 9{,}000 & 7{,}000 + 8{,}000 \\ 8{,}000 + 10{,}000 & 8{,}000 + 9{,}000 \end{bmatrix} \begin{array}{c} Abhiram \\ Gakilal \end{array}$$

$$\begin{array}{cc} \text{Sonalika} & \text{Kalyanasona} \end{array}$$
$$A + B \begin{bmatrix} 14{,}000 & 15{,}000 \\ 18{,}000 & 17{,}000 \end{bmatrix} \begin{array}{c} Abhiram \\ Gakilal \end{array}$$

b) Change in sales from October to November.

$$\begin{array}{cc} \text{Sonalika} & \text{Kalyanasona} \end{array}$$
$$A - B \begin{bmatrix} 5{,}000 - 9{,}000 & 7{,}000 - 8{,}000 \\ 8{,}000 - 10{,}000 & 8{,}000 - 9{,}000 \end{bmatrix} \begin{array}{c} Abhiram \\ Gakilal \end{array}$$

$$\begin{array}{cc} \text{Sonalika} & \text{Kalyanasona} \end{array}$$
$$A - B \begin{bmatrix} -4{,}000 & -1{,}000 \\ -2{,}000 & -1{,}000 \end{bmatrix} \begin{array}{c} Abhiram \\ Gakilal \end{array}$$

Production of both Sonalika and Kalyanasona have decreased from October to November for both farmers.

Illustration 3.5.2

Suppose company A produces soap. The cost per soap can be denoted in three aspects such as wage, cost of raw materials and packing cost. The cost per soap (in rupee) is given in matrix as,

$$\text{Matrix A} = \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix} \begin{array}{c} Wage \\ Cost\ of\ raw\ materials \\ PackingCost \end{array}$$

Number of labourers employed, unit of raw materials used and unit of packing covers used in two firms X and Y is given by,

$$\begin{array}{ccc} Labour & \begin{array}{c} Unit\ of \\ raw\ materials \end{array} & Packing\ materials \end{array}$$
$$\text{Matrix B} = \begin{bmatrix} 100 & 300 & 200 \\ 300 & 200 & 200 \end{bmatrix} \begin{array}{c} X \\ Y \end{array}$$

Find the total amount spent by the company in the two firms X and Y.

Solution

We have,

$$\text{Matrix A} = \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix} \begin{array}{c} Wage \\ Cost\ of\ raw\ materials \\ Packing\ Cost \end{array}$$

And,

$$\text{Matrix B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} Labour & \begin{array}{c} Unit\ of \\ raw\ materials \end{array} & Packing\ materials \\ \left[\begin{matrix} 100 \\ 300 \end{matrix}\right. & \begin{matrix} 300 \\ 200 \end{matrix} & \left.\begin{matrix} 200 \\ 200 \end{matrix}\right] \end{array} \begin{array}{c} \\ X \\ Y \end{array}$$

Thus, the total amount spent by the company in the two firms X and Y will be equal to BA.

$$BA = \begin{bmatrix} 100 & 300 & 200 \\ 300 & 200 & 200 \end{bmatrix} \begin{bmatrix} 3 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \times 3 + 300 \times 12 + 200 \times 1 \\ 300 \times 3 + 200 \times 12 + 200 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 300 + 3600 + 200 \\ 900 + 2400 + 200 \end{bmatrix} = \begin{bmatrix} 4100 \\ 3500 \end{bmatrix}$$

Illustration 3.5.3

A store sells commodities X, Y and Z in two of its branches A and B. The quantities of the commodities sold in A and B in a week are given as,

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 400 & 700 & 200 \\ 500 & 800 & 300 \end{bmatrix} \end{array}$$

The individual prices of the commodities are given as, $\begin{array}{c} X \\ Y \\ Z \end{array} \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$

The costs to the store are given as, $\begin{array}{c} X \\ Y \\ Z \end{array} \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$

Find the firm's profit for a week, using

a) total concepts
b) per-unit concept

Solution

It is given that,

Quantities (Q) of the commodities = $\begin{bmatrix} 400 & 700 & 200 \\ 500 & 800 & 300 \end{bmatrix}$

Selling prices (P) = $\begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$

Costs (C) = $\begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$

a) Total Concepts

Thus, the Total Revenue (TR) in A and B is equal to,

$$TR = QP = \begin{bmatrix} 400 & 700 & 200 \\ 500 & 800 & 300 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 400 \times 4 + 700 \times 8 + 200 \times 10 \\ 500 \times 4 + 800 \times 8 + 300 \times 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1600 + 5600 + 2000 \\ 2000 + 6400 + 3000 \end{bmatrix} = \begin{bmatrix} 9200 \\ 11400 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$$

Therefore, $TR = \begin{bmatrix} 9200 \\ 11400 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$

The Total Cost (TC) in A and B is equal to,

$$TC = QC = \begin{bmatrix} 400 & 700 & 200 \\ 500 & 800 & 300 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 400 \times 3 + 700 \times 6 + 200 \times 8 \\ 500 \times 3 + 800 \times 6 + 300 \times 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1200 + 4200 + 1600 \\ 1500 + 4800 + 2400 \end{bmatrix} = \begin{bmatrix} 7000 \\ 8700 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$$

Therefore, $TC = \begin{bmatrix} 7000 \\ 8700 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$

Thus, Profit (Π) = TR – TC

$$\Pi = \begin{bmatrix} 9200 \\ 11400 \end{bmatrix} - \begin{bmatrix} 7000 \\ 8700 \end{bmatrix}$$

$$= \begin{bmatrix} 9200 - 7000 \\ 11400 - 8700 \end{bmatrix}$$

$$= \begin{bmatrix} 2200 \\ 2700 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$$

Therefore, Profit (Π)= $\begin{bmatrix} 2200 \\ 2700 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$

b) Per unit concept

Per unit profit (U) = P – C

$$P = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Per unit profit (U) = P – C

$$= \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 - 3 \\ 8 - 6 \\ 10 - 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Total Profit (Π) = QU

$$= \begin{bmatrix} 400 & 700 & 200 \\ 500 & 800 & 300 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 400 \times 1 + 700 \times 2 + 200 \times 2 \\ 500 \times 1 + 800 \times 2 + 300 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 400 + 1400 + 400 \\ 500 + 1600 + 600 \end{bmatrix}$$

$$= \begin{bmatrix} 2200 \\ 2700 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$$

Therefore, Profit (Π)= $\begin{bmatrix} 2200 \\ 2700 \end{bmatrix} \begin{matrix} A \\ B \end{matrix}$

Illustration 3.5.4

Demand and supply functions of two commodities are given below. Find the equilibrium price of each commodity.

$Qd_1 = 40 – 2p_1 + 2p_2$

$Qd_2 = 72 – 10p_1 + 8p_2$

$Qs_1 = 12 – 4p_1 + 6p_2$

$Qs_2 = 16 – 2p_1 + 6p_2$

Solution

It is given that the demand and supply of two commodities,

$Qd_1 = 40 – 2p_1 + 2p_2$

$Qd_2 = 72 – 10p_1 + 8p_2$

$Qs_1 = 12 – 4p_1 + 6p_2$

$Qs_2 = 16 – 2p_1 + 6p_2$

In an equilibrium, demand is equal to supply. That is d = s.

$Qd_1 = Qs_1$

$40 – 2p_1 + 2p_2 = 12 – 4p_1 + 6p_2$

$40 - 12 = - 4p_1 + 2p_1 + 6p_2 - 2p_2$

$28 = - 2p_1 + 4p_2$

That is, $- 2p_1 + 4p_2 = 28$ …(1)

Now let us find,

$Qd_2 = Qs_2$

$72 – 10p_1 + 8p_2 = 16 – 2p_1 + 6p_2$

$72 - 16 = -2p_1 + 10p_1 + 6p_2 - 8p_2$

$56 = 8p_1 - 2p_2$

That is, $8p_1 - 2p_2 = 56$ …(2)

Let us write the two equations in matrix form,

$$\begin{bmatrix} -2 & 4 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} P1 \\ P2 \end{bmatrix} = \begin{bmatrix} 28 \\ 56 \end{bmatrix}$$

Now it is in AX = B form. Let us solve it using Cramer's rule. Here we have,

$A = \begin{bmatrix} -2 & 4 \\ 8 & -2 \end{bmatrix}$

$X = \begin{bmatrix} P1 \\ P2 \end{bmatrix}$

$B = \begin{bmatrix} 28 \\ 56 \end{bmatrix}$

$|A| = \begin{vmatrix} -2 & 4 \\ 8 & -2 \end{vmatrix} = (-2\times-2) - (8\times4) = 4 - 32 = -28$

$A_1 = \begin{bmatrix} 28 & 4 \\ 56 & -2 \end{bmatrix}$

$|A1| = \begin{vmatrix} 28 & 4 \\ 56 & -2 \end{vmatrix} = (28\times-2) - (56\times4) = -56 - 224 = -280.$

$A_2 = \begin{bmatrix} -2 & 28 \\ 8 & 56 \end{bmatrix}$

$|A2| = \begin{vmatrix} -2 & 28 \\ 8 & 56 \end{vmatrix} = (-2\times56) - (8\times28) = -112 - 224 = -336.$

Therefore,

$p_1 = \frac{|A1|}{|A|} = \frac{-280}{-28} = 10$

$p_2 = \frac{|A2|}{|A|} = \frac{-336}{-28} = 12$

Therefore, the equilibrium prices are $p_1 = 10$ and $p_2 = 12$.

Recap

- Matrix applications are widely used in mathematics, economics and other disciplines
- Matrix algebra provides the theoretical foundation for many quantitative methods, including linear programming, game theory, input-output models, and various statistical models
- In economics, matrices are used in a wide range of areas like analysing the trends of a business, share prices, and constructing business models…etc
- Matrices are used to represent a complex system of equations and massive amounts of data in a compact form in real-world business situations with more than three variables
- The economic method of "input-output analysis," first suggested by Wassily W. Leotief in the 1930s, can be solved simply by using matrices
- Input-output analysis is built on the foundation of input-output tables.
- Matrix Cramer's Rule and determinants are useful tools for resolving a variety of problems in business and economics involving profit maximization and loss minimization
- Variance and covariance can be calculated using matrices
- Matrices also have the advantage of being able to solve systems of equations fast using a computer once they have been set up in matrix form

Objective Questions

1. Which branch of mathematics provides the theoretical foundation for quantitative methods such as linear programming, game theory, and statistical models through the use of matrix algebra?
2. What are the common uses of matrices in economics?
3. Who first suggested the economic method of "input-output analysis" in the 1930s?
4. What is the fundamental concept underlying input-output analysis in economics?
5. How are input-output tables for industries structured?

Answers

1. Matrix algebra
2. Analysing business trends, share prices, and constructing business models.
3. Wassily W. Leontief
4. Economic interdependence of industries
5. Rows and columns of data quantifying the supply chain for various industries

Assignments

1. Discuss the application of matrix in input-output analysis.

2. Keerthi and Jesna are two friends. Keerthi wants to buy 3 Churidars and 1 Saree, while Jesna needs 2 Churidars and 3 Sarees. They both go to a shop to enquire about the rates which are quoted as follows:

   Churidars - Rs. 1,100 each and Sarees - Rs. 1,000 each.

   How much money does each need to spend? Represent it in matrix form.

3. A company is considering which of the three methods of production it has to use in producing three goods X, Y and Z. The amount of each good produced by each method is shown in the matrix A.

$$\begin{array}{l} \\ Method\ 1 \\ Method\ 2 \\ Method\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 5 & 3 & 9 \end{bmatrix} \end{array}$$

The matrix B = $\begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix}$ shows the profit per unit for the goods X, Y and Z respectively. Find which method is the best profit maximization method.

Suggested Readings

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4

BLOCK

Differential Calculus

UNIT 1

Limits and Differentiation

Learning Outcomes

After reading this unit, the leaner will be able to

- understand the concept of limit
- familiarised with the rules of differentiation
- understand higher order derivatives

Prerequisites

We are familiar with economic concepts like Total Utility, Total Revenue, Total Cost, etc. Consider Total Utility. It is the sum of utility or satisfaction you derive from the consumption of a commodity. For example, the total satisfaction from consuming apples is the sum of utility from each apple. The Total Utility changes with the change in quantity of the commodity consumed. Understanding how Total Utility changes with the change in the consumption of a commodity is important for deciding the amount of commodity to be purchased for maximising utility and satisfaction. This rate of change of utility with changes in the amount of commodity consumed is Marginal Utility. The mathematical tool, differentiation allows us to calculate Marginal Utility from Total Utility. So, differentiation represents the rate of change of a quantity with respect to the change in another quantity. Similarly, the Marginal Revenue and Marginal Cost can be calculated from the respective Total Revenue and Total Cost respectively via differentiation.

Let us consider Total Cost. Suppose cost of production is influenced due to cost of labour alone. Here, the changes in wage changes the total cost of production. When number of output increases due increase in labour, the total cost also increases. The marginal cost represents the rate of change in Total Cost due to change in output. Understanding the relationship between economic variables are possible with the tool of differentiation. The rate of influence of one variable on another helps to us to make decision accurately. Let us see the operation of differentiation in detail.

Keywords

Limits, Differentiation, Substitution, Factorisation, Rationalisation, Higher order derivatives

Discussion

4.1.1 Concept of Limit

The concept of a limit is fundamental in calculus and plays a crucial role in understanding the behaviour of functions, sequences, and series. The calculus is referred as the mathematics of change. A limit gives the value that a function approaches as that function's inputs get closer and closer to some number. The idea of a limit is the basis of all calculus and mathematical analysis. It is used for defining derivatives, integrals, continuity of a function, etc. This can be explained as, if the value of f(x) can be made as close as to l by taking values of x sufficiently close to $a (\neq a)$, then we write $\lim_{x \to a} f(x) = l$, read it as f(x) approaches l as x approaches a.

For example, Given $f(x) = x^2$

Limit of x^2 as x approaches 2 is 4. This is expressed as $\lim_{x \to 2} f(x) = 4$

This is illustrated below

Table 4.1.1 Limit

X	1.4	1.8	1.9	1.94	1.97	2
X^2	1.96	3.24	3.61	3.7636	3.8809	4

The difference between the value of x and 2 is becoming smaller and smaller, the difference between x^2 and 4 is also becoming smaller. Therefore $x^2 \to 4$ as $x \to 2$. This is expressed as $\lim_{x \to 2} f(x) = 4$

4.1.1.1 Different methods to find the limit

The substitution method is mainly used for calculating the limit of the function. If f(a) is determinate, direct substitution method is used. If the f(a) is indeterminate like, $\frac{0}{0}$, $\frac{\infty}{\infty}$, ∞ etc, the method of substitution is done by applying the simplification methods like (a) cancel the common factor from numerator and denominator (b) rationalise the numerator or denominator by a rationalising factor (c) apply the limit form, etc. These methods are explained below-

Method 1- Direct substitution

$f(x) = f(a)$ Substitute $x = a$ in $f(x)$ and will obtain $f(a)$. This is used when f(a) is not indeterminate

For example $\lim_{x \to 2}(5x^2 + 4x - 2)$, substituting 2 for x we will get,

$= 5 \times 2^2 + 4 \times 2 - 2 = 26$

Method 2- Substitution in indeterminate form

When substituting x with a in f(x), if f(a) is in an indeterminate form such as $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc, various methods can be employed to address this. These methods include: (a) cancelling common factors from the numerator and denominator, (b) rationalising the numerator or denominator using a suitable rationalising factor, and (c) applying limit forms.

a. Factorising and Cancelling Common Factor

Following these operations, the given function is factorised, and any common factors in the numerator and denominator are subsequently cancelled.

For example, $Lt_{x \to 2} \frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}$, we cannot apply the direct substitution method.

Therefore, factorise the function as $\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x + 2$

Here, (x-2) in the numerator and denominator can cancel out.

Now use the substitution method as follows

$Lt_{x \to 2} \frac{x^2-4}{x-2} = Lt_{x \to 2}\ x + 2 = 2+2 = 4$

b. Using Rationalising factor

Another method when the limit function has indeterminate form is use a rationalising factor for numerator or denominator and apply the substitution method. This is explained in the following examples.

$Lt_{x \to 1} \frac{x^2+1}{x^2-1} = \frac{1+1}{1-1} = \frac{2}{0}, = \infty$

Here, since the function is indeterminate, rationalisation can be done by dividing both the numerator and denominator by x^2 .

$$Lt_{x\to 2}\ \frac{x^2+1}{x^2-1}\ \div \frac{x^2}{x^2} = Lt_{x\to 2}\ \frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}}\ = \frac{1}{1}\frac{+\frac{1}{4}}{-\frac{1}{4}} = 1$$

c. Apply the limit form

Another method to find the limit of a function is apply the limit form given below

$$Lt_{x\to a}\ \frac{x^n - a^n}{x-a}\ = n\ a^{n-1}$$

4.1.1.2 Continuity of a function

A function $f(x)$ is continuous at $x = a$ provided the following conditions are satisfied

1) $f(a)$ is defined
2) $\lim\limits_{x\to a} f(x)$ exists
3) $\lim\limits_{x\to a} f(x) = f(a)$

Illustration 4.1.1

Find $Lt_{x\to 2}\ \frac{\{x^2-4\}(x+5)}{x-2}$

Solution

$$Lt_{x\to 2}\ \frac{\{x^2-4\}(x+5)}{x-2} = Lt_{x\to 2}\ \frac{(x+2)(x-2)(x+5)}{x-2}\ = Lt_{x\to 2}\ (x+2)\ (x+5)$$

$$= (2+2)(2+5) = 28$$

Here, factorisation is used to find the limit. $(x^2 - 4)$ can be written as $x^2 - 2^2$ so that it is written as $(x+2)\ (x-2)$.

Illustration 4.1.2

Find $Lt_{x\to\infty}\ 1+\frac{1}{n}$

Solution

$$Lt_{x\to\infty}\ 1+\frac{1}{n}\ = 1+\frac{1}{\infty} =\ 1 + 0\ = 1$$

Illustration 4.1.3

Find $Lt_{x\to\infty}\ \frac{n}{n+1}$

Solution

$$Lt_{x\to\infty} \frac{n}{n+1} = \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1$$

Here, rationalisation is done by dividing 'n' in numerator and denominator.

Illustration 4.1.4

Find $Lt_{x\to 2} \frac{x^2-3x+2}{x^2-5x-6}$

Solution

$$Lt_{x\to 2} \frac{x^2-3x+2}{x^2-5x-6} = Lt_{x\to 2} \frac{(x-2)(x-1)}{(x-3)(x-2)} = Lt_{x\to 2} \frac{(x-1)}{(x-3)} = \frac{(2-1)}{(2-3)} = \frac{1}{-1} = -1$$

Here, the factorisation of the quadratic equation is done in numerator and denominator.

Illustration 4.1.5

$Lt_{x\to 36} \frac{\sqrt{x}-6}{x-36}$

Solution

In the above example, the limit of denominator equals zero and it is not possible to apply the method of cancelling of common term. So, rationalisation done for calculating limit as given below. Multiply the numerator and denominator by the rationalising factor √x+6 i.e.,

$$Lt_{x\to 36} \frac{\sqrt{x}-6}{x-36} = Lt_{x\to 36} \frac{\sqrt{x}-6}{x-36} \cdot \frac{\sqrt{x}+6}{\sqrt{x}+6}$$

$$= Lt_{x\to 36} \frac{(\sqrt{x}-6)\,(\sqrt{x}+6)}{(x-36)(\sqrt{x}+6)} = Lt_{x\to 36} \frac{x-36}{(x-36)(\sqrt{x}+6)}$$

Substituting 36 for x, the function becomes

$= Lt_{x\to 36} \frac{1}{(\sqrt{x}+6)}$ Here, (x-36) in numerator and denominator becomes 1.

$$= Lt_{x\to 36} \frac{1}{(\sqrt{36}+6)}$$

$$= \frac{1}{(6+6)} = \frac{1}{12}$$

Illustration 4.1.6

$$Lt_{x\to 0} \frac{\sqrt{x+25}-5}{x}$$

Solution

Rationalising factor $\sqrt{x+25}+5$ is multiplied on both the numerator and denominator .

$$Lt_{x\to 0} \frac{\sqrt{x+25}-5}{x} \cdot \frac{\sqrt{x+25}+5}{\sqrt{x+25}+5} = Lt_{x\to 0} \frac{(x+25)-25}{x(\sqrt{x+25}+5)} = Lt_{x\to 0} \frac{x}{x(\sqrt{x+25}+5)}$$

$$= Lt_{x\to 0} \frac{1}{(\sqrt{x+25}+5)} = Lt_{x\to 0} \frac{1}{(\sqrt{0+25}+5)}$$

$$= \frac{1}{(5+5)} = \frac{1}{10}$$

Illustration 4.1.7

Find $Lt_{x\to 4} \frac{x^2-4^2}{x-4}$

$Lt_{x\to 4} \frac{x^2-4^2}{x-4} = Lt_{x\to 2} \frac{x^2-4^2}{x-4}$ = apply the limit form n a $^{n-1}$ which will give

$2\times 4^{2-1} = 2\times 4 = 8$

Illustration 4.1.8

Find $Lt_{x\to\infty} \frac{2x^2+x}{4x^2-3}$

$$Lt_{x\to\infty} \frac{2x^2+x}{4x^2-3} = Lt_{x\to\infty} \frac{x^2\left(2+\frac{1}{x}\right)}{x^2\left(4-\frac{3}{x^2}\right)}$$

Here, x^2 is factored out from numerator and denominator. When x^2 is factored out from the second term in the numerator, i.e., x, it will be become $\frac{1}{x}$. When x^2 is factored out from the second term in denominator, i.e., 3, it will be become $\frac{3}{x^2}$.

Substituting the value of x gives,

$$Lt_{x\to\infty}\ \frac{(2+\frac{1}{x})}{(4-\frac{3}{x^2})} = \frac{(2+\frac{1}{\infty})}{(4-\frac{3}{\infty})} = \frac{2}{4} = \frac{1}{2}$$

Illustration 4.1.9

Determine whether the function $f(x) = 12x + 6$ is continuous at $x = 5$.

Solution

$\lim_{x\to 5} f(x) = 12 \times 5 = 60$ is defined

$f(5) = 60$ exists

$\lim_{x\to 5} f(x) = 60 = f(5)$

$\therefore$ the function is continuous.

Illustration 4.1.10

Determine whether the function $f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = 2$.

Solution

$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4$ is defined

$f(2) = 4$ exists

$\lim_{x\to 2} f(x) = 4 = f(2)$

$\therefore$ the function is continuous.

Illustration 4.1.11

Find the value of x for which the function $f(x) = \frac{x^2-9}{x^2-5x+6}$ is discontinuous.

Solution

$$f(x) = \frac{x^2-9}{x^2-5x+6} = \frac{(x-3)(x+3)}{(x-3)(x-2)} = \frac{(x+3)}{(x-2)} = \frac{5}{0} = \infty$$

When $x = 2$ the function does not exist or it is indeterminate. So the function is not continuous at $x = 2$.

4.1.2 Differentiation

Differentiation is the process of finding the derivative of a function. Let y is a function of x where x is an independent variable. This is denoted as y = f(x). Then the derivative of this function can be explained as follows:

Let Δ y and Δ x be small increment or change in y and x respectively, then $\frac{\Delta y}{\Delta x}$ is the incremental ratio. The value of incremental ratio when Δ x is very small is called the differential coefficient or derivative of y with respect to x, and is denoted by $\frac{dy}{dx}$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

i.e. differentiation is the process of finding the derivative of a function. $\frac{dy}{dx}$ is also denoted as f' This is the first order derivative and we can derive 2nd, 3th,… order derivative. The second order derivative is denoted as $\frac{d^2y}{dx^2}$, the third order derivative is $\frac{d^3y}{dx^3}$ and so on.

The derivative of a function y = f(x) with respect to x can also be derived as $\frac{dy}{dx}$ = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists and h is a small increment in x

Derivatives

The derivative of a function can be derived as given in the following examples. The derivative is derived using the first principle of differentiation, i.e. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

The derivatives are denoted as $y', f'x, \frac{dy}{dx}$, $\frac{d}{dx}$ f(x), etc

4.1.2.1 Rules of Differentiation

There are certain rules for doing differentiation for particular types of functions.

Rule 1 - Constant Rule

Differential of a constant equals zero

Given y = K, Where K is constant

$$\frac{dy}{dx} = \frac{dK}{dx} = 0$$

For example, Given y = 4, $\frac{dy}{dx} = \frac{d4}{dx} = 0$

Rule 2 - Power Rule

$$\frac{dx^n}{dx} = nx^{n-1}$$

For example, y= $7x^2 = \frac{dy}{dx} = \frac{d(7x^2)}{dx} = 7\frac{d}{dx}(x^2) = 7\,(2x^{2-1}) = 14\,(x^1) = 14(x) = 14x$

Rule 3 - Sum / Difference Rule

Let $y = U + V$

Sum rule of differentiation is $\frac{dy}{dx} = \frac{dU}{dx} + \frac{dV}{dx}$

Differential coefficient of the sum of two function = Differential coefficient of the first term + differential coefficient of second term.

Let $y = U - V$

Difference rule of differentiation is $\frac{dy}{dx} = \frac{dU}{dx} - \frac{dV}{dx}$

Differential coefficient of the difference of two function = Differential coefficient of the first term - Differential coefficient of second term

For example, $y = x + 4$

$$\frac{dy}{dx} = \frac{d(x+4)}{dx} = \frac{d\,(x)}{dx} + \frac{d(4)}{dx} = 1 + 0 = 1$$

Let $y = 2x - x^2$

$$\frac{dy}{dx} = \frac{d(2x-x^2)}{dx} = \frac{d\,(2x)}{dx} - \frac{d(x^2)}{dx} = 2 - 2x$$

Rule 5- Product Rule

If u and v are two functions of x, then the differential coefficient of product UV is

$$\frac{d(uv)}{dx} = u\frac{d(v)}{dx} + v\,\frac{d(u)}{dx}$$

Differential coefficient of the product of two functions = First function (u) × Differential coefficient of the second function (v) + Second term (v) × Differential coefficient of the first term (u)

For example, $y = x^2e^x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2e^x)$$

$$= e^x\frac{d}{dx}(x^2) + x^2.\frac{d}{dx}(e^x)$$

$$= e^x.\,2x + x^2.\,e^x$$

$$= 2x.\,e^x + x^2.\,e^x$$

Rule 6 - Quotient Rule

If u and v are two functions of x, then the differential coefficient of Quotient $\frac{u}{v}$ is

$Y = \frac{u}{v}$,

$$\frac{d(\frac{u}{v})}{dx} = \frac{v\,d(u) - u\,d(v)}{v^2}$$

Differential coefficient of a quotient = (Denominator × Differential coefficient of the numerator - Numerator × Differential coefficients of denominator) ÷ Square of denominator.

For example, given $y = \frac{x+1}{x-2}$

Let U = x + 1 and V= x - 2, then

$$d\left(\frac{x+1}{x-2}\right) = \frac{(x-2)\,d(x+1) - (x+1)\,d(x-2)}{(x-2)^2}$$

$$= \frac{(x-2)\,1-(x+1)\,1}{(x-2)^2}$$

$$= \frac{-3}{(x-2)^2}$$

Rule 7-Chain Rule

Rule: If y is a function u, $y = f(u)$, and u is a function of x, $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For example, If y = u^2, u = (x^2+2), $\frac{dy}{du} = 2u, \frac{du}{dx} = 2x, \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} =$ $4ux = 4\,(x^2 + 2)x$

Basic Rules

1. $\frac{d}{dx}\,(k) = 0$
2. $\frac{d}{dx}\,(x^n) = nx^{n-1}$
3. $\frac{d}{dx}\,(e^x) = e^x$
4. $\frac{d}{dx}\,(logx) = \frac{1}{x}$
5. $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
6. $\frac{d}{dx}\,(sinx) = cosx$
7. $\frac{d}{dx}\,(cosx) = sinx$
8. $\frac{d}{dx}\,(tanx) = \sec^2 x$

Illustration 4.1.12

1. y = x, $\frac{dx}{dx}$ =1, ∴ $x = x^{|}$
2. y = $\frac{1}{2\,x} = \frac{dy}{dx} = \frac{1}{2}\left[\frac{d(\frac{1}{x})}{dx}\right] = \frac{1}{2} \times \frac{-1}{x^2} = -\frac{1}{2x^2}$

3. $y = x^5, \frac{d(x^5)}{dx} = 5x^{5-1} = 5x^4$

4. $y = x^{-4}, \frac{d(x^{-4})}{dx} = -4x^{-4-1} = -4x^{-5}$

5. $y = \sqrt{x}, \frac{d(x)^{\frac{1}{2}}}{dx} = \frac{1}{2\sqrt{x}}$

6. $y = \frac{1}{\sqrt{x}}, y = x^{\frac{-1}{2}} \quad \frac{d\, x^{\frac{-1}{2}}}{dx} = \frac{-1\, x^{-\frac{2}{3}}}{2}$

7. $y = \frac{1}{x}, y = x^{-1}, \frac{d(x^{-1})}{dx} = \frac{-1}{x^2}$

Illustration 4.1.13

$y = 6x^2 + 3x$. Find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{d\,(6x^2 + 3x)}{dx} = \frac{d}{dx}(6x^2) + \frac{d(3x)}{dx} = 6\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x)$$

$= 6\,(2x^{2-1}) + 3(1\,x^{1-1})$

$= 6\,(2x) + 3(x^0) = 12x + 3$

$\therefore x^0 = 1$

Illustration 4.1.14

$y = 12x^3 + 8x^2 + 2x + 5$. Find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(12x^3 + 8x^2 + 2x + 5) = \frac{d}{dx}(12x^3) + \frac{d}{dx}(8x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(5)$$

$= 12\frac{d}{dx}(x^3) + 8\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + \frac{d}{dx}(5)$

$= 12(3x^{3-1}) + 8(2x^{2-1}) + 2(1x^{1-1}) + 0$

$= 36x^2 + 16x^1 + 2x^0 + 0$

$= 36x^2 + 16x + 2$

Illustration 4.1.15

y = $4x^3 + \frac{2}{x}$. Find $\frac{dy}{dx}$

Solution

y = $4x^3 + \frac{2}{x} = 4x^3 + 2x^{-1}$

$\frac{d}{dx}(4x^3) + \frac{d}{dx}(2x^{-1}) = 4\frac{d}{dx}(x^3) + 2\frac{d}{dx}(x^{-1}) = 4(3x^2) + 2\times -1x^{-1-1}$

$= 12(x^2) + 2(-x^{-2}) = 12x^2 - 2x^{-2}$

Illustration 4.1.16

y = b log x. Find $\frac{dy}{dx}$

Solution

y = b log $x = \frac{dy}{dx} = \frac{d(b\log x)}{dx} = b\frac{d}{dx}(\log x) = b(\frac{1}{x}) = b\,\frac{1}{x}$

Illustration 4.1.17

y = $4x^2 - 8x$. Find $\frac{dy}{dx}$.

Solution

$\frac{dy}{dx} = \frac{d(4x^2 - 8x)}{dx} = \frac{d}{dx}(4x^2) - \frac{d(8x)}{dx} = 4\frac{d}{dx}(x^2) - 8\frac{d}{dx}(x)$

$= 4(2x^{2-1}) - 8(1x^{1-1})$

$= 8(x) - 8(x^0) = 8x - 8$

Illustration 4.1.18

$y = x^2 \log x$. Find $\frac{dy}{dx}$

Solution

$\frac{d(uv)}{dx} = u\frac{d(v)}{dx} + v\,\frac{d(u)}{dx}$

$$\frac{d(y)}{dx} = x^2\frac{d(\log x)}{dx} + \log x\ \frac{d(x^2)}{dx}$$

$= x^2\ (\frac{1}{x})$+log$x$ (2x)

= x+2$x$log$x$

Illustration 4.1.19

$y = x^2 e^x$. Find $\frac{dy}{dx}$

Solution

$$\frac{d(uv)}{dx} = \text{u}\frac{d(v)}{dx} + \text{v}\ \frac{d(u)}{dx}$$

$$\frac{d(y)}{dx} = x^2\frac{d(e^x)}{dx} + \text{e}^x\ \frac{d(x^2)}{dx}$$

$$= x^2e^x + e^x \times 2x \ = x^2e^x + 2x\,e^x$$

Illustration 4.1.20

y = $\frac{8}{(3x^2+1)}$. Find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{v\,d(u)-u d(v)}{v^2}$$

$$= \frac{(3x^2+1)\ d(8)-8\ d(3x^2+1)}{(3x^2+1)^2}$$

$$= \frac{(3x^2+1)\times 0-8\times 6x}{(3x^2+1)^2}$$

$$= \frac{-48x}{(3x^2+1)^2}$$

Illustration 4.1.21

y = $\frac{x^2}{e^x}$. Find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{v\,d(u) - u d(v)}{v^2}$$

$$\frac{dy}{dx} = \frac{e^x\,d(x^2) - x^2 d(e^x)}{(\,e^x)^2} = \frac{e^x\,2x - x^2 e^x}{e^{2x}}$$

Illustration 4.1.22

y = $\frac{3\sqrt{x}}{x+2}$. Find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{v\,d(u) - u d(v)}{v^2}$$

$$\frac{dy}{dx} = \frac{x+2\;d(3\sqrt{x}\,) - 3\sqrt{x}\;d(x+2)}{(\,x+2)^2}$$

Here, $v\;d(u) = (x+2)d(3\sqrt{x}) = (x+2)3\;dx\;\left(\frac{1}{2}\right) = (x+2).3.\frac{1}{2}x\left(-\frac{1}{2}\right)$

$$= (x+2).3.\frac{1}{2\sqrt{x}}$$

$$udv = 3\sqrt{x}\;\frac{d(x+2)}{dx} = 3\sqrt{x}\;\times 1 = 3\sqrt{x}$$

$$\frac{dy}{dx} = \frac{x+2\;(3/2\sqrt{x}\,) - 3\sqrt{x}\;(1)}{(\,x+2)^2}$$

$$= \frac{x+2\;(3/2\sqrt{x}\,) - 3\sqrt{x}}{(\,x+2)^2}$$

4.1.3 Higher order differentiation

It is the process of differentiating the successive derivative of a function. Let y = f(x), differentiating y with respect to x we get the differential coefficient of y. This is denoted as $\frac{dy}{dx}$ of $f^1(x)$, This is known as first order derivative of y with respect to x. We can differentiate this derivative again and it is denoted as $\frac{d^2y}{dx^2}$ or $f^{11}(x)$, . This is the second order derivative. We can differentiate it again and is denoted as $\frac{d^3y}{dx^3}$ or $f^{111}(x)$, and so on. Thus we can differentiate a differential coefficient again to find its derivatives and this is known as higher order differentiation or successive differentiation.

Illustration 4.1.23

y = 24 x^4. Find the higher order differential coefficients.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(24\,x^4) = 96\,x^3$$

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(96\,x^3) = 288\,x^2$$

$$\frac{d^3y}{dx^3} = \frac{d^3}{dx^3}(288\,x^2) = 576x$$

$$\frac{d^4y}{dx^4} = \frac{d^4}{dx^4}(576x) = 576$$

$$\frac{d^5y}{dx^5} = \frac{d^5}{dx^5}(576) = 0$$

Illustration 4.1.24

Let y= $5x^4 + 3x^3 - 2x^2 + x + 10$, find the higher order differential coefficients.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(5x^4 + 3x^3 - 2x^2 + x + 10) = 20x^3 + 9x^2 - 4x + 1$$

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(20x^3 + 9x^2 - 4x + 1) = 60x^2 + 18x - 4$$

$$\frac{d^3y}{dx^3} = \frac{d^3}{dx^3}(60x^2 + 18x - 4) = 120x + 18$$

$$\frac{d^4y}{dx^4} = \frac{d^4}{dx^4}(120x + 18) = 120$$

$$\frac{d^5y}{dx^5} = \frac{d^5}{dx^5}(120) = 0$$

Illustration 4.1.25

y= $x^2\log x$, find $\frac{d^2y}{dx^2}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(x^2\log x) = x^2\left(\frac{1}{x}\right) + \log x\,(2x) = \frac{x^2}{x} + 2x\log x = x + 2x\log x$$

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(x + 2x\log x) = 1 + (2x\left(\left(\frac{1}{x}\right) + \log x(2)\right) = 1 + (2 + 2\log x) = 3 + 2\log x$$

Illustration 4.1.26

y= e^{3x} find $\frac{d^2y}{dx^2}$.

Solution

$\frac{dy}{dx} = \frac{d}{dx}(e^{3x}) = e^{3x}(3) = 3e^{3x}$

$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(3e^{3x}) = 3(e^{3x}(3) = 9\ e^{3x}$

Recap

- Limit - gives the value that function approaches as that function's inputs get closer and closer to some number
- Used for defining derivatives, integrals, continuity of a function
- Different methods to find the limit – Direct substitution, substitution in indeterminate form

- $\frac{d}{dx}(k) = 0$
- $\frac{d}{dx}(x^2) = 2x$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(logx) = \frac{1}{x}$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$
- $\frac{d}{dx}(sinx) = cosx$
- $\frac{d}{dx}(cosx) = sinx$
- $\frac{d}{dx}(tanx) = \sec^2 x$
- Rules of differentiation – constant rule, addition/ subtraction rule, power rule, quotient rule, multiplication rule, chain rule
- Higher order derivatives - $\frac{d^2y}{dx^2}$ or $f''_{(x)}$, $\frac{d^3y}{dx^3}$ or $f'''_{(x)}$, and so on

Objective Questions

1. What are the uses of limits?
2. Name two methods to find the limits.
3. What are the methods of finding limit under indeterminate function?
4. What is the derivate of a constant?
5. What is differentiation?
6. Express the general mathematical form of derivate.
7. What is power rule?
8. What is quotient rule?
9. What is chain rule?
10. Give two forms of higher order derivates.

Answers

1. Defining derivatives, integrals, continuity of a function
2. Direct substitution, substitution in indeterminate form
3. Factorisation and cancelling common factors, rationalisation, applying limit
4. Zero
5. Differentiation is the process of finding the derivative of a function
6. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$
7. $\frac{dx^n}{dx} = nx^{n-1}$
8. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\,d(u) - u\,d(v)}{v^2}$
9. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
10. $\frac{d^2y}{dx^2}$ or $f''_{(x)}$, $\frac{d^3y}{dx^3}$ or $f'''_{(x)}$

Assignments

1. Differentiate the following

   a. $\frac{7}{x^7}$ b. 3 log x, c. $8e^x$

2. Differentiate with respect to x

   a. x^2 - 4x +3 b. 6x^2 - 9x +3 c. (3x-1)(x-1)

3. Differentiate log functions

   a. $x^2 \log x$ b. $\frac{(x+2)}{\log x}$ c. $\frac{2x}{\log x}$

4. If y = $\frac{(x+6)}{2}$. Find $\frac{dy}{dx}$

5. Explain the different rules of differentiation.

Suggested Readings

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2

UNIT

Partial Differentiation

Learning Outcomes

After reading this unit, the learner will be able to

- familiarise with partial derivatives
- understand the rules of partial differentiation
- learn total differentiation

Prerequisites

We have discussed differentiation in Unit 1. It measures the change in dependent variable due to changes in independent variable. In economics, we have many variables affecting a single dependent variable. Take the case of Cost of production. Cost includes wage, rent, profit, interest. Change in these variables changes the cost of production. So, when there is more than one independent variable affecting the dependent variable, effect of change of each independent variable needs to be calculated to take the total change on the dependent variable. Partial derivative measures the effect of change in each of the independent variable. Let us explain this in detail.

Keywords

Partial Differentiation, Partial Derivative, Independent variable, Total Differentiation

Discussion

4.2.1 Partial differentiation

In this section differentiation of a function which has more than one independent variables are explained. A variable may depend upon more than one independent variables. For example, consumption is a function of income, employment, education, area, taste etc. In this function, consumption function can be written as C = f(a,b,c,d,e). a,b,c,d,e are the five independent variables specified above. Similarly, we can write the functions as Z= f(x,y), Z= f(x,y,z) and so on. Partial differentiation is used for the differentiation of the functions which has more than one independent variables. The differential coefficient of the dependent variable with respect to one of the independent variable keeping other independent variables as constant is called partial differential coefficient.

4.2.1.1 Partial derivative

Let Z = f (x,y), In this function Z is a function of x and y. The Z may change if x only changes or y only changes or both x and y changes simultaneously. In this function, we can derive two partial derivatives. The derivative of Z with respect to x and derivative of Z with respect to y.

The derivative of Z with respect to x is derived by keeping y as fixed and it is denoted as $\frac{\partial z}{\partial x}$. Similarly, the derivative of Z with respect to y is written as $\frac{\partial z}{\partial y}$, keeping x as constant. Partial differentiation is denoted by ∂. Therefore given Z= f(x.y), the two partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Similarly let Z = f(a,b,c), we can derive three partial derivative. The partial derivatives are $\frac{\partial z}{\partial a}$ keeping b and c as fixed, $\frac{\partial z}{\partial b}$ keeping a nd c as fixed and, $\frac{\partial z}{\partial c}$ keeping a and b as fixed. The partial derivatives with respect to x and with respect y can be written as:

$$\frac{\partial z}{\partial x} = lt_{\Delta x \to 0} \frac{(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly

$$\frac{\partial z}{\partial y} = lt_{\Delta y \to 0} \frac{(x, y+\Delta y) - f(x,y)}{\Delta y}$$

In the partial differentiation the impact of one variable keeping other variable as fixed is measured. For partial differentiation also the same differentiation rules are used.

4.2.1.2 Rules of partial differentiation

1. Product rule - Given Z= g(x,y) h(x,y)

$$\frac{\partial z}{\partial x} = \mathrm{g}(x,y)\frac{\partial h}{\partial x} + \mathrm{h}(x,y)\frac{\partial g}{\partial x} \text{ and}$$
$$\frac{\partial z}{\partial y} = \mathrm{g}(x,y)\frac{\partial h}{\partial y} + \mathrm{h}(x,y)\frac{\partial g}{\partial y}$$

2. Quotient Rule - Given Z = $\frac{g(x,y)}{h(x,y)}$ and h(x,y) ≠ 0

$$\frac{\partial z}{\partial x} = \frac{h(x,y)\left(\frac{\partial g}{\partial x}\right) - g(x,y)\left(\frac{\partial h}{\partial x}\right)}{h(x.y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{h(x,y)\left(\frac{\partial h}{\partial y}\right) - g(x,y)\left(\frac{\partial g}{\partial y}\right)}{h(x.y)^2}$$

3. Generalised power function rule - Given Z= $[g(x,y)]^n$

$$\frac{\partial z}{\partial x} = n[g(x,y)]^{n-1}\frac{\partial g}{\partial x}$$

$$\frac{\partial z}{\partial y} = n[g(x,y)]^{n-1}\frac{\partial g}{\partial y}$$

4. Exponential functions – Given z = $e^{g(x,y)}$

$$\frac{\partial z}{\partial x} = \mathrm{e}g(x,y).\frac{\partial g}{\partial x}$$

$$\frac{\partial z}{\partial y} = \mathrm{e}g(x,y).\frac{\partial g}{\partial y}$$

Illustration 4.2.1

Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, Z= $(4xy)$

Solution

$\frac{\partial z}{\partial x}$ = 4y (the derivative of x only is derived keeping y as fixed, derivative of x = 1 and therefore 4y),

Similarly keeping x as fixed the partial derivative of y can be denoted as $\frac{\partial z}{\partial y}$ = 4x (since derivative of y =1)

Illustration 4.2.2

Find the partial derivatives of Z = $3x^2 + 4xy^2 - wy$

Solution

In this function, we have three variables x,y, and w, therefore three partial derivatives can be derived. They are $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\delta z}{\delta w}$

$$\frac{\partial z}{\partial x} = \frac{\partial\,(3x^2)}{\partial x} + \frac{\partial(4xy^2)}{\partial x} - \frac{\partial\,(wy)}{\partial x} = 6x + 4y^2 - 0 = 6x + 4y^2 \text{ (keeping y and w as fixed)}$$

$\frac{\partial z}{\partial y} = \frac{\partial\ (3x^2)}{\partial y} + \frac{\partial\ (4xy^2)}{\partial y} - \frac{\partial (wy)}{\partial y} = 0 + 8xy - 3wy^2 = 8xy - w$ (keeping x and w as fixed)

$\frac{\partial z}{\partial w} = \frac{\partial\ (3x^2)}{\partial w} + \frac{\partial\ (4xy^2)}{\partial w} - \frac{\partial\ (wy)}{\partial w} = -y$

(keeping x and y as fixed)

Illustration 4.2.3

Find the partial derivatives of $Z = \frac{x^2y}{x+y}$

Solution

Apply the quotient rule,

$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\delta}{\delta x}(x^2y) - x^2y\frac{\delta(x+y)}{\delta x}}{(x+y)^2} = \frac{(x+y)2xy - x^2y\ (1+0)}{(x+y)^2}$$

$$= \frac{(x+y)2xy - x^2y}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)\frac{\delta}{\delta y}(x^2y) - x^2y\frac{\delta(x+y)}{\delta y}}{(x+y)^2} = \frac{(x+y)x^2 - x^2y\ (0+1)}{(x+y)^2}$$

$$= \frac{(x+y)x^2 - x^2y}{(x+y)^2}$$

Illustration 4.2.4

Find the partial derivatives of $z = (x^4 + 5y^2)^3$

Solution

$\frac{\partial z}{\partial x} = 3(x^4 + 5y^2)^2.4x^3 = 12x^3(x^4 + 5y^2)^2$

$\frac{\partial z}{\partial y} = 3(x^4 + 5y^2)^2\ (10y) = 30y(x^4 + 5y^2)^2$

Illustration 4.2.5

Find the partial derivatives of $y = e^{3xy^2}$

Solution

$\frac{\partial z}{\partial x} = e^{3xy^2}.\ 3y^2 = 3y^2e^{3xy^2}$

$$\frac{\partial z}{\partial y} = e^{3xy^2}.\ 6xy = 6xye^{3xy^2}$$

4.2.2 Higher Order Partial Derivatives

Higher order partial derivatives are obtained in the same manner as the higher derivatives. In the higher order derivative the successive derivatives are derived for the given function. For example the second order derivative of z with respect to x is obtained by differentiating the first order derivative of z with respect to x. The first order partial derivative of z with respect to x is denoted as $\frac{\partial z}{\partial x}$. The second order partial derivative of z with respect to x is denoted as $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

The second order partial derivative of z with respect to y is denoted as,

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Similarly, the higher order derivatives $\frac{\partial^3 z}{\partial x^3}, \frac{\partial^4 z}{\partial x^4}$ can be derived through successive differentiations.

The higher order partial derivatives can also be derived by differentiating a function, z, with respect to x keeping y as constant and then differentiating the result with respect to y keeping x as constant. It is denoted as $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$. If we differentiate with respect to y keeping x as fixed and then differentiate the result with respect to x keeping y as constant, it is denoted as $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$. Generally, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Illustration 4.2.6

$Z = 6x^4 + 8x^3y - 3x^2 + 4y^3 + 10$ find the first and second order derivatives.

Solution

$$\frac{\partial z}{\partial x} = 24x^3 + 24x^2y - 6x + 0 + 0 = 24x^3 + 24x^2y - 6x$$

$$\frac{\partial z}{\partial y} = 0 + 8x^3 - 0 + 12y^2 + 0 = 8x^3 + 12y^2$$

$$\frac{\partial^2 z}{\partial x^2} = 72x^2 + 48xy - 6$$

$$\frac{\partial^2 z}{\partial y^2} = 0 + 24y = 24y$$

$$\frac{\partial^2 z}{\partial y \partial x} = 24x^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 24x^2$$

Illustration 4.2.7

Derive the higher order partial derivatives of Z= $x^2 + y^2 + 2x^2y^2 + 6xy - 5y^2 - 7$

Solution

$\frac{\partial z}{\partial x} = 2x + 0 + 4xy^2 + 6y – 0 - 0 = 2x + 4xy^2 + 6y$

$\frac{\partial z}{\partial y} = 0 +2y + 4x^2y + 6x - 10y - 0 = 2y + 4x^2y + 6x -10y$

$\frac{\partial z}{\partial x^2} = 2 + 4y^2 + 0 = 2 + 4y^2$

$\frac{\partial^2 z}{\partial y^2} = 2 + 4x^2 + 0-10 = 4x^2- 8$

$\frac{\partial^3 z}{\partial x^3} = 0$

$\frac{\partial z}{\partial y^3} = 0$

$\frac{\partial^2 z}{\partial y\partial x} = 8xy + 6$

$\frac{\partial^2 z}{\partial x\partial y} = 8xy + 6$

4.2.3 Total Differentiation

If z is a continuous function of x and y and x and y are the continuous function of some other variable u, the total derivative of z with respect to u can be obtained as

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

This can be written in the differential form as

dz $= \frac{\partial z}{\partial x}$ dx$+ \frac{\partial z}{\partial y}$ dy, where dz is the total differential of z

Illustration 4.2.8

Let z = $6x^2+3y^2$ Find the total differential function

Solution

Total differential is

dz = $\frac{\partial z}{\partial x}$ dx+ $\frac{\partial z}{\partial y}$ dy

$\frac{\partial z}{\partial x} = 12x$

$\frac{\partial z}{\partial y}$ = 6y

dz = 12$x$dx + 6ydy

Illustration 4.2.9

Z= $4x^2 + 2xy + y^3$ Find the total differential function.

Solution

dz = $\frac{\partial z}{\partial x}$ dx+ $\frac{\delta z}{\delta y}$ dy

$\frac{\partial z}{\partial x}$ = 8x+2y

$\frac{\partial z}{\partial y}$ = $2x+3y^2$

dz = (8x +2y) dx+ ($2x+3y^2$) dy

Recap

- Partial differentiation - function which has more than one independent variables
- Differential coefficient of the dependent variable with respect to one of the independent variable keeping other independent variables as constant
- For a function (Z) with two independent variable - derivative of Z with respect to x and derivative of Z with respect to y.
- Partial derivatives with respect to x - $\frac{\partial z}{\partial x} = lt_{\Delta\, x\to 0}\, \frac{(x+\Delta x, y)-f(x,y)}{\Delta x}$
- Partial derivatives with respect to y - $\frac{\partial z}{\partial y} = lt_{\Delta\, y\to 0}\, \frac{(x,y+\Delta y)-f(x,y)}{\Delta y}$
- Product rule - $\frac{\partial z}{\partial x} = g(x,y)\,\frac{\partial h}{\partial x} + h(x,y)\,\frac{\partial g}{\partial x}$ and

  $\frac{\partial z}{\partial y} = g(x,y)\,\frac{\partial h}{\partial y} + h(x,y)\,\frac{\partial g}{\partial y}$

- Higher order derivative - $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

  $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$
- $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ - Differentiate with respect to y keeping x as fixed and then differentiate the result with respect to x keeping y as constant.
- Generally $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$
- Total Differentiation - dz $= \frac{\partial z}{\partial x}$ dx$+$ $\frac{\partial z}{\partial y}$ dy

Objective Questions

1.What is the use of partial differentiation?

2. How many derivates are possible when there are two independent variables?

3. What are the partial derivates when x and y are independent variables and Z is the dependent variable?

4. What will be the case of the independent variable when the partial derivate of function with respect to the other independent variable is done?

5. What is the higher order derivate when x is the independent variable?

6. What does the equation $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ means?

7. Express the mathematical form of total differentiation.

8. Express the mathematical form of Quotient Rule in partial differentiation.

9. Express the partial differentiation with respect to x and y

10. What is the total differentiation of z = $6x^2 + 3y^2$

Answers

1. Partial differentiation is used when the function has more than one independent variables.
2. 2
3. The derivative of Z with respect to x and derivative of Z with respect to y.
4. Fixed
5. Higher order derivative - $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$
6. Differentiate with respect to y keeping x as fixed and then differentiate the result with respect to x keeping y as constant.
7. $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$
8. Given $Z = \frac{g(x,y)}{h(x,y)}$ and $h(x,y) \neq 0$

$$\frac{\partial z}{\partial x} = \frac{h(x,y)\left(\frac{\partial g}{\partial x}\right) - g(x,y)\left(\frac{\partial h}{\partial x}\right)}{h(x.y)^2}$$

9. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
10. $dz = 12x dx + 6y dy$

Assignments

1. Find the partial derivative of
   a. $Z = 2xy$ b. $Z = (x^2 + y^{2)}$

2. Find the 1st and 2nd order partial derivatives for the following

   a. $f(x,y,z) = 4x^3y^2 - e^z y4 + z^3x^2 + 4y - x^{16}$
   b. $f(u,v,p,t) = 8u^2t^3p - \sqrt{v}p^2t^{-5} + 2u^2t + 3p^4 - v$

3. Find the partial derivatives

a. f(x.y) = $\frac{x-3}{1+y^2}$

b. f(x.y) = $\frac{x+y}{x}$

c. f(x.y) = $(x^2 + y^2)^2$

4. Find the total derivatives dz for the following

a. z = $x^4 e^{3y}$
b. f(x,y)=xy+3y^2

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5

BLOCK

Applications in Economics

1

UNIT

Application of calculus in Economics

Learning Outcomes

After completing this unit, the learner will be able to

- familiarise with the application of calculus in economics
- understand increasing and decreasing functions
- know the convexity and concavity of functions
- solve optimisation of functions
- learn maxima and minima of functions

Prerequisites

Before knowing the application of calculus in economics, let's first understand a few basic concepts. Imagine you're a farmer trying to make the most profit from your crops. First, you need to know how to calculate the rate of change of your crop yields. Calculus helps you do that by finding the increasing and decreasing functions, which tell you whether your crop yield is rising or falling over time. This is essential to make informed decisions about what to plant and when.

In the case of convexity and concavity, think of your profits as a rollercoaster ride. If the track is curving upward (convex), it means your profits are increasing at a faster rate. This is good news for an entrepreneur. On the other hand, if the track is curving downward (concave), your profits are growing at a slower pace, which is not as desirable.

In summary, before you learn how calculus applies to economics, we need to grasp the concepts of increasing and decreasing functions, convexity, concavity, and maxima and minima. These concepts are like tools in your toolbox, helping you make the right economic decisions, just as they assist the farmer and entrepreneur in their respective endeavours.

Before learning the application of calculus in economics, it's essential to understand some basic concepts. Firstly, you should have a good understanding of algebra, as calculus builds upon algebraic principles. Learners should be comfortable in manipulating equations, solving for unknowns, and understanding the relationships between variables.

Keywords

Calculus, Convexity, Concavity, Optimisation, Maxima, Minima

Discussion

5.1.1 Application of Calculus in Economics

Calculus serves as a vital mathematical tool in economics, enabling the analysis and modeling of a wide range of economic phenomena. Its applications in economics encompass various areas as follows:

- **Marginal Analysis:** Calculus is fundamental in understanding marginal concepts like marginal cost, marginal revenue, and marginal utility. These concepts are important in microeconomics, aiding businesses and consumers in decision-making related to pricing, production, and consumption.

- **Optimisation:** Calculus is instrumental in finding optimal solutions to economic problems. Firms use it to maximise profits by identifying the point where marginal cost equals marginal revenue. Consumers employ calculus to determine the most utility-maximising combination of goods within their budget.

- **Production and Cost Analysis:** Calculus is utilised to analyse production and cost functions. This assists firms in minimising production costs and maximising output for a given set of inputs.

- **Elasticity:** Calculus comes into play when calculating elasticity, which measures how responsive supply and demand are to changes in price or income. Understanding elasticity is critical for setting pricing strategies, tax policies, and predicting market responses to economic changes.

- **Integration:** Integration is employed to calculate cumulative economic values over time, such as total revenue, total cost, and total utility. This is vital for long-term economic planning and decision-making.

- **Differential Equations:** Differential equations are used to model dynamic economic systems, especially in macroeconomics, where they analyse economic growth, inflation, and interest rates over time.

- **Consumer and Producer Surplus:** Calculus is used to compute consumer surplus (the difference between willingness to pay and actual payment) and producer surplus (the difference between production costs and selling prices). These concepts are crucial for assessing the welfare implications of different market conditions and policies.

- **Utility and Indifference Curves:** Calculus aids in the analysis of utility functions and indifference curves, which are central to consumer theory. These curves illustrate how consumers make choices based on their preferences and budget constraints.

- **Game Theory:** Calculus is a key tool in modeling and analysing strategic interactions among rational decision-makers in game theory. This is applicable to scenarios like oligopolistic competition, bargaining, and auctions.

- **Economic Growth Models:** Calculus is employed in developing and scrutinising economic growth models such as the Solow growth model, helping economists and policymakers understand factors influencing long-term economic growth.

- **Environmental Economics:** Calculus is utilised in environmental economics to evaluate the trade-offs between economic growth and environmental preservation. It assists in determining optimal strategies for pollution control and resource management.

- **Financial Economics:** In financial economics, calculus is essential for pricing financial derivatives, managing risk, and optimising investment portfolios.

To conclude, calculus is a cornerstone of economic analysis, providing economists with the tools to model, analyse, and make predictions about economic behaviour and outcomes. Its applications span microeconomics, macroeconomics, and various specialised economic fields, making it an indispensable part of economic theory and policy analysis.

5.1.2 Increasing and Decreasing Function

A function f(x) is said to be increasing or decreasing at x = a if in the immediate vicinity of the point [a, f(a)] the graph of the function rises or falls as it moves from left to right. The first derivative measures the rate of change and slope of a function. A positive first derivative at x = a indicates the function is increasing. A negative first derivative indicates it is decreasing. The function is increasing at x = a when $f'(a) > 0$ and decreasing when $f'(a) < 0$. When a function increases or decreases over the entire region it is known as monotonic function. It increases or decreases monotonically.

Therefore condition for increasing and decreasing function are

A function is increasing $\frac{dy}{dx} > 0$ or $f'(x) > 0$

A function is decreasing $\frac{dy}{dx} < 0$ or $f'(x) < 0$

A function is stationary $\frac{dy}{dx} = 0$ or $f'(x) = 0$

The increasing and decreasing functions are represented below.

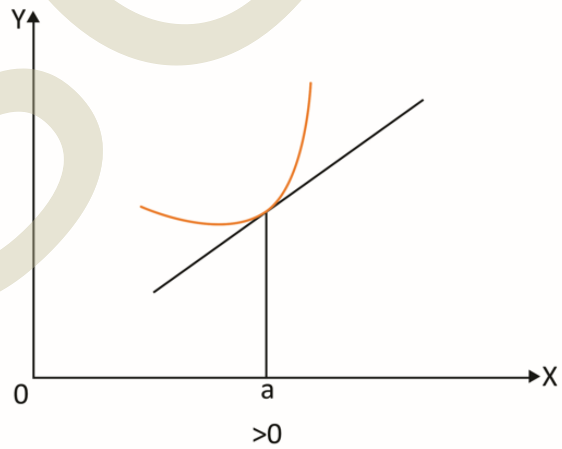

Fig 5.1.1 Increasing function

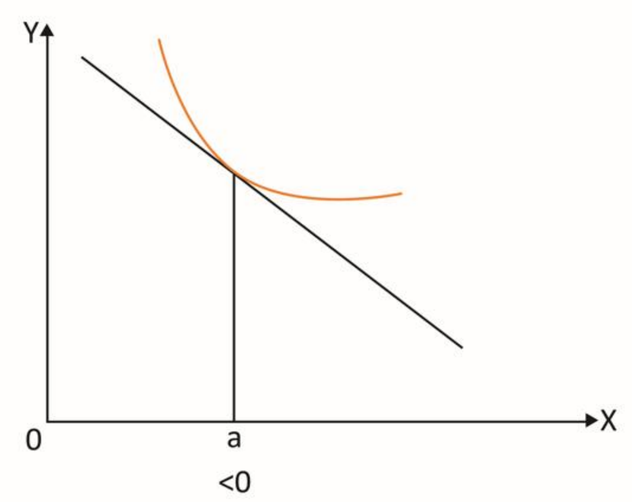

Fig 5.1.2 Decreasing function

Illustration 5.1.1

Check whether the functions are increasing or decreasing:

1) $y = 140 + 5x$
2) $y = 120 - 6x$
3) $y = 2x^2 - 30\ at\ x = 3$
4) $y = 4x^2 - 5x + 2\ at\ x = 2$
5) $y = x^3 - 7x^2 + x - 5\ at\ x = 3$
6) $y = x^4 - 6x^3 + 4x^2 - 13$ at x =4
7) $y = \frac{7x-9}{2x}\ at\ x = 3$

Solution

1) $\frac{dy}{dx} = \frac{d}{dx}(140 + 5x) = 5 > 0$. Therefore, the function is increasing.
2) $\frac{dy}{dx} = \frac{d}{dx}(120 - 6x) = -6 < 0$. The given function is decreasing.
3) $\frac{dy}{dx} = \frac{d}{dx}(2x^2 - 30) = 4x$. At $x = 3$, $\frac{dy}{dx} = 4(3) = 12 > 0$. The given function is increasing.
4) $\frac{dy}{dx} = \frac{d}{dx}(4x^2 - 5x + 2) = 8x - 5$. At $x = 2$, $\frac{dy}{dx} = 8(2) - 5 = 11 > 0$. Therefore, the function is an increasing function

5) $\frac{dy}{dx} = \frac{d}{dx}(x^3 - 7x^2 + x - 5) = 3x^2 - 14x + 1$. At $x = 3$, $\frac{dy}{dx} = 3(3)^2 - 14(3) + 1 = 27 - 42 + 1$

$= -14 < 0$ therefore the function is a decreasing function

6) $\frac{dy}{dx} = \frac{d}{dx}(x^4 - 6x^3 + 4x^2 - 13) = 4x^3 - 18x^2 + 8x$. At $x = 4$, $\frac{dy}{dx} = 4(4)^3 - 18(4)^2 + 8(4) = 0$

Therefore, the function is stationary

5.1.3 Convexity and Concavity Function

The calculus is used for deciding the convexity and concavity of a function. A function is said to be concave upward or convex at x = a, if the second derivative of the function is positive at x = a. If the second derivative is negative, then the function is concave downward (Concave). The slope of the first derivative is irrelevant for concavity. Therefore, the condition for convex function is f'' (x) >0 and for concave is f'' (x) < 0. Graph of the convex function lies above the tangent and for concave function lies below the tangent.

The conditions for a convex curve are that $f'(x) > 0$ and $f''(x) > 0$. $f'(x) > 0$ means that the value of the function tend to increase or the rate of change of y with respect to x is positive. f'' (x) >0 means the slope of the curve tend to increase or is positive. It is given in the figure.5.1.3 and 5.1.4. A function is concave when the following conditions are satisfied, $f'(x) < 0$ and f'' (x) < 0. $f'(x) < 0$ means that the value of the function tend to decrease or the rate of change of y with respect to x is negative. f'' (x) < 0 means that the slope of the curve tend to decrease. It is given in figure 5.1.3 and 5.1.4 . The sign of the second order condition determines the convexity or concavity of the function.

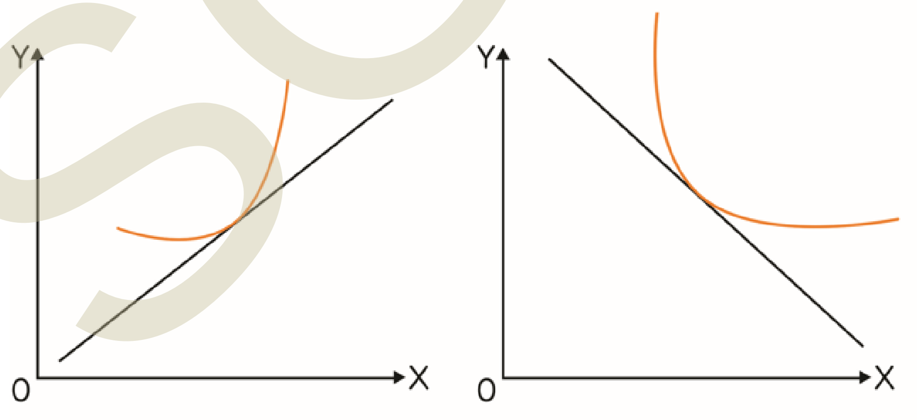

Fig 5.1.3 Convex functions

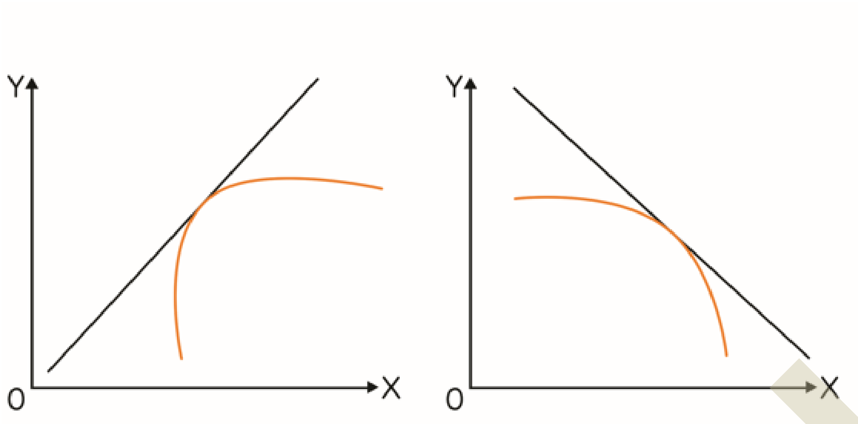

Fig 5.1.4 Concave functions

Illustration 5.1.2

Examine whether the functions are convex or concave:

1) $y = -x^2 - 120x - 32$
2) $y = x^2 + 12x - 32$
3) $y = 2x^3 - 30x^2 + 126x + 59$

Solution

1) $\frac{dy}{dx} = \frac{d}{dx}(-x^2 - 120x - 32) = -2x - 120 = 0;\ x = -60$

$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(-2x - 120) = -2 < 0$ the second derivative is negative, therefore the given function is concave

2) $\frac{dy}{dx} = \frac{d}{dx}(x^2 + 12x - 32) = 2x + 12 = 0,\ x = 6$

$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(2x + 12) = 2 > 0$ the second derivative is positive, therefore the given function is convex

3) $\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 30x^2 + 126x + 59) = 6x^2 - 60x + 126 = 0 = 6(x-3)(x-7) = 0$

$x = 3$ and $x = 7$

$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(6x^2 - 60x + 126) = 12x - 60$

at $x = 3$, $\frac{d^2y}{dx^2} = 12(3) - 60 = -24 < 0$, concave at $x = 3$

at $x = 7$ $\frac{d^2y}{dx^2} = 12(7) - 60 = 24 > 0$, convex at $x = 7$

5.1.4 Optimisation of Function (with one independent variable)

The optimisation is the method of finding the maximum or minimum values of a function. This is one of the important objective of a decision maker. In the case of utility, profit and revenue, the objective is the maximisation. Objective of a decision maker with respect to cost is minimisation. The optimisation technique make use of calculus method for finding the maximisation or minimisation decisions of the maker. The optimisation of a function can be done with one independent variable and more than one independent variables. In this section, optimisation with one independent variable is explained.

There are two conditions for the maximisation of a decision, they are denoted as first order and second order conditions. The first order conditions or necessary condition is that the first derivative equals zero, ie, $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0 and second order condition or sufficient condition is that the second derivative must be negative i.e., $\frac{d^2y}{dx^2}$ < 0 or f''(x) <0. Similarly for minimisation, the conditions are the first order conditions or necessary condition is that the first derivative equals zero, ie, $\frac{dy}{dx}$ = 0 or f'(x) = 0 and second order condition or sufficient condition is that the second derivative must be positive i.e.,

$\frac{d^2y}{dx^2}$ > 0 or $f''(x)$ >0.

5.1.5 Maxima and Minima of Function

A function f(x) is said to have a maximum at x = a if f(x) ceases to increase and begins to decrease at this value of x. A function is said to be minimum at x = a if f(x) ceases to decrease and begin to increase at this value of x. This is explained in the following figure 5.1.5

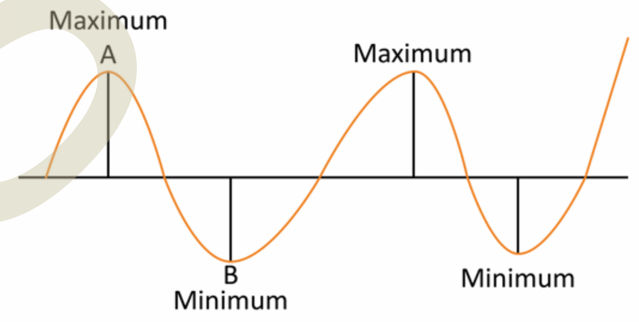

Fig 5.1.5 Maxima and Minima of Function

In the figure, point A is the maximum point since it is the highest value than any value on either side of the A. This means that value increases upto A and it must fall after point A has been reached. At point A, $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0. After point A, the slope of the curve is decreasing as x increases. That is $\frac{d^2y}{dx^2}$ < 0 or $f''(x)$ <0 at point A.

Point B is the minimum point, since it is the lowest value than any value on either side of B. The value decreases upto point B and it must increase after point B. At point B also, $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0. After point B, the slope of the curve is increasing. That is $\frac{d^2y}{dx^2}$ > 0 or $f''(x)$ >0 at point B. Point A and B are the turning points. At these points, the rate of change is constant and therefore $\frac{dy}{dx}$ = 0.

Thus, a function should satisfy two conditions in order to decide about maximum or minimum value at a particular point. These conditions are known as order conditions. The order conditions for maximum and minimum are given below.

Conditions for Maximum value

(1) First order condition or Necessary condition - $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0

(2) Second order condition or Sufficient condition - $\frac{d^2y}{dx^2}$ < 0 or $f''(x)$<0

Conditions for Minimum value

(1) First order condition or Necessary condition - $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0

(2) Second order condition or Sufficient condition - $\frac{d^2y}{dx^2}$ > 0 or $f''(x)$ > 0

Steps to find the Maximum and Minimum values of a function

1. Find the first derivative and equate it to zero $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0
2. Solve the equation $\frac{dy}{dx}$ = 0 or $f'(x)$ = 0. Find the values of x. One or more values we may get.
3. Find the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$
4. Substitute the values of x from step 2 in the second derivative
5. Values of x which makes the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$ negative, the function is maximum. Values of x which makes the second derivative $\frac{d^2y}{dx^2}$ or $f''(x)$ positive, the function is minimum
6. Substitute the respective values of x in the function to get the maximum and minimum values.

Illustration 5.1.3

Find the maximum and minimum values of the function y = $x^3 - 3x+1$

Solution

$\frac{dy}{dx}$ or $f'(x) = \frac{d}{dx}(x^3 - 3x + 1) = 3x^2 - 3 = 0$---------------- (1)

From equation (1), $3x^2 - 3 = 0$, $3(x^2 - 1) = 0$, $x^2 - 1 = 0$,

$(x + 1)(x - 1) = 0$. So, $x = +1, -1$

$f''(x) = \frac{d^2}{dx^2}(3x^2 - 3) = 6x$ ……… (1)

Substitute $x = +1, -1$ in equation (1)

When $x = +1$, $6x = 6 > 0$. Therefore, the given function is minimum at $x = 1$

When $x = -1$, $6x = -6 < 0$. Therefore, the given function is maximum at $x = -1$

Substitute $x = 1$ in the function y = $x^3 - 3x + 1$ to get the minimum value of function

$x = 1$, $y = 1^3 - 3(1) + 1 = -1$

Substitute $x = -1$ in the function y = $x^3 - 3x + 1$ to get the maximum value of function

$x = -1$, $y = -1^3 - 3(-1) + 1 = 3$

Illustration 5.1.4

Total revenue function of a firm is given by TR= $40x - x^2$, x is the output. Find the output at which total revenue is maximum.

Solution

TR = $40x - x^2$

$\frac{dTR}{dx} = \frac{d(40x - x^2)}{dx} = 40 - 2x = 0$, $x = \frac{40}{2} = 20$

$\frac{d^2TR}{dx^2} = \frac{d^2}{dx^2}(40 - 2x) = -2 < 0$

The first and second order conditions for maximum are satisfied. Therefore, the profit maximising level of output is 20

Illustration 5.1.5

The cost function of a firm producing x commodities is given as $5x^2 - 60x + 100$. How many units of x to be produced to minimise the cost?

Solution

$\frac{dTc}{dx} = \frac{d(5x^2 - 60x + 100)}{dx}$ = 10x - 60 = 0, x = $\frac{60}{10}$ = 6

$\frac{d^2Tc}{dx^2} = \frac{d^2}{dx^2}$(10$x$ - 60) = 10 > 0

The first and second order conditions for minimisation function are satisfied and therefore the cost minimising output is at x = 6

Illustration 5.1.6

The total cost function of a firm is given as TC = 200 + 20x^2 and the total revenue function is given as TR =10x. Find the profit maximising level of output?

Solution

Profit= TR-TC

TR = 80x

TC = 200 + 80x^2

Profit p = 80x – (200 + 20x^2) = 80x – 200 - 20x^2

$\frac{dp}{dx} = \frac{d(80x - 200 - 20x^2)}{dx}$ = 80 - 40x = 0, x = $\frac{80}{40}$ = 2

$\frac{d^2p}{dx^2} = \frac{d^2}{dx^2}$(80 - 40$x$) = -40 < 0

The first and second order conditions for maximisation function are satisfied and therefore the profit maximising output is at x =2

Recap

- Calculus - Analyse economic phenomena involving change, such as cost, revenue, and profit functions
- In calculus, functions can be categorised as increasing or decreasing
- Increasing function - Value of function rises as its input (independent variable) increases
- Decreasing function - Value of function falls as its input increases
- Convex functions have a curved upward shape, and concave functions have a curved downward shape
- Convexity - Diminishing marginal returns or increasing marginal costs
- Concavity - Economies of scale or decreasing marginal costs
- Maxima represent peaks or highest values, while minima are lowest points
- Economists use maxima and minima to find optimal solutions
- Optimal Solutions - Profit-maximising price levels or cost-minimising production quantities

Objective Questions

1. What is the direction of an increasing function?
2. What are the primary goals of optimisation in economics?
3. What is the shape of a concave function?
4. What does the derivative of a function measure?
5. What is the primary goal of a business in terms of economics?
6. Represent second order derivate mathematically
7. In the context of finding maxima and minima, what does the second order derivative implies?
8. What is the first order condition for maximum value?
9. What is the second order condition for minimum value?
10. What happens to the slope of curve just after the maximum point?

Answers

1. Upward
2. Maximise or minimise
3. Downward curve
4. Slope of the curve
5. Maximise profit
6. $\frac{d^2y}{dx^2}$
7. Second order derivative implies whether the function at a point is maximum or minimum
8. $\frac{dy}{dx} = 0$ or $f^{1}(x) = 0$
9. $\frac{d^2y}{dx^2} > 0$ or $f^{11}(x) > 0$
10. Slope of the curve falls

Assignments

1. Define and explain the concept of an increasing function and a decreasing function in the context of economics. Provide examples of each.
2. Describe what it means for a function to be convex and concave. How are these concepts relevant in economics, especially in utility and cost functions?
3. Explain the concept of optimization in economics. How does calculus help in finding the optimal solutions to economic problems?
4. Suppose a restaurant's cost function is given by C(q) = 500 + 5q, where q is the number of meals served, and the revenue function is given by R(q) = 15q. Find the profit-maximising level of output (q) for the restaurant.

Suggested Readings

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2

UNIT

Marginal Concepts and Total Concepts

Learning Outcomes

After going through this unit, the learner will be able to

- familiarise with the mathematical representation of marginal concepts
- understand derivation of marginal concepts from total concepts
- knew the application of differentiation under elasticity

Prerequisites

Imagine you're planning a big picnic, and you want to make sure you have enough food and drinks for everyone. The first thing you need to know is the total concept, which is like adding up all the food and drinks you have in your picnic basket. This tells you how much you have in total. If you don't know the total, you might run out of goods, and your friends won't be too happy.

Now, let's talk about the marginal concept. It's like thinking about each additional friend who joins your picnic. You want to know how much more food and drinks you need for each extra person. This helps you make sure everyone gets enough to eat and drink without overdoing it.

In economics, the total concepts and marginal concepts are essential to make informed decisions especially in production and consumptions.

Keywords

Marginal Utility, Total Utility, Marginal Rate of Substitution, Marginal Propensity to Consume, Marginal Propensity to Save, Marginal Rate of Technical substitution

Discussion

5.2.1 Marginal and Total concepts

Differential calculus is one of the important tool used in economics which deals with the marginal concept. The marginal concept in economics explains the rate of change in the economic activities which occupies an important role in economic analysis. In this section, some of the important marginal and total concepts used in economics are explained. The important marginal concepts are**:**

1. Marginal Utility and Total Utility
2. Marginal Rate of Substitution
3. Marginal propensity to Consume
4. Marginal Propensity to Save
5. Marginal cost and Total cost
6. Marginal Product and Total Product
7. Marginal Rate of Technical substitution
8. Marginal Revenue and Total Revenue

5.2.2 Marginal Utility and Total Utility

The word utility denotes the want satisfying power of a commodity or service. An individual demands a particular commodity because of the satisfaction he could receive from consuming it. A commodity possesses utility if it satisfies an economic want. Goods may be poisonous or dangerous to one's health, but it possesses utility for those who want them. Utility is thus subjective. The utility function expresses utility as a function of consumption of real goods. The utility function expresses utility as a function of goods consumed. $U = f(X_1, X_2, X_3 \ldots\ldots X_n)$.

5.2.2.1 Cardinal and Ordinal Utility

There are two basic approaches to the problem of comparison of utilities. They are the Cardinalist approach and the Ordinalist approach. The cardinalist school postulated that utility can be measured. It means that an individual consumer attach specific values or number of utils from consuming each quantity of good or combination of goods. While some economist suggested the measurement of utility in monetary unit, others suggested its measurement in subjective units called utils. The ordinalist school postulated that utility is not measurable, but is an ordinal magnitude. As per this notion, a consumer need not know in specific units the utility of various commodities to make his choice. While ordinal utility only ranks various consumption bundles, cardinal utility provides an actual index or measure of satisfaction.

5.2.2.2 Total and Marginal Utility

The sum total of utilities obtained by the consumer from different units of a commodity is the total utility. Marginal utility is the net addition to total utility resulting from one unit change in consumption. It is the addition to total utility by having an additional unit of the commodity. In other words marginal utility of a commodity is the loss in utility if one unit less is consumed. If there are n commodities with quantities $X_1, X_2, \ldots\ldots, X_n$, the total utility is $U= f(X_1, X_2 \ldots\ldots X_n)$. Total utility is the sum of utilities from the consumption of different units of the commodities ie $U= U_1X_1 + U_2X_2 + \ldots\ldots U_nX_n$

5.2.2.3 Marginal Utility

Marginal utility means an additional or incremental utility. Marginal utility is the change in the total utility that result from one unit change in consumption of the commodity within a given period of time ie, $MU_n = TU_n - TU_{n-1}$.

Mathematically, marginal utilities are the partial derivative of utility function. For any utility function $U = f(X,Y)$, the marginal utilities of X and Y are $MU_X = \frac{\partial U}{\partial X}$ and $MU_y = \frac{\partial U}{\partial y}$.

The marginal utility of a good diminishes as an individual consumes more unit of a good, i.e., as the consumer takes more unit of a good, the extra utility or satisfaction derived from extra unit are negative. Or $\frac{\delta_U^2}{\delta x^2}$ and $\frac{\delta_U^2}{\delta y^2}$ are negative. This means that the slope of MU curve is negative.

5.2.2.4 Relationship between Total utility and Marginal Utility

The relationship between total utility and marginal utility is showed in the table 5.2.1 and figure 5.2.1. Total utility increases as consumption increases and reaches maximum and starts declining. Marginal utility is the addition made to total utility. Marginal utility declines as more and more units of commodities are consumed. Marginal Utility reaches zero when total utility is maximum and turns negative when total utility starts falling. It is clear from the below table.

Table 5.2.1 Total Utility and Marginal Utility

Units of commodity x	Total Utility in Utils	Marginal Utility in Utils
0	0	0
1	10	10
2	18	8
3	24	6
4	28	4
5	30	2
6	30	0
7	28	-2

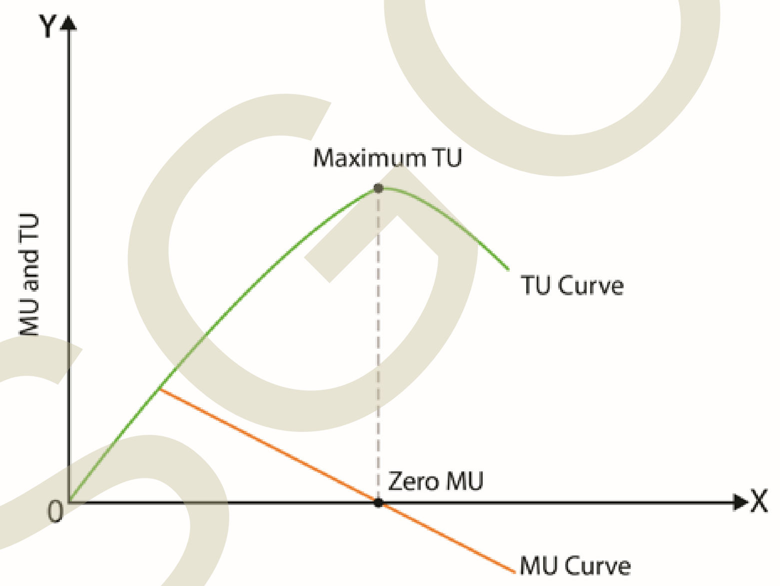

Fig 5.2.1 TU and MU

Illustration 5.2.1

Given the utility function $u = x^2y^3$, find MUx and MUy

Solution

$$MUx = \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) = 2xy^3$$

$MUy = \frac{\partial U}{\partial y} = \frac{\partial}{\partial X}(x^2y^3) = x^2 3y^2$

Illustration 5.2.2

Solution

Given utility function $U = x^2y + 3xy^2 + 4y$, find marginal utilities of good x and good y

$MUx = \frac{\partial U}{\partial X} = \frac{\partial}{\partial X}(x^2y + 3xy^2 + 4y) = 2xy + 3y^2$

$MUy = \frac{\partial U}{\partial y} = \frac{\partial}{\partial X}(x^2y + 3xy^2 + 4y) = x^2 + 6xy + 4$

5.2.3 Marginal Rate of Substitution(MRS)

Marginal Rate of Substitution is the slope of an Indifference Curve (IC). It is the rate at which the consumer is prepared to give up commodity y for one unit gain of *x*, so that the level of satisfaction remains the same. An Indifference Curve is the locus of points of particular combinations of goods, which yield the same utility or level of satisfaction to the consumer. An indifference curve is drawn from the indifference schedule of the consumer. Indifference Curve shows the various combinations of the two commodities such that the consumer is indifferent to those combinations. The important properties of indifference curve are a higher IC to the right of another represents a higher level of satisfaction and preferable combinations of goods; indifference curves are negatively slopped; indifference curves can neither touch nor intersect each other; Indifference curves are convex to the origin; MRS which is the slope of indifference curve is defined as the number of unit of one commodity (y) that must be given up in exchange for an extra unit of another commodity (*x*) so that the consumer maintains the same level of satisfaction. MRS is equal to the ratio of the marginal utilities of commodities involved in the utility function. MRS is the slope of indifference curve at any point. Slope of indifference curve $= -\frac{dy}{dx} = MRSxy$

$MRSxy = \frac{MUx}{MUy}$ or $MRSyx = \frac{MUy}{MUx}$

This can be proved as follows

Let U= f(x, y)

The equation of an indifference curve is U= f(x,y) =K

Total differential of utility function is

$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = MU_x dx + MU_y dy$

This equation shows that total change in U caused by change in y and x is equal to the change in y multiplied by its marginal utility plus change in x multiplied by its marginal utility.

$dU = MU_y dy + Mu_x dx = 0$

$$-\frac{dy}{dx} = \frac{MUx}{MUy} = \text{MRSxy}$$

$$-\frac{dx}{dy} = \frac{MUy}{MUx} = \text{MRSyx}$$

The slope of indifference curve diminishes. The decrease in MRS implies that as the consumer has more and more of good x, he is prepared to forgo less of good Y. It is given in the following figure

.

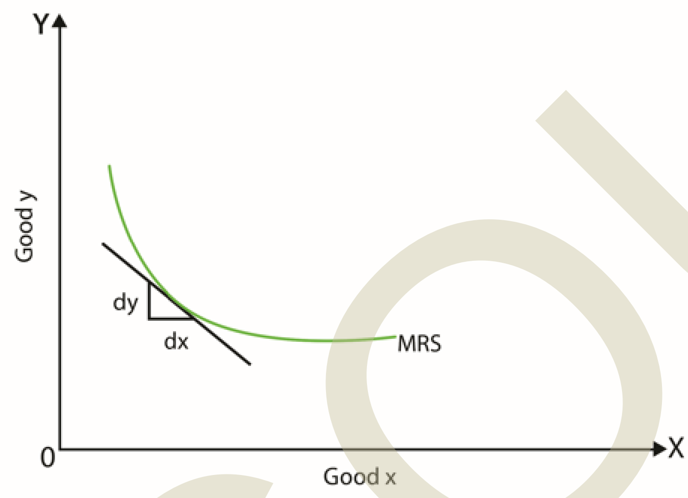

Fig 5.2.2 Marginal Rate of Substitution

Illustration 5.2.3

Given the function U= 3x + y, find MRS xy

Solution

$$\text{MRSxy} = \frac{MUx}{MUy}$$

$$MUx = \frac{\partial u}{\partial x} = 3$$

$$MUy = \frac{\partial u}{\partial y} = 1$$

$$\text{MRSxy} = \frac{MUx}{MUy} = \frac{3}{1} = 3$$

5.2.4 Marginal Propensity to Consume (MPC)

The marginal propensity to consume may be defined as the ratio of change in consumption to the change in income.

MPC quantifies induced consumption. That is, how much increase in income is used for consumption.

$$MPC = \frac{Change\ in\ consumption}{Change\ in\ income}$$

$$MPC = \lim_{\Delta y \to 0} \frac{\Delta C}{\Delta y} = \frac{dC}{dy}$$

Marginal propensity to consume is the slope of consumption line. Consumption line is positively sloped, indicating that greater levels of income generate greater consumption expenditure.

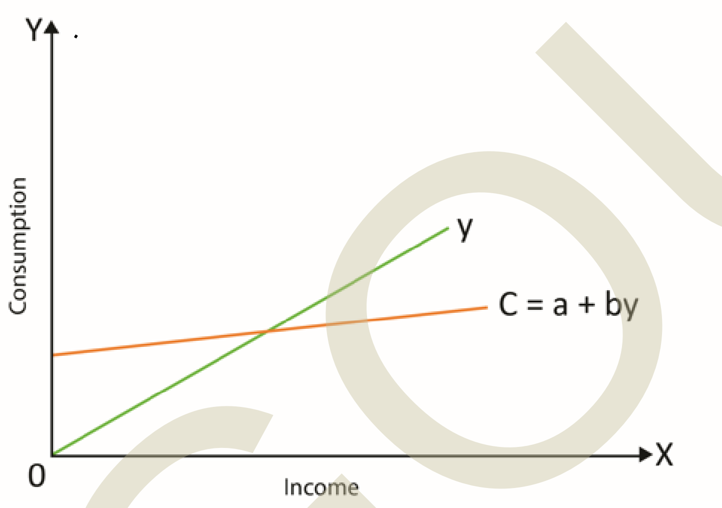

Fig 5.2.3 Consumption Function

Suppose increase in income is Rs. 200 and if Rs.150 of this increase in income is used for consumption, then marginal propensity to consume will be given as:

$$MPC = \frac{150}{200} = 0.75$$

MPC = 0.75. This means that, 75 percent of increase in income goes for consumption. The value of MPC lies between 0 and 1. MPC = 0 implies that the whole increment in income is saved. MPC = 1 implies that the whole increment in income is consumed.

5.2.4.1 Average Propensity to Consume (APC)

APC is the proportion of income used for consumption expenditure. It is defined as the ratio of consumption expenditure to any level of income. It is found by dividing consumption by income. APC = $\frac{\text{consumption}}{\text{income}}$. If APC is 0.9 then 90 percent of income received by the households are used for consumption. APC = $\frac{C}{Y}$

Illustration 5.2.4

Given the consumption function C= 120 + 0.25Y, find MPC.

Solution

MPC = $\frac{dC}{dy}$ = 0.25

5.2.5 Marginal Propensity to Save (MPS)

Marginal Propensity to Save (MPS) refers to the increase in saving that results from an increase in income. It shows the addition of income used for saving. Marginal Propensity to Save is calculated by dividing the change in saving by change in income.

MPS = $\frac{Change\ in\ saving}{Change\ in\ income}$

MPS = $\lim_{\Delta y \to 0} \frac{\Delta S}{\Delta y} = \frac{dS}{dy}$

Marginal Propensity to Save $\frac{\Delta s}{\Delta y}$, is the slope of the saving line and MPS lies between 0 and 1. MPS + MPC = 1

MPS = 1- MPC

MPC = 1- MPS

MPS increases with increase in income. If MPS is 0.30, it means that household will spend 70 paise and save 30 paise.

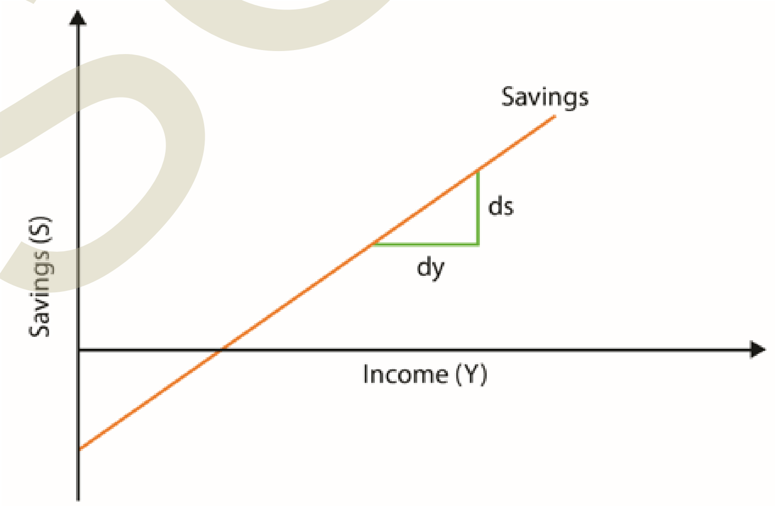

Fig 5.2.4 Marginal Propensity to Save

5.2.5.1 Average Propensity to Save – (APS)

It shows the proportion of income that is saved - APS is also known as saving ratio.

APS = Saving /income i.e., APS = $\frac{S}{Y}$

Income is either consumed or saved, Given

Y= C+S

Divide both side of the equation by Y which gives

$$\frac{Y}{Y} = \frac{C}{Y} + \frac{S}{Y}$$

$$1 = \frac{C}{Y} + \frac{S}{Y} = APC + APS = 1$$

APS = 1- APC

APC =1- APS

5.2.6 Marginal Cost and Total Cost

5.2.6.1 Cost function

The production function and the price of the inputs together determines the cost of a commodity. Therefore, the cost functions are derived from the production function. The cost function can be represented as follows:

C = f (Q, T, P, K)

The relation between the cost and output is technically described as the cost function. The significance of cost-output relationship is so wide. In economic analysis the cost function usually refers to the relationship between cost and rate of output alone and we assume that all other independent variables are kept constant. Mathematically, TC= f (Q) where TC = Total cost and Q stands for output produced.

5.2.6.2 Types of Cost

a. Explicit costs

Explicit costs are the actual out-of-pocket expenditure of the firm to purchase or hire the inputs it requires in production. These expenditures include the wages to hire labour, interest on borrowed capital, rent on land and buildings, and the expenditures on raw and semifinished materials.

b. Implicit costs

Implicit costs refer to the value of the inputs owned and used by the firm in its own production processes. The value of these owned inputs must be imputed or estimated from what these inputs could earn in their best alternative use.

c. The opportunity cost (Economic Cost)
The opportunity cost of producing a particular good is the cost in terms of other goods which might have been produced using the same resource.

1. Total fixed cost (TFC)
TFC refers to total money expenses incurred on fixed inputs like plant, machinery, tools and equipment in the short run. Total fixed cost corresponds to the fixed inputs in the short run production function. TFC remains the same at all levels of output in the short run. It is the same when output is nil. It indicates that whatever may be the quantity of output, TFC remains constant. The TFC curve is horizontal and parallel to X axis, showing that it is constant regardless of output per unit of time. TFC starts from a point on Y-axis indicating that the total fixed cost will be incurred even if the output is zero.

2. Total variable cost (TVC)
TVC refers to total money expenses incurred on the variable factor inputs like raw materials, power, fuel, water, transport and communication etc; in the short run. Total variable cost corresponds to variable inputs in the short run production function. It is obtained by summing up the production of quantities of variable inputs multiplied by their prices.

TVC = TC -TFC
TVC = f (Q)

i.e., TVC is an increasing function of output. In other words TVC varies with output. It is nil, if there is no production. Thus, it is a direct cost of output. TVC rises sharply in the beginning, gradually in the middle, and sharply at the end in accordance with the law of variable proportion.

The law of variable proportion explains that in the beginning to obtain a given quantity of output, relative variation in variable factors needed are in less proportion, but after a point when the diminishing returns operate, variable factors are to be employed in a larger proportion to increase the same level of output. TVC curve slope upwards from left to right. TVC curve rises as output is expanded. When output is zero, TVC also will be zero. Hence, the TVC curve starts from the origin.

3.Total cost (TC)
The total cost refers to the aggregate money expenditure incurred by a firm to produce a given quantity of output. The total cost is measured in relation to the production function by multiplying the factor prices with their quantities. TC = f (Q) which means that the TC varies with the output. Theoretically, TC includes all kinds of money costs, both explicit and implicit cost. Normal profit is included in the total cost as it is an implicit cost. It

includes fixed as well as variable costs. Hence, TC = TFC + TVC. TC varies in the same proportion as TVC.

In other words, a variation in TC is the result of variation in TVC since TFC is always constant in the short run.

4. Average fixed cost (AFC)

Average fixed cost is the fixed cost per unit of output. When TFC is divided by total units of output, AFC is obtained**,** thus, AFC = $\frac{TFC}{Q}$, AFC and output have inverse relationship. Hence, TFC spreads over each unit of output with the increase in output. Consequently, AFC diminishes continuously. The AFC curve has a negative slope. The curve slopes downwards throughout the length. The AFC curve goes very nearer to X axis, but never touches the axis. Graphically it will fall steeply in the beginning, gently in middle, and tend to become parallel to X-axis. Mathematically, as output increases, AFC diminishes.

5. Average variable cost: (AVC)

The average variable cost is variable cost per unit of output. AVC can be computed by dividing the TVC by total units of output. AVC = $\frac{TVC}{Q}$. The AVC will come down in the beginning and then rise as more units of output are produced with a given plant. This is because as we add more units of variable factors in a fixed plant, the efficiency of the inputs first increases and then it decreases.

6. Average total cost (ATC) or Average cost (AC)

AC refers to cost per unit of output. AC is also known as the unit cost since it is the cost per unit of output produced. AC is the sum of AFC and AVC. Average total cost or average cost is obtained by dividing the total cost by total output produced. AC = $\frac{TC}{Q}$, AC is the sum of AFC and AVC. In the short run AC curve also tends to be U-shaped. The combined influence of AFC and AVC curves will shape the nature of AC curve. AC and AVC curves are U shaped.

AC = $\frac{TC}{Q}$

TC = TFC+TVC

AC = $\frac{TC}{Q} = \frac{TFC}{Q} + \frac{TVC}{Q}$ = AFC + AVC

7. Marginal Cost (MC)

Marginal Cost may be defined as the net addition to the total cost as one more unit of output is produced. In other words, it implies additional cost incurred to produce an additional unit. For example, if it costs Rs. 50 to produce 10 units of a commodity and Rs.55 to produce 11 units, then MC would be Rs. 5. It is obtained by calculating the change in total costs as a result of a change in the total output. Also MC is the rate at which total cost changes with output.

$$MC = \frac{\Delta TC}{\Delta Q} \text{ or } MC = \frac{d\ TC}{d\ TQ.}$$

Where d TC stands for change in total cost and d TQ stands for change in total output. It can be also written as, MCn = TCn –TC n-1

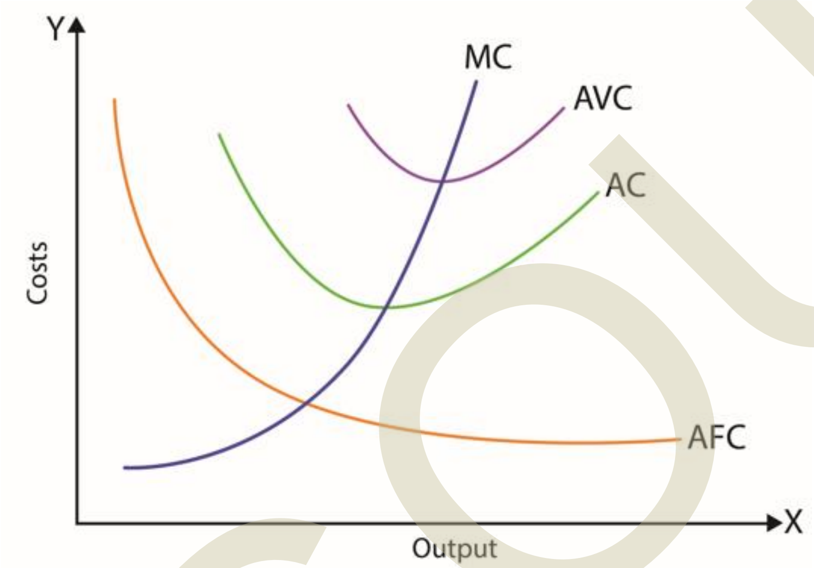

Fig 5.2.5 Cost curves

Illustration 5.2.5

Cost function of a firm is given by TC = $x^2(2x-4)$, Find the marginal cost when production is 2 units.

Solution

TC= $x^2(2x - 4) = 2x^3 - 4x^2$

$$MCx = \frac{dTC}{dX} = \frac{d(2x^3 - 4x^2)}{dX} = 6x^2 - 8x$$

Marginal cost of x, when $x= 2$

$= 6(2^2) - 8(2) = 24 - 16 = 8$

Illustration 5.2.6

Given the total cost function TC = $x^3 + 6xy^2 - y^3$ of firm producing X and Y. Find the marginal cost of X and Y.

Solution

MCx = $\frac{\partial TC}{\partial X}$ = $3x^2 + 6y^2$

MCx = $\frac{\partial TC}{\partial X}$ = $12xy - 3y^2$

Illustration 5.2.7

Given the cost function TC = $7Q^3 - 2Q^2 - 3Q + 10$. Find Total variable cost (TVC), Average Variable cost (AVC), Average Cost (AC), Total Fixed Cost (TFC), Average Fixed Cost (AFC), and Marginal Cost (MC) at Q = 2.

Solution

Total Cost $=7Q^3 - 2Q^2 - 3Q + 10$

Here 10 is the TFC which is not dependent on output, Q

Total variable cost (TVC) = $7Q^3 - 2Q^2 - 3Q$

At Q = 2, TVC = $7(2^3) - 2(2^2) - 3(2) = 56 - 8 - 6 = 42$

Average variable cost (AVC) = $\frac{TVC}{Q}$

$= \frac{7Q^3 - 2Q^2 - 3Q}{Q} = 7Q^2 - 2Q - 3 = 7(2^2) - 2(2) - 3 = 21$ at Q = 2

Average cost (AC) = $\frac{TC}{Q} = \frac{7Q^3 - 2Q^2 - 3Q + 10}{Q} = 7Q^2 - 2Q - 3 + \frac{10}{Q}$

At Q = 2, AC = $7(2^2) - 2(2) - 3 + \frac{10}{2} = 28 - 4 - 3 + 5 = 26$

Total Fixed cost (FC) =10

Average Fixed cost = $\frac{TFC}{Q} = \frac{10}{Q}$ at Q = 2, AFC = $\frac{10}{2} = 5$

Marginal Cost (MC) = $\frac{dTC}{dQ} = \frac{d}{dQ}(7Q^3 - 2Q^2 - 3Q + 10) = 21Q^2 - 4Q - 3$

At Q = 2 MC = $21(2^2) - 4(2) - 3 = 21(4) - 4(2) - 3 = 84 - 8 - 3 = 73$

5.2.6.2 Relationship between Average cost and Marginal cost

The relationship between MC and AC are:

1. When AC falls, MC is less than AC:MC curves lies below AC or Fall in MC is higher than fall in AC
2. When AC is minimum, MC equal AC

3. When AC increases, MC is greater than AC:MC curve lies to the left of AC- Rise in MC is higher than the rise in AC

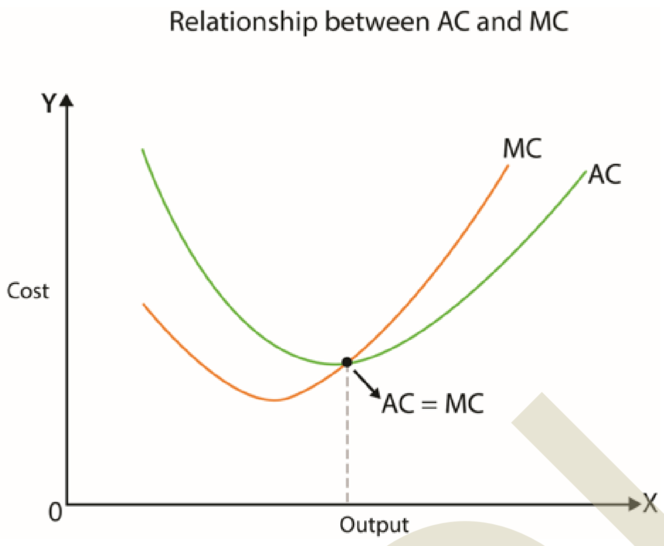

Fig 5.2.6 Relationship between MC and AC

When AC falls, MC is below AC

Condition for fall or decrease is $\frac{dy}{dx} < 0$

AC falls or decreases means that $\frac{d}{dQ}(\frac{C}{Q})$ have negative sign ie $\frac{d}{dQ}(\frac{C}{Q}) < 0$

Differentiate $(\frac{C}{Q})$ wrt to Q.

$$\frac{d}{dQ}(\frac{C}{Q}) = \frac{Q\frac{dC}{dQ} - C\frac{d}{dQ}(Q)}{Q2} < 0$$

$$= \frac{Q\frac{dC}{dQ} - C}{Q2} < 0 \text{ since, } \frac{dQ}{dQ} = 1$$

$$[\frac{1}{Q}.\frac{dC}{dQ} - \frac{C}{Q2}] < 0$$

$\frac{1}{Q}[\frac{dC}{dQ} - \frac{C}{Q}] < 0$ since Q not < 0 or Q is positive

$$[\frac{dC}{dQ} - \frac{C}{Q}] < 0$$

$$\frac{dC}{dQ} < \frac{C}{Q}$$

MC < AC since $\frac{dC}{dQ}$ = MC and $\frac{C}{Q}$ = AC

2. When AC is minimum, MC=AC

$$\frac{dy}{dx} = 0$$

$\frac{d}{dQ}\left(\frac{C}{Q}\right) = 0$ (apply quotient rule of differentiation)

$$\frac{d}{dQ}\left(\frac{C}{Q}\right) = \frac{Q\frac{dC}{dQ} - C\frac{d}{dQ}(Q)}{Q2} = 0$$

$= \frac{Q\frac{dC}{dQ} - C}{Q2} = 0$ since $\frac{dQ}{dQ} = 1$

$$\left[\frac{1}{Q} \cdot \frac{dC}{dQ} - \frac{C}{Q2}\right] = 0$$

$$\frac{1}{Q}\left[\frac{dC}{dQ} - \frac{C}{Q}\right] = 0$$

$$\left[\frac{dC}{dQ} - \frac{C}{Q}\right] = 0$$

$\frac{dC}{dQ} = \frac{C}{Q}$, $\frac{dC}{dQ} = \text{MC}$ $\quad \frac{C}{Q} = \text{AC}$

MC = AC

3. When AC increases, MC greater than AC:MC > AC i.e. $\frac{dy}{dx} > 0$

$$\frac{d}{dQ}\left(\frac{C}{Q}\right) > 0$$

$$\frac{d}{dQ}\left(\frac{C}{Q}\right) = \frac{Q\frac{dC}{dQ} - C\frac{d}{dQ}(Q)}{Q2} > 0$$

$= \frac{Q\frac{dC}{dQ} - C}{Q2} > 0$, since $\frac{dQ}{dQ} = 1$

$$\left[\frac{1}{Q} \cdot \frac{dC}{dQ} - \frac{C}{Q2}\right] > 0$$

$$\frac{1}{Q}\left[\frac{dC}{dQ} - \frac{C}{Q}\right] > 0$$

$$\frac{1}{Q}\left[\frac{dC}{dQ} - \frac{C}{Q}\right] > 0$$

$$\left[\frac{dC}{dQ} - \frac{C}{Q}\right] > 0$$

$$\frac{dC}{dQ} > \frac{C}{Q}$$

MC > AC where $\frac{dC}{dQ}$ = MC and $\frac{C}{Q}$ = AC

5.2.7 Marginal Product and Total Product

Total product (TP) is defined as the total quantity of goods and services produced by a firm with the given inputs during a specific period of time. As the amount of factor increases, TP increases.TP starts from the origin, increases at an increasing rate, increases at a decreasing rate, reaches maximum and starts falling as given in figure 5.2.7.

5.2.7.1 Average Product (AP)

Average product of a factor is the total output produced per unit of the factor employed or total product divided by the factors employed. The firms production function is given as Q = f(K, L) then,

$$AP\,L = \frac{TP}{L} \text{ or } \frac{Q}{L}$$

$$APK = \frac{TP}{K} \text{ or } \frac{Q}{K}$$

An AP curve is shown in the figure 5.2.7, where AP starts from the origin, increases at a decreasing rate reaches maximum, and starts falling as shown in figure.

5.2.7.2 Marginal Product (MP)

Marginal product of an input or factor of production is the extra output that can be produced by using one more unit of the input, keeping all other inputs unchanged. The marginal product of a given input can be expressed as,

$$MP = \frac{\Delta Q}{\Delta X}$$

MP is the mathematical derivative of the production function with respect to that input. Suppose a firm's output Q is given by the production function Q = f (K, L), Marginal product of capital, MP_K, and Marginal product of labour, MP_L, are the derivative of the production function with respect to K and L respectively

$$MPL = \frac{\partial Q}{\partial L}$$

$$MPK = \frac{\partial Q}{\partial K}$$

Graphically, marginal product of labour is shown by the slope of production function keeping capital fixed. Marginal product of capital is the slope of production function keeping labour fixed.

5.2.7.3 Relationship between MP and AP

The relationship between AP and MP can be illustrated with the help of the following figure.

1. Initially TP, MP and AP curves are rising. Increase in MP is greater than increase in AP
2. When AP is maximum, MP = AP
3. When TP is maximum, MP = 0
4. When TP is falling, MP negative
5. MP curve passes through the maximum point of AP
6. So long as AP sloping upward, MP > than AP
7. As, AP slopes downward, MP < than AP

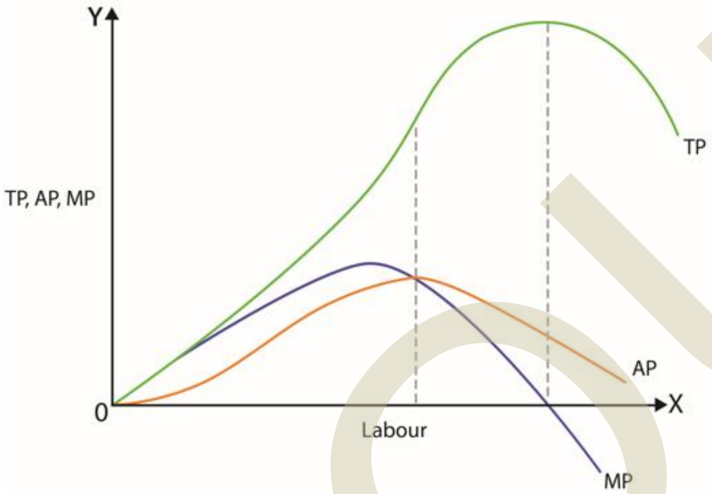

Fig 5.2.7 Relationship between MP and TP

Illustration 5.2.8

Let $Q = 40 - 10L - L^2$ be a production function where L is input and Q is output. Find the marginal productivity, when L= 3.

Solution

$Q = 40 - 10L - L^2$

$MPL = \frac{dQ}{dL} = \frac{d}{dL}(40 - 10L - L^2) = -10 - 2L$

Marginal product when L= 3, 10 - 2(3) = 4

Illustration 5.2.9

Given the production function $Q = 4x^2 + xy^2 + 3y$ for a firm which uses two inputs x and y in the production process, find the marginal products of two inputs.

Solution

$MPx = \frac{\partial Q}{\partial x} = 8x + y^2$

$MPy = \frac{\partial Q}{\partial y} = 2xy + 3$

5.2.8 Marginal rate of technical substitution

Marginal Rate of Technical Substitution (MRTS) is based on the production function where two factors can be substituted in a way as to produce a constant level of output. The MRTS is the amount by which the quantity of one input has to be reduced for one extra unit gain of another input. So that output remains constant. MRTS is the slope of the isoquants. An isoquant represents all those combinations of factors which are capable of producing the same level of output. The word 'iso' is of Greek origin and means 'equal' or 'same'. An isoquant is a curve along which quantity is the same. So the isoquant is also called equal product curve. An isoquant is equivalent to the concept of indifference curve you studied under the analysis of consumer equilibrium. The only difference is that instead of two goods x and y used by a consumer, here we use two inputs labour and capital used by a producer. Properties of Isoquants are they are negatively slopped, convex to the origin, higher the isoquant higher is the level of production, and two isoquants never intersect.

The slope of the isoquant defines the degree of substitutability of factors of production. It means that the marginal rate of technical substitution of labour for capital is the number of units of capital which can be substituted by one units of labour keeping the same level of output. The slope of isoquant decreases as we move downwards along the isoquant.

Therefore, MRTS of input X for input Y is

$$MRTS_{XY} = \frac{MP_X}{MP_Y}$$

MRTS is equal to the ratio of marginal product of two factors

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}}$$

$$MRTS_{KL} = \frac{MP_K}{MP_L} = \frac{\frac{\partial Q}{\partial K}}{\frac{\partial Q}{\partial L}}$$

MRTS as a measure of degree of substitutability of factors depends on the units of measurement of factor only. A better measure of factor substitution is provided by the elasticity of substitution. The elasticity of substitution measures the percentage change in the capital labour ratio divided by percentage change in the rate of technical substitution.

Illustration 5.2.10

Given the Cobb Douglas production function $Q=AK^{\alpha}L^{\beta}$ obtain the $MRTS_{LK}$ and $MRTS_{KL}$

Solution

$Q = AK^{\alpha}L^{\beta}$

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}}$$

$$MRTS_{KL} = \frac{MP_K}{MP_L} = \frac{\frac{\partial Q}{\partial K}}{\frac{\partial Q}{\partial L}}$$

$\frac{\partial Q}{\partial L} = \beta AK^{\alpha}L^{\beta-1} = \beta \frac{A\,K^{\alpha}L^{\beta}}{L} = \beta \frac{Q}{L}$, Note: $L^{\beta-1}$ can be written as $L^{\beta} \times L^{-1}$ and L^{-1} can be written as $\frac{1}{L1}$

$$\frac{\partial Q}{\partial k} = \alpha\ Ak^{\alpha-1}L^{\beta} = \alpha \frac{A\,K^{\alpha}L^{\beta}}{K} = \alpha \frac{Q}{K}$$

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{\beta\frac{Q}{L}}{\alpha\frac{Q}{K}} = \beta\ \frac{Q}{L} \times \alpha\frac{K}{Q} = \frac{\beta}{\alpha}\frac{K}{L}$$

$$MRTS_{KL} = \frac{MP_K}{MP_L} = \frac{\frac{\partial Q}{\partial K}}{\frac{\partial Q}{\partial L}} = \frac{\alpha\frac{Q}{K}}{\beta\frac{Q}{L}} = \frac{\alpha}{\beta}\frac{L}{K}$$

Fig: 5.2.8 MRTS

Illustration 5.2.11

Given a production function $Q = 6x^2 + 3xy + 2y^2$. Find MRTSxy when y = 4 and x = 5

$$MRTS_{xy} = \frac{MP_x}{MP_y} = \frac{\frac{\partial Q}{\partial x}}{\frac{\partial Q}{\partial y}}$$

Solution

$\frac{\partial Q}{\partial x}$ = 12x+3y

$\frac{\partial Q}{\partial y}$ = 3x + 4y

At y = 4 and x = 5, $\frac{\partial Q}{\partial x}$ = 12x + 3y = 12(5) + 3(4) = 72

At y = 4 and x = 5, $\frac{\partial Q}{\partial y}$ = 3x + 4y = 3(5) + 4(4) = 31

$$MRTS_{xy} = \frac{MP_x}{MP_y} = \frac{\frac{\partial Q}{\partial x}}{\frac{\partial Q}{\partial y}} = \frac{72}{31} = 2.32$$

5.2.9 Marginal Revenue and Total Revenue

The term revenue refers to the receipts obtained by a firm from the sale of certain quantities of commodities at various prices.

Revenue = Total revenue, Marginal Revenue, Average Revenue

Total revenue = Total amount of money that a firm receives from the sale of its goods.

It is the total sale proceeds of a firm by selling a commodity at given prices.

TR = P×Q where P = Price, Q = Quantity

Average Revenue (AR) = Total revenue per unit of output sold. It is obtained by dividing the total revenue by the number of output sold.

$AR = \frac{TR}{Q}$

$TR = P \times Q$, $AR = P \times \frac{Q}{Q} = P$

AR = P, Average revenue is identical with price. It is also the demand curve.

Marginal Revenue(MR) = Change in total revenue resulting from selling an additional unit of output.

It is the increase in revenue by selling additional unit of output. MR is the first derivative of TR with respect to Q.

$MR = \frac{dTR}{dQ}$

Since TR= P×Q

$$MR = \frac{dPQ}{dQ} = p.\frac{dQ}{dQ} + Q\frac{dP}{dQ}$$

$$= p + Q\frac{dP}{dQ}$$

AR = P

Therefore $MR = AR + Q\frac{dP}{dQ}$

5.2.9.1 AR and MR under perfect competition

Under perfect competition the price is determined by the market forces. Therefore, only one price prevails in the market or price is same in the whole market. Each firm can sell at the prevailing market price. Thus the demand curve or the average revenue curve is horizontal straight line. Under perfect competition MR = AR

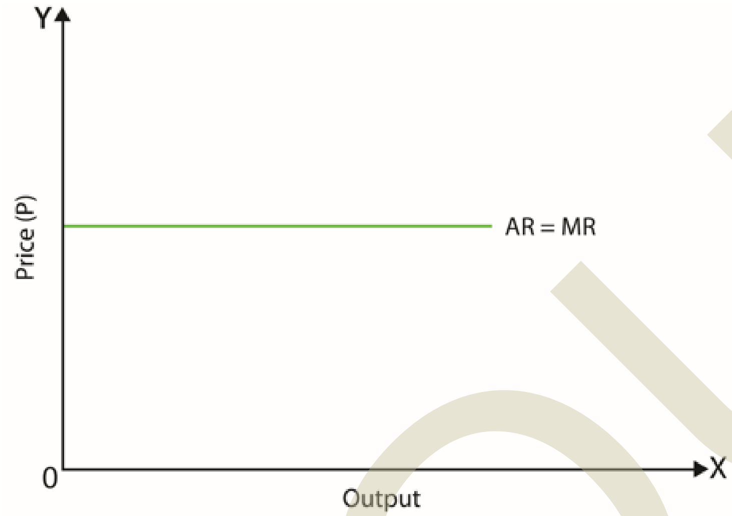

Fig: 5.2.9 AR and MR under perfect competition MR = $\frac{dTR}{dQ}$

Since TR= P×Q

MR = $\frac{dPQ}{dQ}$ = p. $\frac{dQ}{dQ}$ + Q. $\frac{dP}{dQ}$

= p + Q$\frac{dP}{dQ}$

AR = P

Therefore MR = AR + Q$\frac{dP}{dQ}$

Under perfect competition, price does not change with quantity sold. Therefore $\frac{dP}{dQ}$ = 0

So, marginal revenue is equal to price or MR= AR

As long as price is constant, MR= AR

5.2.9.2 AR and MR under imperfect competition

The demand curve facing the monopolists is downward sloping because monopolists is a price taker. Since demand curve is downward sloping, the monopolists has to reduce the price in order to increase sale of extra units. Thus, MR is less than AR. It is shown in the following figure.

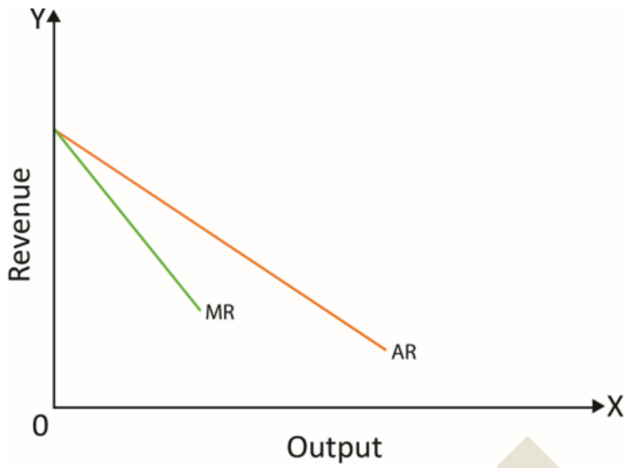

Fig: 5.2.10 MR and AR under imperfect competition

MR= $\frac{dTR}{dQ}$

Since TR= P×Q

MR = $\frac{dPQ}{dQ}$ = p. $\frac{dQ}{dQ}$ + Q$\frac{dP}{dQ}$

= p + Q$\frac{dP}{dQ}$

AR = P

Therefore MR= AR +Q$\frac{dP}{dQ}$

Under monopoly, price received will decline with quantity sold, $\frac{dP}{dQ} < 0$, So the marginal revenue will be less than price, MR < AR. Therefore, MR lies below AR.

5.2.10 Elasticity

One of the important concept used in economics is the term elasticity. It also explains the rate of change. The techniques of differentiation is used to explain the term elasticity. The application of differentiation techniques in the field of elasticity is briefly explained in this section. Elasticity measures the degree of responsiveness or rate of change. Elasticity can be divided into elasticity of demand and Elasticity of supply. Elasticity of demand explains the change in quantity demanded due to changes in factors determining demand like price, price of related commodities, income etc. Elasticity of supply explains the change in quantity supplied due to change in factors determining supply.

5.2.10.1 Elasticity of Demand

The law of demand explains the functional relationship between quantity demanded and price. The relationship between quantity demanded and price is inverse or negative. The quantity demanded increases as price decreases. The law of demand thus explains the direction of change in demand due to changes in price. The law of demand cannot give the rate of change in quantity demanded due to changes in price. The concept of elasticity is used to explain the rate of change in quantity demanded due to change in price. The elasticity or responsiveness of demand describes the responsiveness of the demand for a change in price. Elasticity of demand can also be stated as the percentage change in quantity demanded due to change in price, prices of other commodities, income, etc. The elasticity can be elastic or inelastic. Accordingly, there exists different types of elasticities of demand like (a) price elasticity of demand- change in quantity demanded due to change in price, (b) income elasticity of demand- change in quantity demanded due to change in income, (c)cross elasticity of demand- change in quantity demanded due to change in price of other commodities.

The demand for a commodity is elastic when the quantity demanded increases in greater proportion than the fall in price or quantity demanded decreases by a large amount due to a small increase in price. Inelastic demand or less elastic demand means a small increase in quantity demanded due to great fall in price and vice versa.

5.2.10.2 Types of Elasticity of Demand

The elasticity of demand can be divided into –

(1) Price elasticity of demand
(2) Cross elasticity of demand
(3) Income elasticity of demand

(1) **Price elasticity of demand**

Price elasticity of demand measured the relative change in the quantity demanded of a commodity due to the change in price. It is the ratio of proportionate change in the quantity demanded of a commodity to a proportionate change in its own price. Price elasticity of demand is obtained by dividing percentage change in quantity demanded to percentage change in its price. It is usually denoted as e_d. It is defined as follows.

$$\text{Price elasticity of demand} = \frac{\textit{Percentage change in quantity demanded}}{\textit{percentage change in Price}}$$

Mathematically

$$e_p = \frac{\frac{dQ}{Q} \times 100}{\frac{dP}{p} \times 100}$$

$$= \frac{dQ}{Q} \times \frac{p}{dP}$$

$$= \frac{dQ}{dP} \times \frac{p}{Q}$$

Where P is the price and Q is the quantity demanded

In order to indicate the inverse relationship between quantity demanded and price a negative sign is put before $\frac{dQ}{dP} \times \frac{P}{Q}$

Therefore $e_p = -\frac{dQ}{dP} \times \frac{p}{Q}$

The value of the coefficient lies between zero and infinity, $0 \leq e_p \leq \infty$

5.2.10.3 Types or Degree of Price Elasticity

Price elasticity of demand is generally divided into five types. They are (a) Perfectly elastic demand (b) Perfectly inelastic demand (c) Elastic demand (d) Inelastic demand (e) Unitary elastic demand.

(a) **Perfectly elastic demand** - A small rise in price causes the quantity demanded of a commodity to fall to zero and a small fall in price will cause an infinite increase in the quantity demanded of the commodity. The coefficient of price elasticity of demand will be infinite since dP = 0 i.e., $e_p = -\frac{dQ}{dP} \times \frac{P}{Q} = \infty$. The graph will be a line parallel to X axis as shown in the figure 5.2.11.

(b) **Perfectly inelastic demand** - Quantity demanded of the commodity remains the same whatever be the change in price (price rises or falls). Therefore, it is also known as zero elastic demand. The coefficient of the price elasticity of demand is zero. i.e., $e_p = -\frac{dQ}{dP} \times \frac{P}{Q} = 0$. The graph of perfectly inelastic demand will be parallel to Y axis as given in the 5.2.11.

(c) **Elastic demand** - Proportionate change in the quantity demanded is much greater than the proportionate change in price. It means that a small proportionate fall in price of the commodity is followed by a large proportionate increase in the quantity demanded and a small proportionate rise in price of the commodity is followed by a large proportionate fall in the quantity demanded of the commodity. The coefficient of price elasticity of demand will be greater than one. $e_p = -\frac{dQ}{dP} \times \frac{p}{Q} > 1$ since $\frac{dQ}{Q} > \frac{dP}{P}$. The graph of the curve will be a flatter curve which is downward sloping as given in the figure 5.2.11.

(d) **Inelastic demand** - Proportionate change in the quantity demanded of the commodity is less than the proportionate change in price. The greater proportionate fall in price of a commodity is followed by a smaller proportionate increase in its quantity demanded and vice versa. Numerical value of the coefficient of price elasticity of demand is less than one i.e., $e_p = -\frac{dQ}{dP} \times \frac{P}{Q} = < 1$ since $\frac{dQ}{Q} < \frac{dP}{P}$. The graph will be a steeper downward sloping curve as in figure

(e) **Unitary elastic demand** - Proportionate change in quantity of a commodity will be proportional to change in price. Proportionate fall in price is followed by an equal proportionate increase in quantity demanded. The elasticity of demand is equal to one. $e_p = -\frac{dQ}{dP} \times \frac{P}{Q} = 1$ since $\frac{dQ}{Q} = \frac{dP}{P}$

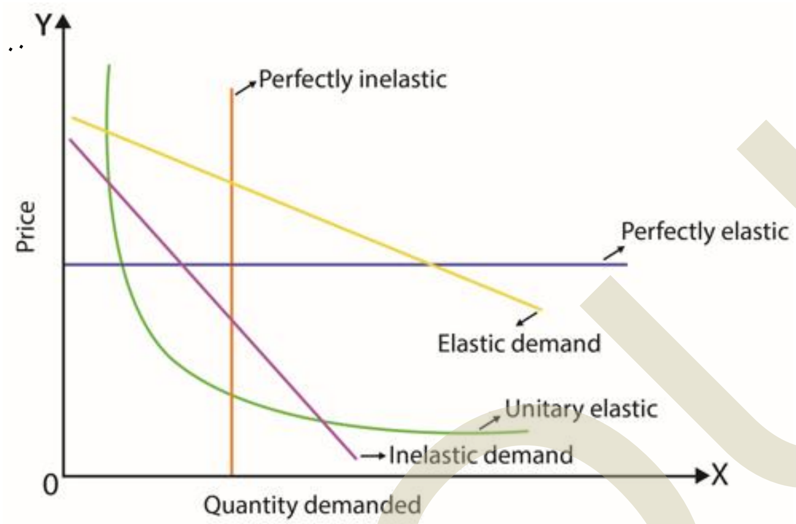

Fig: 5.2.11 Price elasticity

These are the different types of price elasticity of demand. The price elasticity of demand is measured using:

(a) Total Outlay Method or Total Expenditure Method
(b) Arc Method
(c) Point Method
(d) Percentage Method

Illustration 5.2.11

Given the demand function Q = -10P + 100, Find elasticity of demand when price P= 6

Price elasticity of demand = $\frac{dQ}{dP} \times \frac{p}{Q}$

Solution

Given Q = -10P + 100

$\frac{dQ}{dP} = \frac{d(-10P+100)}{dP} = -10$

P = 6

At P = 6, Q = -10(6) +100= 40

Substitute the above results in the formula

Price elasticity of demand = $\frac{dQ}{dP} \times \frac{p}{Q}$ = $-10 \times \frac{6}{40}$ = -10 × 0.15 = -1.5

Illustration 5.2.12

Given the demand function, P = 10 - 5Q, Find elasticity at P = 2

Price elasticity of demand = $\frac{dQ}{dP} \times \frac{P}{Q}$

The demand function is given as P = 10-5Q, rearrange it in term of Q, then we will get

Given P = 10-5Q

5Q = 10 - P

$Q = \frac{10}{5} - \frac{P}{5}$

$Q = 2 - \frac{P}{5}$

$\frac{dQ}{dP} = \frac{d(2-\frac{P}{5})}{dP} = -\frac{1}{5}$

At P= 2, $Q = 2 - \frac{P}{5} = 2 - \frac{2}{5} = \frac{8}{5}$

Substitute in the elasticity formula

$= \frac{dQ}{dP} \times \frac{P}{Q} = -\frac{1}{5} \times \frac{2}{\frac{8}{5}} = -\frac{1}{5} \times 2 \times \frac{5}{8} = -\frac{2}{8} = -0.25$

(2) Cross elasticity of demand

The second type of elasticity of demand is the cross elasticity of demand. The demand for a commodity is determined not only by its own price but also by the price of related commodities. The cross elasticity of demand is defined as ratio of proportionate change in quantity demanded of a commodity (Q) of X due to proportionate change in the price(P)of related goods (Y). The cross elasticity of demand is measured as

Cross elasticity of demand = $\frac{\text{Percentage change in quantity demanded } Qx}{\text{percentage change in Price of } Y}$

$$= \frac{dQx}{Qx} \div \frac{dPY}{PY}$$

$$= \frac{dQx}{Qx} \times \frac{PY}{dPY}$$

$$= \frac{dQx}{dPY} \times \frac{PY}{Qx}$$

$$e_c = \frac{dQx}{dPY} \times \frac{PY}{Qx}$$

The cross elasticity of demand is important in the case of commodities which are substitute and complementary. Tea and coffee are substitute goods, pen and ink are complementary goods which are jointly consumed. If the sign of cross elasticity of demand is negative, the commodities are complementary goods. If the sign of cross elasticity of demand is positive the commodities are substitutes. If cross elasticity of demand is zero, the commodities are unrelated. Cross elasticity will be infinite if two goods are perfect substitutes.

Illustration 5.2.13

Find the cross elasticity for the following demand function
$Q_1 = 10 - 2P_1 + 4P_2$ for given prices $P_1 = 2$ and $P_2 = 3$

Solution

$$e_c = \frac{dQ_1}{dP_2} \times \frac{P_2}{Q_1}$$

$$\frac{dQ1}{dP2} = \frac{d(10-2P_1 + 4P_2)}{dP_2} = 4$$

Given $P_1=2$ and $P_2 = 3$, $Q_1 = 10 - 2(2) + 4(3) = 18$

Substitute the above results into the formula

$$e_c = \frac{dQ_1}{dP_2} \times \frac{P_2}{Q_1} = 4 \times \frac{3}{18} = \frac{12}{18} = 0.66$$

since $e_c > 0$ or is positive, the goods 1 and 2 are substitutes.

Illustration 5.2.14

Find the cross elasticity of demand for the given demand function $Q_1 = 4 - 2P_1 + P_2$ and $Q_2 = 5 + P_1^2 - 2P_2$ at $P_1 = 2$ and $P_2 = 4$

Solution

We have two demand function and two prices. Therefore, two cross elasticities can be calculated, ec_1 and ec_2

$$ec_1 = \frac{dQ_1}{dP_2} \times \frac{P_2}{Q_1}$$

$$ec_2 = \frac{dQ_2}{dP_1} \times \frac{P_1}{Q_2}$$

$$ec_1 = \frac{dQ_1}{dP_2} \times \frac{P_2}{Q_1}$$

$$\frac{dQ1}{dP_2} = \frac{d(4-2P_1+P_2)}{dP_2} = 1$$

At $P_1 = 2$ and $P_2 = 4$, $Q1 = 4 - 2P_1 + P_2 = 4 - 2(2) + 4 = 4 > 0$,

$$ec_2 = \frac{dQ_2}{dP_1} \times \frac{P_1}{Q_2}$$

$$\frac{dQ_2}{dP_1} = \frac{d}{dP_1}(5 + P_1^2 - 2P_2) = 2P_1$$

At $P_1 = 2$ and $P_2 = 4$, $Q_2 = 5 + 2^2 - 2(4) = 1 > 0$

Since ec_1 and ec_2 are positive, good 1 and good 2 are substitutes

(3) Income elasticity of demand

Another important elasticity of demand is the income elasticity of demand. The demand for a commodity may also change due to change in the income of the consumer. Income elasticity of demand is defined as the proportionate change in quantity demanded of a commodity to proportionate change in income of the consumer. Thus the income elasticity of demand can be written as:

$$\text{Income elasticity of demand} = \frac{\textit{Percentage change in quantity demanded } Q}{\textit{Percentage change in income}}$$

$$e_y = \frac{dQ}{Q} \div \frac{dY}{Y}$$

$$= \frac{dQ}{Q} \times \frac{Y}{dY}$$

$$= \frac{dQ}{dY} \times \frac{Y}{Q}$$

$$e_y = \frac{dQ}{dY} \times \frac{Y}{Q}$$

If the income elasticity is positive, such commodities are normal goods. A commodity is luxury good, if income elasticity is greater than one and if the income elasticity is less than one, such commodities are known as necessities.

Illustration 5.2.15

Find the income elasticity of demand given the demand function Q = 20 – 3p + 0.5y, for given p = 2, y = 100

Solution

$$e_y = \frac{dQ}{dY} \times \frac{y}{Q}$$

$$\frac{dQ}{dY} = \frac{d}{dY}(20 - 3p + 0.5y) = 0.5$$

Given p = 2, y = 100, Q = 20 – 3p + 0.5y

Q = 20 – 3p + 0.5y = 20 - 3(2) + 0.5(100)

$= 20 - 6 + 50 = 64$

Substitute the values in the formula

$e_y = \frac{dQ}{dY} \times \frac{y}{Q} = 0.5 \times \frac{100}{64} = 0.5 \times 1.5625 = 0.7813$

Illustration 5.2.16

The demand function for a commodity is $Q = 15 - 2p_1 + p_2 + 0.4y$ for given prices $p_1 = 2$, $p_2 = 3$ and $y = 200$, find the income elasticity of demand.

Solution

$e_y = \frac{dQ}{dY} \times \frac{y}{Q}$

$\frac{dQ}{dY} = \frac{d}{dY}(15 - 2p_1 + p_2 + 0.4y) = 0.4$

Substitute $p_1 = 2$, $p_2 = 3$ and $y = 200$, in the demand function to get Q

$Q = 15 - 2p_1 + p_2 + 0.4y = 15 - 2(2) + 3 + 0.4(200) = 15 - 4 + 3 + 80 = 94$

$e_y = \frac{dQ}{dY} \times \frac{y}{Q} = 0.4 \times \frac{200}{94} = 0.4 \times 2.1276 = 0.85$

5.2.10.4 Elasticity of Supply

The elasticity of supply is the degree of responsiveness of quantity supplied of a good to a change in its own price. It is the response of the producer to a change in the price of the product. It is defined as the ratio of proportionate change in amount supplied to a proportionate change in price.

Elasticity of supply = $\frac{\textit{Percentage change in quantity supplied } Q_s}{\textit{Percentage change in price}}$

$e_s = \frac{dQ_s}{Q_s} \div \frac{dp}{p}$

$= \frac{dQ_s}{Q_s} \times \frac{P}{dP}$

$= \frac{dQ_s}{dP} \times \frac{P}{Q_s}$

$e_s = \frac{dQ_s}{dP} \times \frac{P}{Q_s}$

Illustration 5.2.17

Given the supply function $Q_s = 3 + 2p + 6$, find elasticity of supply at p = 2.

Solution

$$e_s = \frac{dQ_s}{dP} \times \frac{P}{Q_s}$$

$$\frac{dQ_s}{dP} = \frac{d}{dP}(3 + 2p + 6) = 2$$

Given p = 2, $Q_s = 3 + 2p + 6 = 3 + 2(2) + 6 = 13$

Substitute these values in $e_s = \frac{dQ_s}{dP} \times \frac{P}{Q_s}$

$$= 2 \times \frac{2}{13} = 2 \times 0.1538 = 0.3076$$

Illustration 5.2.18

Find the elasticity of supply given $Q_s = 12 + 4p + p^2$ when p = 3.

Solution

$$e_s = \frac{dQ_s}{dP} \times \frac{P}{Q_s}$$

$$\frac{dQ_s}{dP} = \frac{d}{dP}(12 + 4p + p^2) = 4 + 2p$$

At p = 3, $\frac{dQ_s}{dP} = 4 + 2(3) = 10$

When p = 3, Q_s is

$Q_s = 12 + 4p + p^2 = 12 + 4(3) + (3)^2 = 12 + 12 + 9 = 33$

Substitute these values in $e_s = \frac{dQ_s}{dP} \times \frac{P}{Q_s}$

$$= 10 \times \frac{3}{33} = 10 \times 0.0909 = 0.909$$

5.2.10.5 Relationship between AR, MR and Elasticity

$$MR = \frac{dTR}{dQ}$$

Since TR= PQ

$$MR = \frac{dPQ}{dQ} = P.\frac{dQ}{dQ} + Q\frac{dP}{dQ}$$

$$MR = P + Q\frac{dP}{dQ}$$

The price elasticity of demand is defined as $e = \frac{dQ}{dp} \cdot \frac{P}{Q}$

Rearranging we obtain $- e \frac{Q}{P} = \frac{dQ}{dP}$ or $- \frac{P}{eQ} = \frac{dP}{dQ}$

Substitute $- \frac{P}{eQ} = \frac{dP}{dQ}$ in MR, $MR = P + Q\frac{dP}{dQ} = P - Q\frac{P}{eQ} = P - \frac{P}{e}$, $MR = P(1 - \frac{1}{e})$

If elasticity is one, MR=0

$MR = P(1- \frac{1}{e})$

$MR = AR\ (1- \frac{1}{e})$ since P = AR

$MR = AR\ (\frac{e-1}{e})$

$MR\ (\frac{e}{e-1}) = AR$

$e = (\frac{AR}{AR-MR})$

Illustration 5.2.19

The demand curve for commodity x is given as P = 12 - 3x. Find Total revenue, Average Revenue, and marginal revenue at x = 2

Solution

Given P = 12 - 3x

Total revenue (TR) $= PX = \frac{(12-3x)}{x} = 12x - 3x^2$

at x = 2, $TR= 12(2) -3(2^2) = 24 - 12 = 12$

Average Revenue (AR) $= \frac{TR}{X} = \frac{(12x - 3x^2)}{x} = 12 - 3x$

At x = 2, AR = 12 - 3(2) = 6

Marginal revenue (MR) $= \frac{dTR}{dQ} = \frac{d}{dQ}(12x - 3x^2) = 12 - 6x$

At x = 2, MR = 12 - 6(2) = 0, i.e. at x = 2 TR is maximum

Illustration 5.2.20

Total revenue function is $TR= 50Q - Q^2$. Find average and marginal revenue functions.

Solution

$AR = \frac{TR}{Q} = \frac{(50Q - Q^2)}{Q} = 50 - Q$

$$MR = \frac{dTR}{dQ} = \frac{d}{dQ}(50Q - Q^2) = 50 - 2Q$$

Illustration 5.2.21

Given the demand function

Q = -5p + 100. Find price elasticity of demand at p=5.

Solution

$$e_p = \frac{dQ}{dp} \cdot \frac{p}{Q}$$

$$\frac{dQ}{dp} = \frac{d(-5p+100)}{dp} = -5$$

q at p= 5, -5 × 5 + 100 = 75

$$e = -5 \times \frac{5}{75} = \frac{25}{75} = 0.333$$

$$Q\frac{16}{P^3} = = 16P^{-3}$$

$$e = \frac{dQ}{dp} \cdot \frac{p}{Q}$$

$$= - \frac{48}{P^4} \cdot \frac{5}{\frac{16}{P^3}}$$

Rearranging,

$$= - \frac{48}{P^4} \cdot \frac{5.P^3}{16}$$

$$= - \frac{48}{5^4} \cdot \frac{5.5^3}{16}$$

$$= - \frac{48}{16} = - 3$$

Illustration 5.2.22

If MR is 10 and the elasticity of demand with respect to price is 2, find average revenue.

Solution

$$MR = AR(1 - \frac{1}{e})$$

$$\frac{MR}{AR} = 1 - \frac{1}{e}$$

$$\frac{10}{AR} = 1 - \frac{1}{2}$$

$$\frac{10}{AR} = \frac{1}{2}$$

20 = AR

Therefore AR = 20

Recap

- Marginal concepts - studying the effects of small, incremental changes in economic variables
- Marginal cost - additional cost of producing one more unit
- Marginal utility - extra satisfaction from consuming one more unit
- Marginal analysis helps in decision-making
- Total concepts deal with accumulated or overall measures
- Total cost - overall cost of producing a given quantity
- Total utility - total satisfaction from consuming all units
- Total revenue - overall income from selling a certain quantity
- Total analysis helps understand the overall impact of economic decisions
- Total analysis provides a broader perspective in micro and macroeconomics

Objective Questions

1. What does the term "marginal" refer to in economics?
2. What does the marginal cost represent in production economics?
3. Represent MR mathematically
4. What does the total cost represent in economics?
5. What is total utility?
6. Represent TR mathematically.
7. What is the total product of labour?
8. What is the formula for calculating total profit?
9. If marginal cost is less than average total cost, what happens to total cost as more units are produced?

10. How do you find the total utility when given the marginal utility for each unit of a product consumed?
11. Give the mathematical equation for price elasticity of demand.

Answers

1. Small incremental changes in a variable.
2. The additional cost of producing one more unit of a good or service.
3. MR = $\frac{dTR}{dx}$
4. The overall cost of producing a certain quantity of goods or services.
5. TU is sum of utilities.
6. TR = P × Q
7. The total quantity of output produced by all units of labour employed.
8. Total Profit = Total Revenue - Total Cost.
9. Total cost decreases.
10. Sum all the marginal utilities for each unit consumed.
11. $\frac{dQ}{dP} \cdot \frac{-P}{Q}$

Assignments

1. Define and explain the concept of Marginal Cost (MC) and Marginal Revenue (MR) in economics. Why are these concepts important for firms?
2. Describe the relationship between Marginal Product (MP), Average Product (AP), and Total Product (TP) in the production process. How can calculus be used to analyse these concepts?
3. Given a demand function for a product, D(p) = 100 - 2p, calculate the marginal revenue function and explain how it relates to the demand function.

Suggested Readings

1. N. Gregory Mankiw (2022): "Principles of Economics", Cengage publication,8th edition
2. Hal R. Varian (2001): "Intermediate Microeconomics: A Modern Approach", W.W Norton & Co Inc
3. Carl P. Simon, Lawrence Blume (2018): "Mathematics for Economists", Viva books

References

1. N. Gregory Mankiw (2015): "Intermediate Macroeconomics", Worth publication
2. Geoffrey A. Jehle and Philip J. Reny (2008): "Advanced Microeconomic Theory", Pearson publication, second edition
3. Paul Krugman, Robin Wells (2018): “Microeconomics" and "Macroeconomics", Worth publication, 5th edition

6

BLOCK

Integral Calculus

UNIT 1

Meaning and Rules of Integration

Learning Outcomes

After completing this unit, the learner will be able to

- understand the fundamental concept of integration
- understand the importance of integration in economic analysis
- learn the rules for integration
- apply substitution methods for integration

Prerequisites

Let us imagine economics as a bustling market. Prices fluctuate, and commodities are constantly changing hands. To navigate this complex marketplace successfully, you need a tool—a tool that can help you calculate the total value of transactions, assess profits, and understand market dynamics. This tool is integration. It helps us add up lots of small changes in a way that makes sense, like adding up all the prices in our market. It allows you to sum up even minute changes, combining them into a comprehensive whole. But to wield this wand effectively, you must first understand the rules of integration. Now, think of integration substitution as a special trick. It is like looking at things from a different angle. When we use substitution, it is like using a secret code that helps us simplify complex problems. It is like making the puzzle pieces fit together better.

By understanding integration and substitution, you get the keys to unlock the secrets of economics. You can figure out how things in the economy work, make predictions, and smart decisions. It is like becoming a great navigator in the world of economics, guiding yourself through the twists and turns of economic data, finding patterns, and helping societies and businesses thrive. So, get ready for this adventure with an open mind, because with integration and substitution, you are about to solve the puzzle of economics and make smart choices.

Keywords

Integrals, Rules of Integration, Substitution

Discussion

6.1.1 Meaning of Integrals

In the study of changing economic views, we are concerned with achieving equilibrium positions for certain economic phenomena through adjustments over time. This examination of economic changes involves considering time in two ways: (a) as a continuous variable and (b) as a discrete variable. In the case of a continuous variable, the variable changes at each point in time, while with a discrete variable, the variable changes only once within a period of time. When dealing with continuous variables, we should utilise the mathematical technique of integration. In this section, we will discuss the meaning and rules of integration, as well as integration by substitution.

In simple terms, integration is the precise opposite of differentiation. Integration or antidifferentiation is reversing the process of differentiation and finding the original function from the derivative. When a function is y = f (x) is differentiated, we get f'(x) or $\frac{dy}{dx}$. For example, $y = f(x) = x^2 - 4x + 20$, on differentiation, it gives f'(x) or $\frac{dy}{dx} = 2x - 4$. Integration as the inverse operation of differentiation and it is the technique of finding the function y = f (x) from a given derivative f'(x) or $\frac{dy}{dx}$. As per the above example, the integration of $f'(x) = 2x - 4$ should give the original function

$y = f(x) = x^2 - 4x + 20$.

Now, a problem arises that the same derivative $f'(x) = 2x - 4$ may be obtained from the differentiation of many functions such as,

$f(x) = x^2 - 4x + 50$, $f(x) = x^2 - 4x - 500$, $f(x) = x^2 - 4x + 104$ etc. and in general term

$f(x) = x^2 - 4x + c$

Where c is an arbitrary constant and may assume any value. So when we integrate f'(x) = 2x – 4, we can only get back the function $y = f(x) = x^2 - 4x + c$ instead of $y = f(x) = x^2 - 4x + 20$. So in order to obtain the original function $y = f(x) = x^2 - 4x + 20$ from f' (x) = 2x – 4, we must have additional information, known as initial condition, that will give the definite value of c as c = 20. As per the above example, we need to have an initial information that when x = 0, then f (x) = 20, i.e.;

$$f(x) = x^2 - 4x + 20$$
$$f(x) = f(0) = (0)^2 - 4(0) + c = 20$$
$$\text{or } c = 20$$

In the above discussion, we have used the symbol f (x) as original function and f' (x) as its derivative. But in this section, i.e., integral calculus or in the case of integration we need to modify the symbols.

Normally in integral calculus the original and derivative functions are denoted by F (x) and f (x) respectively instead of f (x) and f' (x). This means that the differentiation of F (x) will yield F' (x) = f (x) and integration of F' (x) = f (x) will give F (x) + c, where c is an arbitrary constant. The function F (x) is referred to as integral or antiderivative of the function f (x) and c is called constant of integration and a symbol " ∫ " is used to indicate the integration.

Letting f (x) = F' (x) for simplicity, the antiderivative of f(x) is expressed mathematically as;

$$\int f(x)\,dx = F(x) + c$$

Here, the symbol ∫ is an integral sign, f (x) is the integrand, dx is the integration operation to be performed with respect to x variable and c is the constant of integration. Here, the left hand side of the equation is called "the indefinite integral f(x) with respect to x", because $\int f(x)\,dx = F(x) + c$, we can refer to $\int f(x)\,dx$ as an integral of f (x). Since integral $\int f(x)\,dx$ is expressed as a function of x in the form F (x) + c with no definite numerical value.

6.1.2 Rules of Integration

The rules of integration depend on the rules of differentiation. In the coming section, we will discuss the basic rules of integration that will be obtained by reversing the corresponding rules of differentiation. Their accuracy is easily checked since the derivative of the integral must equal the integrand.

Rule 1. The integral of a constant k is

$$\int k\,dx = kx + c$$

Illustration 6.1.1

Determine the following integrals and check the answer on your own by making sure that the derivative of the integral equals the integrand.

(i) $\int 5\, dx$

Solution: $\int 5\, dx = 5x + c$

(ii) $\int 8.5\, dx$

Solution: $\int 8.5\, dx = 8.5x + c$

Rule 2. The integral of 1, written simply as dx, not 1 dx, is $\int dx = x + c$

Rule 3. Power Rule of Integration

The integral of a power function x^n, where $n \neq -1$ is given by:

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + c, \ (n \neq -1)$$

Illustration 6.1.2

Determine the following integrals

(i) $\int x^4\, dx$

Solution: $\int x^4\, dx = \frac{1}{4+1} x^{4+1} + c = \frac{1}{5} x^5 + c$

(ii) $\int x^{\frac{2}{3}}\, dx$

Solution: $\int x^{\frac{2}{3}}\, dx = \frac{1}{\frac{2}{3}+1} x^{\frac{2}{3}+1} + c = \frac{3}{5} x^{\frac{5}{3}} + c$

(iii) $\int 5x^{-3}\, dx$

Solution: $\int 5x^{-3}\, dx = \frac{5}{-3+1} x^{-3+1} + c = -\frac{5}{2} x^{-2} + c$

(iv) $\int x^{\frac{-7}{2}}\, dx$

Solution: $\int x^{\frac{-7}{2}}\, dx = \frac{1}{-\frac{7}{2}+1} x^{\frac{-7}{2}+1} + c = -\frac{2}{5} x^{\frac{-5}{2}} + c$

(v) $\int \sqrt{x}\, dx$

Solution: $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \frac{2}{3} x^{\frac{3}{2}} + c$

It should be noted that we cannot use the power rule in case of $\int \frac{1}{x} dx$ since $\int \frac{1}{x} dx$ = $\int x^{-1} dx$ and n = -1. When n = -1, the power rule cannot be applied.

Rule 4. Logarithmic Rule of Integration

The integral of x^{-1} is (or $\frac{1}{x}$) is

$$\int x^{-1} dx = \ln x + c \qquad (x > 0)$$

The condition x > 0 is added because only positive numbers have logarithms. For negative numbers,

$$\int x^{-1} dx = \ln |x| + c \qquad (x \neq 0)$$

Illustration 6.1.3

Determine the following integral

(i) $\int \frac{dx}{x}$

Solution: $\int \frac{dx}{x} = \int \frac{1}{x}\ dx = \ln |x| + c$

Rule 5. Exponential Rule of Integration

The integral of an exponential function is

$$\int a^{kx} dx = \frac{a^{kx}}{k \ln a} + c$$

Illustration 6.1.4

Determine the following integral

(i) $\int 3^{4x} dx$

Solution: $\int 3^{4x} dx = \frac{3^{4x}}{4 \ln 3} + c$

Rule 6. The integral of a natural exponential function is

$$\int e^{kx} dx = \frac{e^{kx}}{k} + c \quad \text{since } \ln e = 1$$

Illustration 6.1.5

Determine the following integral

(i) $\int 12e^{-4x}\,dx$

Solution: $\int 12e^{-4x}\,dx = \frac{12e^{-4x}}{-4} + c = -3e^{-4x} + c$

Rule 7. The Constant Multiple Rule of Integration

The integral of a constant time a function equals the constant times the integral of the function

$\int k\, f(x)\,dx = k \int f(x)\,dx$

Illustration 6.1.6

(i) Find $\int 20x^4\,dx$

Solution: $\int 20x^4\,dx = 20 \int x^4\,dx = \frac{20}{5} x^5 + c = 4x^5 + c$

(ii) Find $\int 10\sqrt{x}\,dx$

Solution: $\int 10\sqrt{x}\,dx = 10 \int x^{\frac{1}{2}}\,dx = 10\, \frac{1}{\frac{1}{2}+1}\, x^{\frac{1}{2}+1} + c = \frac{20}{3} x^{\frac{3}{2}} + c$

Rule 8. The Sum Rule of Integration

The integral of the sum or difference of two or more functions equals the sum or difference of their integrals

$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Illustration 6.1.7

(i) Find $\int (4x^2 + x^4)\,dx$

Solution: $\int (4x^2 + x^4)\,dx = \int 4x^2\,dx + \int x^4\,dx = \frac{4}{3} x^3 + \frac{1}{5} x^5 + c$

(ii) Find $\int (10x + \frac{4}{x})\,dx$

Solution: $\int (10x + \frac{4}{x})\,dx = \int 10x\,dx + \int 4\frac{1}{x}\,dx = \frac{10}{2} x^2 + 4 \ln x + c$

Rule 9. The negative Rule of Integration

The integral of the negative of a function equals the negative of the integral of that function

$\int -f(x)\,dx = -\int f(x)\,dx$

Illustration 6.1.8

Find $\int -\frac{1}{4}\,dx$

Solution: $\int -\frac{1}{4}\,dx = \int -\frac{1}{4}\,dx = -\frac{1}{4}x + c$

In the above example, we have applied the rules 1 and 9.

Example 10. (i) Find $\int 7x^{-1}\,dx$

Solution: $\int 7x^{-1}\,dx = 7\ln|x| + c$

In the above example, we have applied the rules 7 and 4.

Illustration 6.1.9

Find $\int (10x^6 + 4x^4 + 6x)\,dx$

Solution

$$\int (10x^6 + 4x^4 + 6x)\,dx = 10\int x^6\,dx + 4\int x^4\,dx + 6\int x\,dx$$

$$= 10\left(\frac{1}{7}x^7\right) + 4\left(\frac{1}{5}x^5\right) + 6\left(\frac{1}{2}x^2\right) + c$$

$$= \frac{10}{7}x^7 + \frac{4}{5}x^5 + \frac{6}{2}x^2 + c$$

In the above example, we have applied the rules 7, 8 and 3.

Illustration 6.1.10

Find $\int (4x^6 - 6x^4)\,dx$

Solution

$\int (4x^6 - 6x^4)\,dx = \frac{4}{7}x^7 - \frac{6}{5}x^5) + c$

In the above example, we have applied the rules 3, 7, 8 and 9.

6.1.3 Integration by Substitution

The integration by substitution is one of the techniques of transforming a complex integrand into a standard form to use the standard rules of integration. We have already discussed many rules of integration in the above section and that rules of integration may not have direct application in case of complex variety of integrands such as product or quotient of two functions. Although it is quite easy to differentiate a function involving the product or quotient of two functions using standard rules, but it is not an easy task to integrate an expression which is the product or quotient of two functions using the rules of integral in a upfront way. So, we need to transform such an expression of the integrand into a standard form to use the standard rules of integration. That process can be done through the technique of integration by substitution. Another technique of transforming a complex integrand into a standard form to use the standard rules of integration is integration by parts.

Integration of a product or quotient of two differentiable functions of x, such as
$\int 24x^2(x^4 + 3)\, dx$

If the integrand can be expressed as a constant multiple of another function u and its derivative $\frac{du}{dx}$, integration by substitution is possible. By expressing the integrand f(x) as a function of u and its derivative $\frac{du}{dx}$ and integrating with respect to x,

$\int f(x)\, dx = \int (u \frac{du}{dx})\, dx$

$\int f(x)\, dx = \int u\, du = F(u) + c$

The substitution method reverses the operation of the chain rule and the generalized power function rule in differential calculus.

Illustration 6.1.11

The substitution method is used to determine the indefinite integral
$\int 24x^3(x^4 + 3)\, dx$

Solution:

Steps :

1. Be sure that the integrand can be converted to a product of another function u and its derivative $\frac{du}{dx}$ times a constant multiple.

    a) Let u equal the function in which the independent variable is raised to the higher power in terms of absolute value: here let $u = x^4 + 3$

    b) Take the derivative of u; $\frac{du}{dx} = 4x^3$

    c) Solve algebraically for dx; $dx = \frac{du}{4x^3}$

    d) Then substitute u for $x^4 + 3$ and $\frac{du}{4x^3}$ for dx in the original integrand:

    $$\int 24x^3(x^4 + 3)\, dx = \int 24x^3 \,.\, u. \frac{du}{4x^3} = \int 6u\, du = 6 \int u\, du$$

Where 6 is a constant multiple of u.

2. Integrate with respect to u, using Rule 3 and ignoring c in the first step of the calculation.

$$\int 6u\, du = 6\,(\tfrac{1}{2}\, u^2) = 3u^2 + c$$

3. Convert back to the terms of the original problem by substituting $x^4 + 3$ for u.

$$\int 24x^3(x^4 + 3)\, dx = 3u^2 + c = 3\,(x^4 + 3)^2 + c$$

4. Check the answer by differentiating with the generalized power function rule or chain rule.

$$\frac{d}{dx}\,[3(x^4 + 3)^2 + c] = 6\,(x^4 + 3)(4x^3) = 24\, x^3\,(x^4 + 3)$$

Illustration 6.1.12

Determine the following integral, using the substitution method. Check the answer on your own.

$\int 12x\,(x^2 + 5)^6\, dx$

Solution

Let $u = x^2 + 5$. Then $\frac{du}{dx} = 2x$ and $dx = \frac{du}{2x}$. Substituting in the original integrand to reduce it to a function of u $\frac{du}{dx}$.

$\int 12x\,(x^2 + 5)^6\, dx = \int 12x\, u^6 \frac{du}{2x} = 6 \int u^6\, du$

Integrating by the power rule, $6 \int u^6\, du = 6\,(\frac{1}{7}\, u^7)$

Substituting $u = x^2 + 5$, $\int 12x\,(x^2 + 5)^6\, dx = \frac{6}{7}\,(x^2 + 5)^7 + c$

Illustration 6.1.13

Determine the integral $\int \frac{x^2}{(6x^3+9)^2}\, dx$

Solution

$\int \frac{x^2}{(6x^3+9)^2}\, dx = \int x^2\,(6x^3 + 9)^{-2}\, dx$

Let $u = 6x^3 + 9$. Then $\frac{du}{dx} = 18x^2$ and $dx = \frac{du}{18x^2}$. Substituting,

$\int x^2 (6x^3 + 9)^{-2}\, dx = \int x^2 \; u^{-2} \frac{du}{18x^2} = \frac{1}{18} \int u^{-2}\, du$

Integrating, $\frac{1}{18} \int u^{-2}\, du = - \frac{1}{18} \; u^{-1} + c$

Substituting $u = 6x^3 + 9$, $\int \frac{x^2}{(6x^3+9)^2} dx = - \frac{1}{18(6x^3+9)} + c$

Illustration 6.1.14

Determine the integral $\int \frac{6x^2+4}{8x^3+16x} dx$

Solution

Let $u = 8x^3 + 16x$, Then $\frac{du}{dx} = 24x^2 + 16$ and $dx = \frac{du}{24x^2+16}$. Substituting,

$\int 6x^2 + 4)\, u^{-1} \frac{du}{24x^2+16} = \frac{1}{4} \int u^{-1}\, du$

Integrating, $\frac{1}{4} \int u^{-1}\, du = \frac{1}{4} \ln|u| + c$

Substituting $\int \frac{6x^2+4}{8x^3+16x} dx = - \frac{1}{4} \ln|8x^3 + 16x| + c$

Illustration 6.1.15

Determine the integral $\int 18e^{3x+9}\, dx$

Solution

Let $u = 3x + 9$, Then $\frac{du}{dx} = 3$ and $dx = \frac{du}{3}$. Substituting,

$\int 18\, e^u \frac{du}{3} = 6 \int \; e^u\, du = 6\, e^u + c$

Substituting $\int 18e^{3x+9}\, dx = 6\, e^{3x+9} + c$

Recap

- Integration is the precise opposite of differentiation
- The integral of a constant k is $\int k\,dx = kx + c$
- The integral of 1, written simply as dx, not 1 dx, is $\int dx = x + c$
- The integral of a power function x^n, where $n \neq -1$ is given by $\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c,\ (n \neq -1)$
- The integral of x is $^{-1}$ (or $\frac{1}{x}$) is $\int x^{-1}\,dx = \ln x + c,\ (x > 0)$. The condition $x > 0$ is added because only positive numbers have logarithms. For negative numbers, $\int x^{-1}\,dx = \ln|x| + c,\ (x \neq 0)$
- The integral of an exponential function is $\int a^{kx}\,dx = \frac{a^{kx}}{k \ln a} + c$
- The integral of a natural exponential function is $\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$ since ln e =1
- The integral of a constant times a function equals the constant times the integral of the function $\int k\,f(x)\,dx = k \int f(x)\,dx$
- The integral of the sum or difference of two or more functions equals the sum or difference of their integrals $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
- The integral of the negative of a function equals the negative of the integral of that function $\int -f(x)\,dx = -\int f(x)\,dx$
- The integration by substitution is one of the techniques of transforming a complex integrand into a standard form to use the standard rules of integration
- Another technique of transforming a complex integrand into a standard

Objective Questions

1. Find $\int 6x^{-2}$ dx.
2. Find $\int \frac{dx}{\sqrt[3]{x}}$
3. Find $\int 10^x$ dx.
4. Find $\frac{1}{x^2}$ dx.
5. Determine the integral using the substitution method, $\int 4\,(e^{2x} + x)\,(e^{2x} + x^2)$ dx.
6. Find $\int (x^{\frac{2}{3}} + 1)\, dx$
7. Find $\int (4x^3 + 6)\, dx$
8. Find $\int (3x^2 + 4x^3)\, dx$
9. What is integration?

Answers

1. $\frac{6}{x} + c$
2. $\frac{3}{2} x^{\frac{2}{3}} + c$
3. $\frac{10^x}{\ln 10} + c$
4. $-\frac{1}{x} + c$
5. $\frac{2}{3}(e^{2x} + x^2)^2 + c$
6. $\frac{3}{5} x^{\frac{5}{3}} + x + c$
7. $x^4 + 6x + c$
8. $x^3 + x^4 + c$
9. Integration or antidifferentiation is reversing the process of differentiation and finding the original function from the derivative.

Assignments

1. Explain the fundamental concept of integration by substitution. Provide a step-by-step description of the process.
2. Solve the following
   a. $\int x^2(1 - \frac{1}{x^2})\, dx$
   b. $\int \frac{x^3 + 5x^2 - 4}{x^2}\, dx$
   c. $\int (1 - x)\sqrt{x}\, dx$
3. Explain the rules of integration with examples

Suggested Readings

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UNIT 2

Integration by Parts

Learning Outcomes

After completing this unit, the learner will be able to

- understand the integral of products of functions
- know how to use the integration by parts formula
- learn the properties of integrals
- evaluate definite integrals

Prerequisites

Imagine you are a chef preparing a grand feast. In your kitchen, you have various ingredients and recipes at your disposal. However, to create a masterpiece, you need to understand the precise measurements, cooking techniques, and how each ingredient blends. Just like cooking, economics requires a set of skills, and that's where Integration by Parts and Definite and Indefinite Integrals come into play. Integration by Parts is your recipe for dissecting economic problems. It helps you break down complex economic scenarios into manageable steps. You will learn how to peel away layers of data, revealing the hidden flavours of economic relationships. Much like a chef uncovering the perfect blend of spices, you will expose the essential components of economic systems. Definite and Indefinite Integrals are your kitchen tools, much like a chef's knives. They allow you to slice through the raw data and compute precise results. With them, you can measure the economic performance, understand the growth of investments, and calculate the impact of policies.

So, why learn these topics? Because, just like a chef who needs the right skills to create a culinary masterpiece, you need Integration by Parts and Definite and Indefinite Integrals to excel in economics. They are the tools that help you create a detailed economic analysis, make informed decisions, and discover the intricacies of economic phenomena. As you embark on this journey, you will be equipped with the skills to investigate economic challenges, predict market trends, and unravel the stories hidden in economic data. These tools are your key to mastering the art of economic analysis and making a significant impact in the world of economics. Your journey through Integration by Parts and Definite and Indefinite Integrals is like crafting a unique, data-driven dish that leaves a lasting impression on the economic landscape.

Keywords

Integration by parts, Definite Integral, Indefinite Integral

Discussion

6.2.1 Introduction by parts

In the above section we have discussed the integration by substitution, which is one of the techniques of transforming a complex integrand into a standard form to use the standard rules of integration. Another technique of transforming a complex integrand into a standard form to use the standard rules of integration is integration by parts. If an integrand is a product or quotient of differentiable functions of x and cannot be expressed as a constant multiple of u du/dx, integration by parts is frequently useful. The method is derived by reversing the process of differentiating a product. From the product rule of differentiation (f'(x) = g(x).h'(x) + h (x). g'(x)).

$$\frac{d}{dx}[f(x)\, g(x)] = f(x)\, g'(x) + g(x)\, f'(x)$$

Taking the integral of the derivative gives

$$f(x)\, g(x) = \int f(x)\, g'(x)\, dx + \int g(x)\, f'(x)\, dx$$

Then solving algebraically for the first integral on the right-hand side,

$$\int f(x)\, g'(x)\, dx = \mathbf{f(x)\, g(x) - \int g(x)\, f'(x)\, dx}$$

The integration of (First Function x Second Function) = (First Function) x (Integration of Second Function) – Integration of (Differentiation of First Function x Integration of Second Function).

In the context of using the integration by parts formula, the choice of which function to designate as the first function and which to designate as the second function is guided by a specific priority order, often referred to as "LIATE." Here is the order one should consider:

- Logarithmic functions (L)
- Inverse trigonometric functions (I)
- Algebraic functions (A)
- Trigonometric functions (T)
- Exponential functions (E)

The integration by parts is explained through the following examples.

Example 1. The Integration by parts is used below to determine $\int 8x(x + 3)^3\, dx$

Solution:

Steps :

1. Separate the integrand into two parts amenable to the formula $\int f(x)dx = F(x) + c$. Generally, consider first the simpler function for f(x) and the more complicated function for g'(x). By letting $f(x) = 8x$ and $g'(x) = (x + 3)^3$, then $f'(x) = 8$ and $g(x) = \int (x + 3)^3\, dx$, which can be integrated by using the simple power rule (Rule 3):

   $g(x) = \int (x + 3)^3\, dx = \frac{1}{4}(x + 3)^4 + c_1$

2. Substitute the values for f(x), f'(x) and g(x) in $\int f\,(x)\,dx = F\,(x) + c$; and note that g'(x) is not used in the formula

   $\int 8x(x + 3)^3\, dx = f(x).\, g(x) - \int [g(x).\, f'(x)]\, dx$

   $= 8x\, [\frac{1}{4}\,(x + 3)^4 + c_1] - \int[\frac{1}{4}(x + 3)^4 + c_1]\,(8)\, dx$

   $= 2x\,(x + 3)^4 + 8c_1\, x - \int[\frac{1}{4}(x + 3)^4 \times 8 + c_1 \times 8]\, dx$

   $= \int\, [2(x + 3)^4 + 8c_1]\, dx$

3. Use Rule 3 to compute the final integral and substitute.

   $\int 8x(x + 3)^4\, dx = \; 2x\,(x + 3)^4 + 8c_1\, x - 2\times \frac{1}{5}(x + 3)^5 - 8c_1\, x + c$

   $=2x(x + 3)^4 - \frac{2}{5}(x + 3)^5 + c$

Note that the c_1 term does not appear in the final solution. Since this is common to integration by parts, c_1 will henceforth be assumed equal to 0.

4. Check the answer by letting $y(x) = 2x\,(x+3)^4 - \frac{1}{5}(x+3)^5 + c$ and using the product and generalized power function rules.
   $= y'(x) = [\,x.\,8\,(x+3)^3 + (x+3)^4.1] - (x+3)^4 = 8x\,(x+3)^3$

Illustration 6.2.1

(i) Use integration by parts to evaluate the following integral. Check the answer on your own.

$\int 25x\,(x+6)^{\frac{3}{2}}\,dx$

Solution

Let $f(x) = 25x$, then $f'(x) = 25$. Let $g'(x) = (x+6)^{\frac{3}{2}}$, then $g(x) = \int (x+6)^{\frac{3}{2}}dx$

$= \frac{2}{5}(x+6)^{\frac{5}{2}}$

Substituting, $\int 25x\,(x+6)^{\frac{3}{2}}dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx$

$= 25x\,[\,\frac{2}{5}(x+6)^{\frac{5}{2}}] - \int \frac{2}{5}(x+6)^{\frac{5}{2}}\,25dx = 10x(x+6)^{\frac{5}{2}} - 10\int (x+6)^{\frac{5}{2}}\,dx$

Evaluating the remaining integral,

$\int 25x(x+6)^{\frac{3}{2}}dx = 10x\,(x+6)^{\frac{5}{2}} - \frac{20}{7}(x+6)^{\frac{7}{2}} + c$

(ii) $\int \frac{7x}{(x-1)^2}\,dx$

Let $f(x) = 7x$, then $f'(x) = 7$. Let $g'(x) = (x-1)^{-2}$, then $g(x) = \int (x-1)^{-2}\,dx = -(x-1)^{-1}$

$\int \frac{7x}{(x-1)^2}\,dx = 7x\,[-\,(x-1)^{-1}] - \int [-(x-1)^{-1}\,7\,dx = -7x\,(x-1)^{-1} + 7\int [(x-1)^{-1}\,dx$

Integrating again

$\int \frac{7x}{(x-1)^2}\,dx = -7x\,(x-1)^{-1} + 7\ln|x-1| + c = \frac{-7x}{x-1} + 7\ln|x-1| + c$

6.2.2 The Definite Integral

In this section we are going to discuss about the definite integral. Definite Integrals are integrals which have definite numerical values. If we take a particular value of x, say x = a, in the domain of f (x), then substitution of a into right hand side of the below equation.

$\int f(x)\,dx = F(x) + c$

i.e.; F (a) + c which is also not definite in value since the constant (c) is unknown. Similarly, if we take x = b (assuming that b > a), in the domain of f (x), then the substitution of x = b into the right-hand side of the above equation gives F (b) + c. Thus, the difference of these two values can be expressed as:

$$\{F(b) + c\} - \{F(a) + c\} = F(b) - F(a)$$

And we get a definite numerical value as the arbitrary constant of integration (c) vanishes. This definite numerical value is called definite integral of f (x) from a to b and is denoted by,

$$\int_a^b f(x)dx$$

Where a is called lower limit of the integration and b is called upper limit of the integration. The definite integral is a real number which can be evaluated by using the fundamental theorem of calculus.

The fundamental theorem of calculus states that the numerical value of the definite integral of a continuous function f (x) over the interval from a to b is given by the indefinite integral F (x) + c evaluated at the upper limit of the integration b, minus the same indefinite integral F (x) + c evaluated at the lower limit of the integration a. Since c is common to both, the constant of integration is eliminated in subtraction.

The procedure for evaluating a definite integral of f(x) from a to b is as follows:

(i) Integrate f (x) to get F(x) + c

(ii) Evaluate the values of F (x) at x = a and x = b;

(iii) Obtain the difference of F(b) – F(a) which is denoted by $\int_a^b f(x)dx$.

Expressed mathematically.

$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Where the symbol I_a^b, $]_a^b$, or $[\ldots.]_a^b$ indicates that b and a are to be substituted successively for x.

Let us now take a few numerical examples.

Illustration 6.2.2

The definite integrals given below.

(i) $\int_3^6 20x\,dx$ (ii) $\int_2^5 (8x^7 + 6x)$ (iii) $\int_3^6 2x^2\,dx$

Solution:

(i) $\int_3^6 20x\,dx = 10x^2]_3^6 = 10(6)^2 - 10(3)^2 = 360 - 90 = 270$

(ii)$\int_2^5 (8x^7 + 6x)$

$\int_2^5 (8x^7 + 6x)\,dx = [x^8 + 3x^2]_2^5 = [(5)^8 + 3(5)^2] - [(2)^8 + 3(2)^2] = 390700 - 268 = 390432$

(iii) $\int 2x^2\,dx = 2\,\frac{x^{2+1}}{2+1} = \frac{2}{3}x^3$

$\int_3^6 2x^2\,dx = \frac{2}{3}(6)^3 - \frac{2}{3}(3)^3$

$= \frac{2}{3}(6)^3 - \frac{2}{3}(3)^3$

$= \frac{2}{3}[216 - 27] = \frac{378}{3} = 126$

Properties of Definite Integrals

1. Reversing the order of the limits changes the sign of the definite integral.
   $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero.
   $\int_a^a f(x)dx = F(a) - F(a) = 0$
3. The definite integral can be expressed as the sum of component sub integrals.
   $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \quad a \leq b \leq c$
4. The sum of difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.
   $\int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^b [f(x) \pm g(x)]\,dx$
5. The definite integral of a constant time a function is equal to the constant times the definite integral of the function.
   $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

Illustration 6.2.3

Evaluate the definite integrals given below.

(i) $\int_1^3 3x^5 dx$ (ii) $\int_5^5 (4x+3)dx$ (iii) $\int_0^4 12x\,dx$

Solution

(i) $\int_1^3 3x^5 dx = -\int_1^3 3x^5 dx$

$\int_1^3 3x^5 dx = \frac{1}{2} x^6]_1^3 = \frac{1}{2} (3)^6 - \frac{1}{2} (1)^6 = \frac{1}{2} 729 - \frac{1}{2} 1 = \frac{728}{2} = 364$

Checking this answer,

$\int_3^1 3x^5 dx = \frac{1}{2} x^6 I_3^1 = \frac{1}{2} (1)^6 - \frac{1}{2} (3)^6 = -364$

(ii) $\int_5^5 (4x + 3)dx = 0$

Checking this answer,

$\int_5^5 (4x + 3)dx = [x^4 + 3x]_5^5 = [(5)^4 + 3(5)] – [(5)^4 + 3(5)] = 0$

(iii) $\int_0^4 12x\, dx = \int_0^3 12x\, dx + \int_3^4 12x\, dx$

$\int_0^4 12x\, dx = 6x^2]_0^4 = 6(4)^2 - 6(0)^2 = 96$

$\int_0^3 12x\, dx = 6x^2]_0^3 = 6(3)^2 - 6(0)^2 = 54$

$\int_3^4 12x\, dx = 6x^2 I_3^4 = 6(4)^2 - 6(3)^2 = 42$

Checking this answer, 96 = 54 + 42

6.2.3 The Definite Integral as an area under the curve

The definite integral can also be interpreted geometrically as an area under a given curve. There is no geometric formula for the area under an irregularly shaped curve, such as y = f(x) between x = a and x = b in the figure shown below (a). If the interval [a, b] is divided into n subintervals [x_1, x_2], [x_2, x_3] etc., and rectangles are constructed such that the height of each is equal to the smallest value of the function in the subinterval, as shown in the figure (b) then the sum of the areas of the rectangles $\sum_{i=1}^{n} f(x_i)\, \Delta x_i$, called a Reimann sum, will approximate, but underestimate, the actual area under the curve.

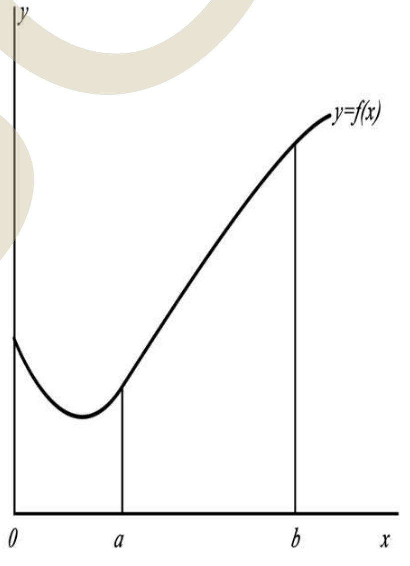

Fig 6.2.1 Area under the curve (a)

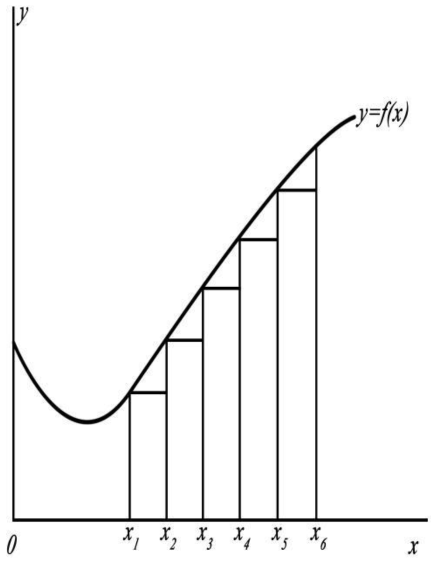

Fig 6.2.2 Area under the curve(b)

The smaller the subintervals (the smaller the Δx_i), the more rectangles are created and the closer the combined area of the rectangles $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$, approaches the actual area under the curve. If the number of subintervals is increased so that n → ∞ each subinterval becomes infinitesimal ($\Delta x_i = dx_i = dx$) and the area A under the curve can be expressed mathematically as

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

The area under a graph of a continuous function such as that in the above figure from a to b (a < b) can be expressed more concisely as the definite integral of f(x) over the interval a to b. Put mathematically,

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

The Indefinite Integral

The standard notation used to denote the integration of f (x) to derive F (x) + c is

$$\int f(x)\, dx = F(x) + c$$

Here, the symbol ∫ is an integral sign, f (x) is the integrand, dx is the integration operation to be performed with respect to x variable and c is the constant of integration. Here, the left-hand side of the equation is called "the indefinite integral of f of x with respect to x", because $\int f(x)\,dx = F(x) + c$, we can refer to $\int f(x)\,dx$ as an integral of f (x). Since integral $\int f(x)\,dx$ is expressed as a function of x in the form $F(x) + c$ with no definite numerical value.

The more details regarding the indefinite integrals have already discussed in Unit 1 in detail with suitable examples.

Recap

- If an integrand is a product or quotient of differentiable functions of x and cannot be expressed as a constant multiple of u du/dx, integration by parts is frequently useful.
- Integrals known as definite integrals which have definite numerical values.
- The definite numerical value is called definite integral of f (x) from a to b and is denoted by $\int_a^b f\,(x)dx,$ Where a is called lower limit of the integration and b is called upper limit of the integration.
- Reversing the order of the limits changes the sign of the definite integral $\int_a^b f\,(x)dx = -\int_a^b f\,(x)dx$
- If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero $\int_a^a f\,(x)dx$ = F (a) – F (a) = 0
- The definite integral can be expressed as the sum of component sub integrals $\int_a^c f\,(x)dx = \int_a^b f\,(x)dx + \int_b^c f\,(x)dx \quad a \leq b \leq c$
- The sum of difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions $\int_a^b f\,(x)dx \pm \int_a^b g(x)dx = \int_a^b [f\,(x) \pm g\,(x)]\,dx$
- The definite integral of a constant times a function is equal to the constant times the definite integral of the function $\int_a^b kf\,(x)dx = k\int_a^b f\,(x)dx$
- The definite integral can be interpreted geometrically as an area under a given curve.
- The standard notation used to denote the integration of f (x) to derive F (x) + c is $\int f\,(x)\,dx = F\,(x) + c$

Objective Questions

1. Find ∫ ln x dx.
2. Find $\int \frac{2x}{(x-8)^3}\,dx$

3. Find $\int x\, e^{x}\, dx$.
4. Find $\int 6x\, e^{x+7}\, dx$.
5. Find $\int_1^3 x\,(x+1)^3\, dx$.
6. Evaluate $\int_1^{10} 3x^2\, dx$.
7. Write the equation for integration by parts.
8. Write the standard notation for indefinite integrals
9. How is definite integral interpreted geometrically?
10. What is a and b in $\int_a^b f(x)dx$?

Answers

1. $x(\ln x - 1) + c$
2. $\int \frac{2x}{(x-8)^3}\, dx = -x\,(x-8)^{-2} - (x-8)^{-1} + c = \frac{-x}{(x-8)^2} - \frac{1}{x-8} + c$
3. $e^{x}(x-1) + c$
4. $\int 6x\, e^{x+7}\, dx = 6x\, e^{x+7} - 6x\, e^{x+7} + c$
5. 138.4
6. 999
7. $\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx$
8. $\int f(x)\, dx = F(x) + c$
9. As an area under the curve
10. a is called lower limit of the integration and b is called upper limit of the integration

Assignments

1. Explain the properties of definite integrals with examples
2. Discuss about definite integrals and explain definite integral as an area under a curve.
3. Differentiate between definite and indefinite integrals.

Suggested Readings

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2. Edward T. Dowling Ph.D (2004), "Introduction to Mathematical Economics", Third Edition, Schaum's Outlines, Tata McGraw-Hill Publishing Company Limited
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5. Allen R G D, "Mathematical Analysis for Economists", MACMILLAN India Limited

UNIT 3

Economic Applications of Integral Calculus

Learning Outcomes

After completing this unit, the learner will be able to

- use integration for economic analysis
- calculate total revenue using integration
- compute the total cost of production
- apply integral calculus to solve real-world economic problems

Prerequisites

Imagine you are the captain of a ship navigating the vast ocean of economics. To sail these intricate waters, you need a set of tools, and Integral Calculus is your compass and sextant. These tools are your guide to understanding the depths of economic concepts such as Total Cost, Total Revenue, Total Utility, Consumer Surplus, and Producer Surplus.

Think of Total Cost as your ship's expenses. Just like a captain accounting for every piece of cargo, you will learn how to calculate all the costs involved in producing goods and services. Integral Calculus helps you break down these costs and get an accurate picture of what it takes to keep your economic ship afloat. Now, consider Total Revenue as the earnings from your voyage. Integral Calculus allows you to measure the income generated by selling your products or services. It is like keeping track of the treasures you have collected along your economic journey. Total Utility is the satisfaction you and your crew derive from your economic voyage.

So, why learn these economic applications of Integral Calculus? Because, as the captain of the economic ship, these tools are your way to navigate the waters of production, revenue, and consumer and producer satisfaction. They help you optimize your economic journey, maximise profits, and steer your ship through the complexities of market dynamics. With these tools in your arsenal, you will not only have a comprehensive understanding of the economic seascape but also the ability to make informed decisions that can lead to prosperous voyages and successful economic adventures. Your journey through the realm of Economic Applications of Integral Calculus is a thrilling expedition into the heart of economics, where every calculation and decision can make or break your voyage to economic success.

Keywords

Total Cost, Total Revenue, Total Utility, Consumer Surplus, Producer Surplus

Discussion

6.3.1 Introduction

As we have already discussed the definite and indefinite integrals in the last unit. Now, we are going to discuss about the important economic applications of integral calculus. The derivation of total function from marginal function in economic analysis is one of the important applications of integral. We need to find out the total cost function from marginal cost function or total revenue function from marginal revenue function or saving function from marginal propensity to save function or total utility function from marginal utility function.

6.3.2 Total Cost

If C is the cost of producing an output Q then marginal cost function, MC = $\frac{dC}{dQ}$., i.e., the differentiation of a total cost function gives marginal cost. In the same way the integration of a marginal cost function will give the total cost function, C = $\int$ (MC) dQ + c. Thus, if the marginal cost is given by,

MC = 2aQ + b

Where Q is the output, the total cost will be given by

TC = $\int$ (2aQ + b) dQ

$$= 2a\frac{Q^2}{2} + bQ + c$$

Or $TC = aQ^2 + bQ + c$

Here, c is an arbitrary constant,

In the above equation, ($TC = aQ^2 + bQ + c$) consist of total fixed costs and total variable costs. Here, the first part of the equation $aQ^2 + bQ$ represents the variable costs and the constant, c, represents the total fixed cost, i.e; when the firm does not produce anything, it must bear the fixed cost and so the costs at Q = 0 represents the fixed cost. Thus C (0) = c.

Average Cost function, $AC = \frac{C}{Q}$, $Q \neq 0$

Now let us discuss few numerical examples,

Illustration 6.3.1

If the Marginal cost (MC) function of a firm is given by MC = C'(Q) = 7Q + 1 and the total fixed cost is 40, Find the total cost.

Solution:

The total cost will be obtained by integrating C'(Q) with respect to Q. Thus

$TC = C(Q) = \int (7Q + 1)\, dQ$

$= \int 7Q + 1 + \int dQ$

$= \frac{7}{2}Q^2 + Q + c.$

When Q = 0, C (0) = c = 40.

$\therefore TC = \frac{7}{2}Q^2 + Q + 40$

Illustration 6.3.2

The marginal cost function of manufacturing Q units of a commodity is $9 + 20x - 9x^2$. Find the total cost and average cost, given that the total cost of producing 1 unit is 25.

Solution:

The total cost will be obtained by integrating MC'(Q) with respect to x. Thus

$MC = 9 + 20x - 9x^2$

$TC = \int (MC)\, dx + c$

$= \int (9 + 20x - 9x^2)\, dx + c$

$= 9x + \frac{20x^2}{2} - \frac{9x^3}{3} + c$

$= 9x + 10x^2 - 3x^3 + c$ ---------------------------------------(1)

Given, when $x = 1$, $TC = 25$

$\therefore (1) \rightarrow 25 = 9 + 10 - 3 + c$

$\rightarrow c = 9$

$\therefore$ Total Cost function, $C = 9x + 10x^2 - 3x^3 + 9$

Average Cost function, $AC = \frac{C}{Q}$, $Q \neq 0$

$$= 9 + 10x - 3x^2 + \frac{9}{Q}$$

Illustration 6.3.3

The marginal cost function of manufacturing Q units of a commodity is $6x^2 - 4x + 12$. If there is no fixed cost find the total cost and average cost function.

Solution:

Given that,

$MC = 6x^2 - 4x + 12$

$C = \int (MC)\, dx + c$

$= \int (6x^2 - 4x + 12)\, dx + c$

$= \frac{6}{3}x^3 - \frac{4}{2}x^2 + 12x + c$

$= 2x^3 - 2x^2 + 12x + c$

No fixed cost $\rightarrow c = 0$

$\therefore$ Total cost, $C = 2x^3 - 2x^2 + 12x$

Average Cost function, $AC = \frac{C}{Q}$, $Q \neq 0$

$= 2x^2 - 2x + 12$

Illustration 6.3.4

The marginal cost function is given by $MC = C'(Q) = 24Q^2 - 8Q + 7$ and fixed cost is 200. Find the total cost function.

Solution:

Total Cost function is given by integration of C'(Q) function.

$\therefore TC = C(Q) = \int C^1 (Q)\, dQ$

$= \int (24Q^2 - 8Q + 7)\, dQ$

$= \int 24Q^2\, dQ - \int 8Q\, dQ + \int 7dQ$

$$= \frac{24}{3} Q^3 - \frac{8}{2} Q^2 + 7Q + c$$

Or $C(Q) = 8Q^3 - 4Q^2 + 7Q + c$

When $Q = 0$, $C(0) = c = 200$

$\therefore TC = 8Q^3 - 4Q^2 + 7Q + 200$

Illustration 6.3.5

The marginal cost function of manufacturing Q units of a commodity is $5 - 4x - x^2$. If the fixed cost is 100, find the total cost and average cost functions.

Solution:

Given that,

$MC = 5 - 4x - x^2$

$C = \int (MC)\, dx + c$

$= \int (5 - 4x - x^2)\, dx + c$

$= 5x - 2x^2 + \frac{x^3}{3} + c$

Given that fixed cost C = 100

$\therefore (1) \rightarrow c = 100$

$\therefore$ Total cost, $C = 5x - 2x^2 + \frac{x^3}{3} + 100$

Average Cost function, $AC = \frac{C}{Q}$, $Q \neq 0$

$= 5 - 2x + \frac{x^2}{3} + \frac{100}{x}$

Illustration 6.3.6

Given the marginal cost function, $MC = C'(Q) = Q^2 - 8Q + 5$, Find the level of output (Q) at which the Average Variable Cost (AVC) will be minimum?

Solution:

The total cost will be obtained by integrating C'(Q) with respect to Q. Thus

$\therefore TC = C(Q) = \int (Q^2 - 8Q + 5)\, dQ$

$$= \frac{1}{3} Q^3 - 8 \frac{1}{2} Q^2 + 5Q + c$$

The Total Variable Cost (TVC) $= \frac{1}{3} Q^3 - 4Q^2 + 5Q$

$\therefore$ Average variable Cost (AVC) = $\frac{TVC}{Q} = \frac{1}{3}Q^2 - 4Q + 5$

To minimise AVC, we need to do the following steps.

$\frac{d}{dQ}$ (AVC) = 0 and $\frac{d^2}{dQ^2}$ (AVC) > 0

Now $\frac{d}{dQ}$ (AVC) = 0 gives

$\frac{2}{3}Q - 4 = 0$

Or Q = 6

Second order condition gives

$$\frac{d^2\,(AVC)}{dQ^2} = \frac{2}{3} > 0$$

$\therefore$ Q = 6 minimises the Average Variable Cost.

6.3.3 Total Revenue

If R is the revenue of producing an output Q then marginal revenue function, MR = $\frac{dR}{dQ}$.,

i.e., the differentiation of a total revenue function gives marginal revenue. In the same way the integration of a marginal revenue function will give the total revenue function, R = $\int$ (MR) dQ + c. Let us consider a linear marginal revenue function of the form

$$MR = R'(Q) = a - bQ$$

Where Q is the quantity; a and b are parameters. Thus, total revenue will be given by

$$TR = R(Q) = \int R'(Q)\, dQ$$

$$\text{Or } R(Q) = \int (a - bQ)\, dQ = aQ - \frac{b}{2}Q^2 + c \text{ ------------------- (1)}$$

It should be noted that total revenue is always zero when output is zero.

$$\therefore R(0) = 0 = c$$

Thus, the arbitrary constant, c is always zero while integrating a marginal revenue function.

From (1)

$$\therefore R(Q) = aQ - \frac{b}{2}Q^2$$

If the total revenue R = 0, when Q = 0,

Demand function, p = $\frac{R}{Q}$, $Q \neq 0$

Now let us discuss few numerical examples,

Illustration 6.3.7

The marginal revenue function is given by R'(Q) = 60 – 0.8Q. Find the total revenue function?

Solution:

The total revenue function is given by the integral of R' (Q)

$$TR = R(Q) = \int R'(Q)\, dQ$$

$$\int (60 - 0.8Q)\, dQ = 60Q - 0.8\frac{Q^2}{2} + c$$

$$= 60Q - 0.4\, Q^2$$

Illustration 6.3.8

If the marginal revenue for a commodity is $MR = 12 - 9x^2 + 4x$. Find the total revenue and demand function.

Solution:

Given that, $MR = 12 - 9x^2 + 4x$

$R = \int (MR)\, dx + c$

$= \int (12 - 9x^2 + 4x)\, dx + c$

$= 12x - 3x^3 + 2x^2 + c$

Since R = 0 when Q = 0, c = 0

$\therefore R = 12x - 3x^3 + 2x^2$

$p = \frac{R}{Q}, Q \neq 0 \rightarrow p = \frac{R}{x} = \frac{12x - 3x^3 + 2x^2}{x} = 12 - 3x^2 + 2x$

Illustration 6.3.9

Given the marginal function MR = R' (Q) = 40 - Q. Find out the price of the product when Q = 20. Also find how much price will change when Q increases to 30.

Solution:

The total revenue function is given by the integral of R' (Q)

$$TR = R(Q) = \int R'(Q)\, dQ$$

$$\int (40 - Q)\, dQ = 40Q - \frac{Q^2}{2} + c$$

Since c = 0 when Q = 0,

$$\therefore TR = 40Q - \frac{Q^2}{2}.$$

When Q = 20, TR = 40 × 20 – $\frac{(20)^2}{2}$ = 800 - $\frac{400}{2}$ = 800 - 200 = 600

∴ The price will be AR = $\frac{TR}{Q}$ = $\frac{600}{20}$ = 30

When Q = 30, TR = 40×30 – $\frac{(30)^2}{2}$ = 1200 - $\frac{900}{2}$ = 1200 - 450 = 750

∴ The price will be AR = $\frac{TR}{Q}$ = $\frac{750}{30}$ = 25

When there is an increase in quantity from 20 to 30 units, the price reduces to 25 from 30.

Illustration 6.3.10

For the marginal revenue function MR = 5 – 4x – x^2. Find the revenue function and demand function.

Solution:

Given that, MR = 5 – 4x – x^2

R = ∫ (MR) dx + c

= ∫ (5 – 4x – x^2) dx + c

= 5x - $2x^2$ + $\frac{x^3}{3}$ + c

Since R = 0 when Q = 0, c = 0

∴ R = 5x - $2x^2$ + $\frac{x^3}{3}$ + c

p = $\frac{R}{Q}$, Q ≠ 0 → p = 5 - 2x + $\frac{x^2}{3}$

Illustration 6.3.11

The marginal revenue function is given by MR = R'(Q)=100–8Q. Find the point elasticity of demand when Q = 10

Solution:

The point elasticity is given by

$$\mathbf{e = \frac{AR}{AR - MR}}$$

The TR is given by integration of MR

TR = R(Q) = ∫ (100-8Q) dQ

$$= 100Q - 8\frac{Q^2}{2} + c$$

$$= 100Q – 4Q^2 \ (\therefore c = 0)$$

$$\therefore AR = \frac{TR}{Q} = + \frac{100Q - 4Q^2}{Q} = 100 – 4Q.$$

Thus, when Q = 10,

MR = 100 – 80 = 20

AR = 100 – 40 = 60

$$\therefore e = \frac{60}{60-20} = \frac{3}{2} = 1.5$$

Illustration 6.3.12

The marginal revenue and marginal cost function of a firm are given by

$MR = R'(Q) = 100 - \frac{1}{2}Q$

$MC = C'(Q) = 0.4\,Q^2 - \frac{1}{3}Q + 4$ and the total fixed cost is 20. Find out total profit when the firm produces and sells 20 units of output.

Solution:

The total revenue is given by the integral of marginal revenue function.

$$TR = R(Q) = \int (100 - \frac{1}{2}Q)\,dQ$$

$$= 100Q - \frac{1}{2}\frac{1}{2}Q^2 + c = 100Q - \frac{Q^2}{4} \ (\therefore c = 0 \text{ when } Q = 0)$$

$$TC = C(Q) = \int (0.4Q^2 - \frac{1}{3}Q + 4)\,dQ$$

$$= \frac{0.4}{3}Q^3 - \frac{1}{3}\frac{1}{2}Q^2 + 4Q + c$$

$$= \frac{0.4}{3}Q^3 - \frac{1}{6}Q^2 + 4Q + 20 \ (\therefore \text{fixed cost} = c = 20)$$

The production function is defined as

Profit = TR – TC

$$= 100Q - \frac{Q^2}{4} - \frac{0.4}{3}Q^3 + \frac{1}{6}Q^2 - 4Q - 20$$

When Q = 20,

$$\text{Profit} = 100 \times 20 - \frac{(20)^2}{4} - \frac{0.4}{3}(20)^3 + \frac{1}{6}(20)^2 - 4 \times 20 - 20$$

= 800

6.3.4 Total Utility

If the total utility function is q^2, then the marginal utility function is 2q. i.e;

$$U = q^2, \text{ then } \frac{du}{dq} = 2q$$

It is the rate of change of total utility (U) with respect to quantity consumed (q) and is the marginal utility function. Now, the marginal utility function is 2q, then the total utility function $U = q^2$

Illustration 5.2.22

Suppose the marginal utility function for good 'X' is given as: MU(x) = 10 – x, where x is the quantity of good 'X' consumed. Find the TU.

Solution

$MU = 10 - x$

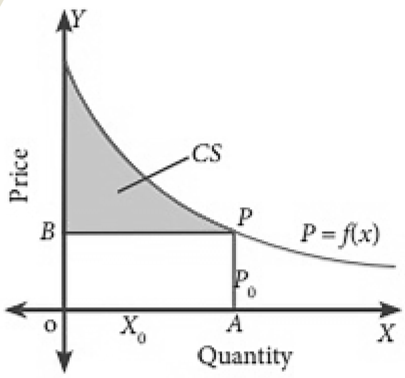
$TU = \int MU\, dx$

$= \int (10 - x)\, dx$

$= 10x - \frac{x^2}{2} + c$

6.3.5 Consumer surplus

Consumer surplus represents the benefit or gain that consumers receive when they can purchase a good or service at a price lower than the maximum price, they are willing to pay. The demand function reveals the relationship between the quantities that the people would buy at a given price. It can be expressed as P = f (x). Let us assume that the demand of the product $X = X_0$ when the price is P_0. But there can be some consumer who is ready to pay more than P_0 for the same quantity X_0. Any consumer who is ready to pay the price more than P_0 gains from the fact that the price is only P_0. This gain is called the consumer's surplus.



Mathematically the Consumer's Surplus (CS) can be defined as

CS = (Area under the demand curve from X = 0 to X = X_0) – (Area of the rectangle OAPB)

CS = $\left[\int_0^{X_0} f(x)dx\right]$ - X_0P_0

Illustration 6.3.14

The demand function of a commodity is y = 36 − x^2. Find the consumer's surplus for

y_0 = 11

Solution:

Given y = 36 − x^2 and y_0 = 11

Let us first find the value of x

When y = 11 the value of x will be,

11 = 36 - x^2

x^2 = 36 – 11

x^2 = 25

i.e., x = 5 or x = -5

since x cannot be negative, we take x = 5

CS = $\left[\int_0^{X_0} f(x)dx\right]$ - X_0P_0

CS = $\left[\int_0^{5} 36 - x^2 dx\right] - 5 \times 11$

CS = $[36x - \frac{x^3}{3}]_0^5 - 5 \times 11$

$= 36 \times 5 - \frac{125}{3} - 55$

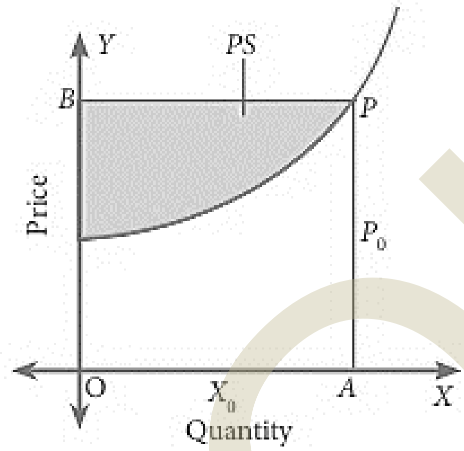
$= 180 - \frac{125}{3} - 55$

$= \frac{250}{3}$

Hence the consumer surplus is $\frac{250}{3}$ units.

6.3.6 Producer surplus

Producer surplus represents the benefit or gain that producers receive when they can sell a good or service at a price higher than their minimum acceptable price. A supply function g(x) represents the quantity that can be supplied at a price P. Let P_0 be the market price for the corresponding supply X_0. But there can be some producers who are willing to supply the commodity below the market price gain from the fact that the price is P_0. This gain is called the producer's surplus. It is represented in the following diagram.



Mathematically, producer's surplus (PS) can be defined as,

PS = (Area of the rectangle OAPB) − (Area below the supply function from $X = 0$ to $X = X_0$)

$$PS = X_0P_0 - \left[\int_0^{X_0} g(x)dx\right]$$

Illustration 6.3.15

Find the producer's surplus defined by the supply curve when $X_0 = 5$.

Solution: g(x) = 4x + 8 and $X_0 = 5$

$P_0 = 4 \times 5 + 8$

$P_0 = 28$

$$PS = X_0P_0 - \left[\int_0^{X_0} \mathrm{g(x)dx}\right]$$

$$PS = (5 \times 28) - \left[\int_0^{5} 4x + 8\, dx\right]$$

$= 140 - [\,2\,x^2 + 8x]_0^5$

$= 140 - [\,50 + 40]$

$= 50$

Hence the producer's surplus= 50 units.

Recap

- The derivation of total function from marginal function in economic analysis is one of the important applications of integral.
- The total cost function is derived from marginal cost function or total revenue function from marginal revenue function or saving function from marginal propensity to save function or total utility function from marginal utility function.
- If C is the cost of producing an output Q then marginal cost function, MC = $\frac{dC}{dQ}$., i.e., the differentiation of a total cost function gives marginal cost.
- If R is the revenue of producing an output Q then marginal revenue function, MR = $\frac{dR}{dQ}$., i.e., the differentiation of a total revenue function gives marginal revenue.
- The integration of a marginal revenue function will give the total revenue function.
- If the total utility function is q^2, then the marginal utility function is 2q. i.e; U = q^2, then $\frac{du}{dq}$ = 2q

Objective Questions

1. Marginal Cost is given by MC = $\frac{dTC}{dQ}$ = 35 + 30Q - 9Q^2. Fixed Cost is 65. Find the (a) Total Cost (b) Average Cost (c) Variable Cost functions
2. Marginal Revenue is given by MR = $\frac{dTR}{dQ}$ = 90 - 2Q - 2Q^2. Find (a) The TR function and (b) The demand function P = f(Q)
3. Find (a) Total Revenue Function and (b) The Demand function, given MR = 95 - 4Q - Q^2
4. The Marginal Revenue Function is given by MR = R'(Q) = 50 - 4Q. Find the point elasticity of demand when Q = 10
5. If MC = 2aQ + b, what is TC?
6. If MR = a – bQ, what is TR?
7. If MC = 2q, then what is TU?
8. What is the benefit or gain that producers receive when they can sell a good or service at a price higher than their minimum acceptable price?

Answers

1. (a) 35Q + 15Q^2 - 3Q^3 + 65

   (b) 35 + 15Q - 3Q^2 + 65/Q

   (c) 35Q + 15Q^2 - 3Q^3

2. (a) 90Q - Q^2- $\frac{2}{3}$ Q^3

   (b) 90 - Q - $\frac{2}{3}$ Q^2

3. (a) 95Q - 2Q^2 - $\frac{1}{3}$ Q^3

   (b) 95 - 2Q - $\frac{1}{3}$ Q^2

4. 1.5

5. $TC = aQ^2 + bQ + c$
6. $R(Q) = aQ - \frac{b}{2}Q^2$
7. $U = q^2$
8. Producer surplus

Assignments

1. The demand and supply function of a commodity are $P_d = 18 - 2x - x^2$ and $P_s = 2x - 3$. Find the consumer's surplus and producer's surplus at equilibrium price.
2. How we can find TC, TR, and TU from their respective marginal functions? Explain with examples.
3. Explain with examples on how consumer surplus and producer surplus is calculated using integration

Suggested Readings

1. Carl P. Simon and Lawrence Blume (2014): Mathematics for Economists, Viva Books.
2. Chiang and Wainwright (2018): Fundamental Methods of Mathematical Economics,4th Ed, McGraw-Hill.
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2. Edward T. Dowling Ph.D (2004), “Introduction to Mathematical Economics”, Third Edition, Schaum’s Outlines, Tata McGraw-Hill Publishing Company Limited
3. Monga, Gopal Sohanlal (2001), “Mathematics and Statistics for Economics.”, Vikas Publishing House
4. Gupta, S. C., and V. K. Kapoor (2020), “Fundamentals of Mathematical Statistics”, Sultan Chand & Sons
5. Allen R G D, “Mathematical Analysis for Economists”, MACMILLAN India Limited

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Model Question Paper- I

SECOND SEMESTER - BA ECONOMICS EXAMINATION

DISCIPLINE CORE - 02- B21EC02DC - MATHEMATICS FOR ECONOMICS

(CBCS - UG)

2022-23 - Admission Onwards

Time: 3 Hours   Max Marks: 70

Section A - Objective Type Questions

Answer any 10 questions. Each question carries 1 mark

(10 X 1=10 marks)

1. Find the L C M of 3 and 12.
2. Form the quadratic equation with roots 2 and 3 ?
3. If $\begin{bmatrix} 2 & y \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ x & 4 \end{bmatrix}$, find x and y.
4. Which arithmetic operation does not satisfy the closure property in the set of all positiveintegers?
5. Define null matrix.
6. Find the value of the determinant $\begin{vmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 1 & 6 & 0 \end{vmatrix}$.
7. Define eigen values.
8. What are the conditions for finding the maximum value of a function?
9. Define linear equation.

10. Find the partial derivative of the function $z = 4xy + x$ with respect to x.

11. What is associative law of matrix addition?

12. Find the marginal cost of x, if the total cost function is $TC = x^3 + 6xy$.

13. Define MPS.

14. Evaluate the integral $\int(x^4 + 2)\ dx$.

15. If the upper limit of integration equal to the lower limit of integration, what is the value of the definite integral?

Section B- Very Short Questions

Answer any 10 questions. Each question carries 2 marks

(10x2=20 marks)

16. List out the four methods to factorize numbers and algebraic expressions.

17. Define transpose of a matrix with examples.

18. The ratio between two quantities is 9:8. If the first quantity is 45, find the other quantity.

19. Multiply the matrix $A = \begin{bmatrix} 2 & -5 \\ 3 & 4 \\ -5 & 6 \\ 7 & -1 \end{bmatrix}$ by a scalar 3 and find the new matrix.

20. Define adjoint of a matrix with an example.

21. Let the matrix $A = \begin{bmatrix} 5 & 3 & 7 \\ 1 & 4 & 2 \\ 0 & 6 & x \end{bmatrix}$ and the trace of matrix A is 6. Find the value of x.

22. Find the derivative of $y = x^2 \log x$

23. Find the co factor of the elements 5 and 8 of the matrix $\begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 6 \\ 3 & 7 & 8 \end{bmatrix}$.

24. Define partial derivatives.

25. Find the eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

26. Explain first order derivative and second order derivative.

27. What is Average Propensity to Consume?

28. The marginal cost function of manufacturing Q units of a commodity is $5- 4x-x^2$. If the fixed cost is 100, find the total cost and average cost functions.

29. Explain the Logarithmic Rule of Integration.

30. Evaluate the definite integral $\int_1^5 (x^2 + 4)\, dx$.

Section C- Short Answer

Answer any 5 questions. Each question carries 4 marks

(5x4=20 marks)

31. Explain the law of exponents.
32. . Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. Prove that A(B + C) = AB + AC
33. What are the properties of matrix addition?
34. Find the partial derivatives of z = $\frac{x^2 y}{x+y}$ with respect to x and y.
35. If A = $\begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, then what is the value of x?
36. Explain the various uses of matrices in economics.
37. Find $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x^2 - 5x + 6}$.
38. Explain the rules of partial derivative.
39. Explain in detail about MRTS.
40. Evaluate the integral $\int_1^3 x\,(x+1)^3$ dx.

Section D- Long Answer/Essay

Answer any 2 questions. Each question carries 10 marks

(2x10=20 marks)

41. Examine the properties of arithmetic operations with suitable examples.
42. Solve the linear equations using Cramer's Rule.
    6x + 4y + 2z = 12, 4x - 6y + 6z = 4 and 2x + 2y + 2z = 8.
43. Mathematically illustrate and explain the concept of MRS.
44. The marginal revenue and marginal cost function of a firm are given by $MR = R'(Q) = 40 - \frac{1}{2} Q$, $MC = C'(Q) = 0.5\, Q^2 - \frac{1}{2} Q + 5$ and the total fixed cost is
    Find out total profit when the firm produces and sells 30 units of output.

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Model Question Paper- II

SECOND SEMESTER - BA ECONOMICS EXAMINATION

DISCIPLINE CORE - 02- B21EC02DC - MATHEMATICS FOR ECONOMICS

(CBCS - UG)

2022-23 - Admission Onwards

Time: 3 Hours Max Marks: 70

Section A - Objective Type Questions

Answer any 10 questions. Each question carries 1 mark

(10x1=10 marks)

1. Which is the identity element under addition that does not belong to the set of natural numbers?
2. Write the trace of the matrix $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 2 & 1 \\ -4 & 1 & 1 \end{bmatrix}$.
3. What is the difference between rational and irrational number?
4. What is the unique feature of a diagonal matrix?
5. Examine whether the function $y = 120 - 6x$ is convex or concave.
6. What condition must be met for the multiplication of two matrices to be possible?
7. What is the value of determinant for a triangular matrix?
8. Find the Marginal utility, , MU_x, for the function $U = x^2y^3$.
9. What is the nature of the quadratic equation if the discriminant is zero?
10. What do you mean by input-output analysis?

11. Find $Lt_{n \to \infty} \frac{n}{n+1}$.
12. Find the derivative $\frac{dy}{dx}$ of $y = 6x^2 + 3x$.
13. Find the partial derivative of the function $z = xy + 2x$ with respect to y.
14. Evaluate the integral $\int 3^{4x}\,dx$. .
15. Find the value of $\int_5^5 (4x+3)dx$.

Section B- Very Short Questions

Answer any 10 questions. Each question carries 2 marks

(10x2=20 marks)

16. Distinguish between LCM & HCF.
17. If a bus travels 160 km in 4 hours and a train travels 320 km in 5 hours at uniform speed, then find the ratio of the distances travelled by them in one hour.
18. Prove that the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$ is symmetric.
19. Define Crammer's rule.
20. . Let matrix $X = \begin{bmatrix} -4 & 8 \\ 4 & 9 \\ 2 & -1 \\ 3 & 5 \end{bmatrix}$ and Y = 3X. Find Y.
21. Distinguish between a row matrix and a column matrix.
22. Find the adjoint of a matrix $\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$.
23. What is eigen decomposition?
24. Find the first and second order partial derivative of the function $z = x^3 + xy$ with respect to x.
25. Write the relationship between AC and MC.
26. Find the values of x for which the function $f(x) = \frac{x^2-9}{x^2-5x+6}$ is discontinuous.
27. Explain total differentiation.
28. The total cost function of a firm is given as $TC = 200+20x^2$ and the total revenue function is given as TR= 10x. Find the profit maximising level of output.

29. Evaluate $\int x(x^2+5)^3 dx$ by substitution method.

30. How are definite and indefinite integrals different? Illustrate with examples.

Section C- Short Answer

Answer any 5 questions. Each question carries 4 marks

(5x4=20 marks)

31. Discuss the properties of addition in real numbers.

32. Let matrix A $\begin{bmatrix} 7 & 9 & -8 \\ 3 & 1 & 12 \\ 6 & 0 & 7 \end{bmatrix}$ and matrix B = A + Unit Matrix of 3 x 3. Find matrix B.

33. Find the total derivative of the function z = $6x^2 + 3y^2$

34. Let matrix A = $\begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & 0 \\ 4 & -2 & -3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ -1 & 3 & 3 \end{bmatrix}$. Find AB.

35. Explain any 5 types of Matrices.

36. A company is considering three methods of production it has to use in producing three goods X, Y and Z. The amount of each good produced by each method is shown in the matrix A.

$$\begin{array}{l c} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{l} Method1 \\ Method2 \\ Method3 \end{array} & \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 5 & 3 & 9 \end{bmatrix} \end{array}$$

The matrix B = $\begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix}$ shows the profit per unit for the goods X, Y and Z respectively. Find which method is the best profit maximization method.

37. Explain the Properties of Definite Integrals.

38. Find the second order derivative $\frac{d^2y}{dx^2}$, of the function y = $x^2 \log x$.

39. Explain in detail the different methods to find limit.

40. Differentiate between APC and MPC.

Section D- Long Answer/Essay Question

Answer any 2 questions. Each question carries 10 marks

(2x10=20 marks)

41. Solve the simultaneous equation
    $x + y + z = 6,\ -x + 2y + 3z = 14,\ -x + y - z = -2.$
42. Explain the properties of determinants with examples.
43. The marginal revenue function is given by $MR = R^{I}(Q) = 100 - 4Q$. Find the revenue function and demand function. Also find the point elasticity of demand when $Q = 20$.
44. Explain in detail the types or degree of price elasticity.

സർവ്വകലാശാലാഗീതം

വിദ്യയാൽ സ്വതന്ത്രരാകണം
വിശ്വപൗരരായി മാറണം
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ഗുരുപ്രകാശമേ നയിക്കണേ

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നീതിവൈജയന്തി പാറണം

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ജ്ഞാനകേന്ദ്രമേ ജ്വലിക്കണേ

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Regional Centres

Kozhikode
Govt. Arts and Science College
Meenchantha, Kozhikode,
Kerala, Pin: 673002
Ph: 04952920228
email: rckdirector@sgou.ac.in

Thalassery
Govt. Brennen College
Dharmadam, Thalassery,
Kannur, Pin: 670106
Ph: 04902990494
email: rctdirector@sgou.ac.in

Tripunithura
Govt. College
Tripunithura, Ernakulam,
Kerala, Pin: 682301
Ph: 04842927436
email: rcedirector@sgou.ac.in

Pattambi
Sree Neelakanta Govt. Sanskrit College
Pattambi, Palakkad,
Kerala, Pin: 679303
Ph: 04662912009
email: rcpdirector@sgou.ac.in

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Sreenarayanaguru Open University
Kollam, Kerala Pin- 691601, email: info@sgou.ac.in, www.sgou.ac.in Ph: +91 474 2966841